COMPUTATIONAL MATHEMATICS

MSC 35A01

THE NUMERICAL SOLUTION OF SOME CLASSES OF THE SEMILINEAR SOBOLEV-TYPE EQUATIONS

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A unique solvability of the Cauchy problem for a class of semilinear Sobolev type equations of the second order is proved. The ideas and techniques, developed by G.A. Sviridyuk for the investigation of the Cauchy problem for a class of semilinear Sobolev type equations of the first order and by A.A. Zamyshlyaeva for the investigation of the high-order linear Sobolev type equations are used. We also used theory of the differential manifolds which was finally formed in S. Leng's works. In article we considered two cases. The first one is when an operator at the highest time derivative is continuously invertible. In this case for any point from a tangent bundle of an original Banach space there exists a unique solution lying in this space as trajectory. The second case when the operator isn't continuously invertible is of great interest for us. Hence we used the phase space method. Besides the Cauchy problem we considered the Showalter – Sidorov problem. The last generalizes the Cauchy problem and is more natural for Sobolev-type equation. In the last section described an algorithm of the numerical solution of Showalter – Sidorov problem for Sobolev-type equation of the second order.

Keywords: Sobolev-type equation, phase space, Showalter – Sidorov problem, algorithm of the numerical solution.

Introduction

Mathematical models of different physical processes, for instance, filtration of a viscoelastic liquid, creep buckling, vibration of a molecule DNA, shallow-water waves propagation, and the propagation of longitudinal deformation waves in an elastic rod, ion-acoustic waves are described by initial problems for Sobolev-type equations, which are frequently nonlinear. Sviridyuk G.A. and Zagrebina S.A. wrote a good review about nonclassical mathematical models in [1]. The initial-boundary value problems for nonlinear equations often don't have analytic solutions, thus necessity of development of algorithms a numerical method was appeared.

The Cauchy problem for the Sobolev-type equation is not solvable for arbitrary initial values. Before finding a numerical solution we have to define conditions of existence a uniqueness of solution. The one way for investigation of these equations is the phase space method, which was proposed by G.A. Sviridyuk and T.G. Sukacheva in the study of the semilinear Sobolev type equations of the first order [2]. Essence of the method is in reducing of a singular equation to a regular one defined on a subset of original Banach space consisting of admissible initial values, which is understood as a phase space of given equation.

In addition we use the relatively polynomially bounded operator pencil theory, which was developed by A.A. Zamyshlyaeva, and theory of p-bounded operators, developed by G.A. Sviridyuk [3].

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ with boundary $\partial\Omega$ of C^∞ class. Consider *the mathematical model of shallow-water waves propagation*, provided that motion is potential and the law of conservation of mass in layer is fulfilled:

$$(\lambda - \Delta)\ddot{u} = \alpha^2 \Delta u + \Delta f(u), \tag{1}$$

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3}$$

Function $u(x,t)$ defines the height of the wave at time t in point x. Coefficients λ, α are responsible for gravitational constant, deep and Bond number [4, 5].

The mathematical model of *propagation of longitudinal deformation waves in an elastic rod (Boussinesq – Löve model)* is described by following initial-boundary value problem

$$(\lambda - \Delta)\ddot{u} = \alpha(\Delta - \lambda')\dot{u} + \beta(\Delta - \lambda'')u + \Delta f(u), \tag{4}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{5}$$

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x). \tag{6}$$

Function $u(x,t)$ defines the longitudinal deformation of an elastic rod, $\alpha, \beta, \lambda, \lambda', \lambda''$ characterize properties of the rod material such as density, Young's modulus, Poisson's ratio and coefficient of elasticity. If $f(u) = u^3$ then equation (4) is called a damped IMBq equation, it describes the damping shallow-water waves propagation, where α is the coefficient of hydrodynamical damping. A.A. Zamyshlyeva studied linearization equation (4), G. Chen studied non-degenerate case [6].

We consider these mathematical models with Showalter – Sidorov and Cauchy conditions. The Showalter – Sidorov conditions are more general then the Cauchy conditions and are more natural for Sobolev-type equations. Moreover, there is no need for checking that initial values lie in the phase space. Nonlinear nonclassical mathematical models with Cauchy conditions were studied by N.A. Manakova and E.A. Bogatyreva [7].

1. Mathematical model of shallow-water waves

Consider the Cauchy problem

$$u(0) = u_0, \dot{u}(0) = u_1 \tag{7}$$

for the Sobolev-type equation

$$L\ddot{u} = Mu + N(u), \tag{8}$$

where operators $L,\ M \in \mathcal{L}(\mathfrak{U};\ \mathfrak{F})$, $N \in C^\infty(\mathfrak{U};\ \mathfrak{F})$, moreover operator M is $(L,\ 0)$-bonded.

Definition 1. *If a vector-function $u \in C^2((-\tau,\tau);\mathfrak{U})$, $\tau \in \mathbb{R}_+$ satisfies equation* (8) *then it is called a solution of this equation. If it in addition satisfies condition* (7) *then it is called a solution of the problem* (7), (8).

Definition 2.

The set $\mathfrak{P}$ is called a phase space of (8) *if*
(i) for all $(u_0, u_1) \in T\mathfrak{P}$ there exists a unique solution of (7), (8)*;*
(ii) the solution $u = u(t)$ of (8) *lies in $\mathfrak{P}$ as trajectory, i.e. $u(t) \in \mathfrak{P}$ for all $t \in (-\tau, \tau)$.*

Let $\ker L \neq \{0\}$ and operator M be $(L, 0)$-bounded, then due to splitting theorem [2] equation (8) can be reduced to the equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(M + N)(u^0 + u^1), \\ \ddot{u}^1 = L_1^{-1} Q(M + N)(u^0 + u^1), \end{cases} \tag{9}$$

where $u^1 = Pu, u^0 = (\mathbb{I} - P)u$.

Consider the set $\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(M + N)(u^0 + u^1) = 0\}$. Let u_0 be a point of $\mathfrak{M}$. Denote $u_0^1 = Pu_0 \in \mathfrak{U}^1$.

Let the following condition be fulfilled:

$$(\mathbb{I} - Q)(M - N'_{u_0}) : \mathfrak{U}^0 \to \mathfrak{F}^0 \text{is a toplinear isomorphism.} \tag{10}$$

According to implicit function theorem [8] there exist neighborhoods $\mathcal{O}^0 \subset \mathfrak{U}^0$ and $\mathcal{O}^1 \subset \mathfrak{U}^1$ of points $u_0^0 = (\mathbb{I} - P)u_0$, $u_0^1 = Pu_0$ respectively and the operator $B \in C^\infty(\mathcal{O}^1;\ \mathcal{O}^0)$ such that $u_0^0 = B\big(u_0^1\big)$. Construct the operator $\delta = \mathbb{I} + B : \mathcal{O}^1 \to \mathfrak{M}$, $\delta\big(u_0^1\big) = u_0$. The operator δ^{-1} together with the set $\mathcal{O}^1$ makes a map of $\mathfrak{M}$ and is a restriction of P on $\delta[\mathcal{O}^1] = \mathcal{O} \subset \mathfrak{M}$. Thus holds

Lemma 1. *If condition* (10) *is fulfilled, the set $\mathfrak{M}$ is a C^∞-manifold at the point u_0.*

Act with the Frechet derivative of the second order $\delta''_{(u_1^1, u_0^1)}$ onto the second equation of system (9). Then, since

$$\delta''_{(u_1^1, u_0^1)} \ddot{u}^1 = \frac{d^2}{dt^2}\big(\delta(u^1)\big) \text{ и } \delta(u^1) = u,$$

we obtain the equation

$$\frac{d^2 u}{dt^2} = \delta''_{(u_1^1, u_0^1)} L_1^{-1} Q(M - N)(u),$$

defined on $\mathcal{O}$. By the theorem for nondegenerate equation [9] we obtain

Theorem 1. *Let the operator M be $(L, 0)$-bounded and operator $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$. Then for any pair $(u_0, u_1) \in T\mathfrak{M}$ under the condition* (10) *there exists a unique solution of the problem* (7)–(8) *lying in $\mathfrak{M}$ as trajectory.*

In order to reduce the problem (1)–(3) to problem (7)–(8) set

$$\mathfrak{U} = \{u \in W_2^{m+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}, \quad \mathfrak{F} = W_2^m(\Omega). \tag{11}$$

Operators L, M, N are defined by formulas:

$$L = \lambda - \Delta, \quad M = \alpha^2 \Delta, \quad N(u) = \Delta f(u).$$

For any $m \in \{0\} \cup \mathbb{N}$ operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Denote by $\sigma(\Delta) = \{\lambda_l\}$ the eigenvalues of the Dirichlet problem for the Laplace operator Δ, numbered nonincreasingly taking into account their multiplicity. Denote by $\{\varphi_k\}$ corresponding eigenfunctions orthonormal in the sense of the scalar product in $L^2(\Omega)$.

The regularity lemma holds [10]

Lemma 2. *Let $f \in C^\infty(\mathbb{R})$ and $k > n/2 - 2$. Then $N \in C^\infty\big(W_2^k(\Omega)\big)$ where $N : u \to \Delta f(u)$.*

Constructs projector

$$P = \begin{cases} \mathbb{I}, \text{ if } \lambda \notin \sigma(\Delta), \\ \mathbb{I} - \sum\limits_{\lambda = \lambda_l} \langle \cdot, \varphi_l \rangle \varphi_l, \text{ if } \lambda \in \sigma(\Delta). \end{cases}$$

Projector Q has the same form but it is given on the space $\mathfrak{F}$.

Fix $m > n/2 - 2$ and construct the set

$$\mathfrak{M} = \begin{cases} \mathfrak{U}, \text{ if } \lambda \notin \sigma(\Delta), \\ \{u \in \mathfrak{U} : \langle Mu + N(u), \varphi_l \rangle = 0, \lambda = \lambda_l\} \end{cases}$$

Thus reduction is finished and due to abstract theorem 1 there is

Theorem 2. *(i) For all $\lambda \notin \sigma(\Delta)$, $m > n/2 - 2$, $u_0, u_1 \in \mathfrak{U}$ and $\tau = \tau(u_0, u_1) > 0$ there exists a unique solution $u \in C^\infty\big((-\tau, \tau), \mathfrak{U}\big)$ of the problem (1) – (3).*
(ii) Let $\lambda \in \sigma(\Delta)$, $m > n/2 - 2$, $(u_0, u_1) \in T\mathfrak{M}$ and condition (10) be fulfilled. Then there exists a unique local solution $u \in C^\infty\big((-\tau, \tau), \mathfrak{M}\big)$ of the problem (1)–(3).

Proof.

In the first case, when $\lambda \notin \sigma(\Delta)$, since the set $\mathfrak{M}$ coincides with space $\mathfrak{U}$ it is C^∞-manifold. In the second case, when $\lambda \in \sigma(\Delta)$, if condition (10) is fulfilled then by lemma 1 $\mathfrak{M}$ is a Banach C^∞-manifold at point $u_0 \in \mathfrak{M}$.

□

Now consider mathematical model of shallow-water waves propagation with Showalter – Sidorov conditions

$$P(u(x,0) - u_0(x)) = 0, \quad P(\dot{u}(x,0) - u_1(x)) = 0, x \in \Omega, \tag{12}$$

where P is a projector along the kernel of L.

Using procedure described above mathematical model (2), (3), (12) can be reduced to the Showalter – Sidorov problem

$$P(u(0) - u_0) = 0, \quad P(\dot{u}(0) - u_1) = 0 \tag{13}$$

for incomplete Sobolev-type equation of the second order

$$L\ddot{u} = Mu + N(u). \tag{14}$$

Theorem 3. *Let $m > n/2 - 2$ and condition (10) be fulfilled. For all $u_0, u_1 \in \mathfrak{U}$ and $\tau = \tau(u_0, u_1) > 0$ there exists a unique solution $u \in C^\infty((-\tau, \tau), \mathfrak{U})$ of the problem (1), (3), (12).*

2. Boussinesq – Löve mathematical models

Investigate mathematical model (4)–(6) in the frame of the relatively polynomially bounded operator pencil theory [3]. By $\overrightarrow{B}$ denote the pencil of operators B_1, B_0. The sets $\rho^A(\overrightarrow{B}) = \{\mu \in \mathbb{C} : (\mu^2 A - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and $\sigma^A(\overrightarrow{B}) = \overline{\mathbb{C}} \setminus \rho^A(\overrightarrow{B})$ are called *A-resolvent set* and *A-spectrum* of pencil $\overrightarrow{B}$ respectively. The operator-function $R^A_\mu(\overrightarrow{B}) = (\mu^2 A - \mu B_1 - B_0)^{-1}$ with domain $\rho^A(\overrightarrow{B})$ is called *A-resolvent* of pencil $\overrightarrow{B}$.

If $\exists\, a \in \mathbb{R}_+ \forall \mu \in \mathbb{C} : \quad (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F}; \mathfrak{U}))$ then operator pencil $\vec{B}$ is called *polynomially A-bounded.*

Introduce the additional condition [3]

$$\int_\gamma R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O}, \tag{A}$$

where the circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 3. *[3] Let the pencil $\vec{B}$ be polynomially A-bounded and condition (A) be fulfilled. The operators*

$$P = \frac{1}{2\pi i} \int\limits_{\gamma} R^A_\mu(\vec{B}) \mu A d\mu, \quad Q = \frac{1}{2\pi i} \int\limits_{\gamma} \mu A R^A_\mu(\vec{B}) d\mu$$

are projectors in spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively.

Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \operatorname{im} P$, $\mathfrak{F}^1 = \operatorname{im} Q$. According to lemma 3 $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By symbols A^k (B^k_l) denote restriction of operators A (B_l) on $\mathfrak{U}^k$, $k = 0, 1;$ $l = 0, 1$.

Consider the Cauchy problem

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \tag{15}$$

for semilinear Sobole-type equation

$$A\ddot{u} = B_1 \dot{u} + B_0 u + N(u) \tag{16}$$

provided that the operator A has a nontrivial kernel, operators pencil $\vec{B}$ is $(A, 0)$-bounded and the condition (A) is fulfilled. Then due to splitting theorem [2] equation (16) can be reduced to equivalent system of equations

$$\begin{cases} 0 = (\mathbb{I} - Q)(B_0 + N)(u^0 + u^1), \\ \ddot{u}^1 = A_1^{-1} Q B_1 (\dot{u}^0 + \dot{u}^1) + A_1^{-1} Q (B_0 + N)(u^0 + u^1), \end{cases} \tag{17}$$

where $u^1 = Pu, u^0 = (\mathbb{I} - P)u$.

Introduce the set $\mathfrak{M} = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(B_0 u + N(u)) = 0\}$.

Let $\mathfrak{M} \neq \varnothing$, i.e. exists a point $u_0 \in \mathfrak{M}$ and the following condition be fulfilled:

$$(\mathbb{I} - Q)(B_0 + N'_{u_0}) : \mathfrak{U}^0 \to \mathfrak{F}^0 \text{ is a toplinear isomorphism.} \tag{18}$$

We can show that the set $\mathfrak{M}$ is a C^∞-manifold at the point u_0 like in previous section. The following theorem holds due to classical theorem about existence of unique local solution of nondegenerate differential equation [9]

Theorem 4. *Let the operator pencil $\vec{B}$ be $(A, 0)$-bounded, operator $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$ and condition* (18) *be fulfilled. Then for any pair $(u_0, u_1) \in T\mathfrak{M}$ there exists a unique solution of the problem* (15), (16) *lying in $\mathfrak{M}$ as trajectory.*

Now reduce mathematical model (4)–(6) to the Cauchy problem (15) for equation (16). Introduce spaces as in (11) and define

$$A = \lambda - \Delta, \quad B_1 = \alpha(\Delta - \lambda'), \quad B_0 = \beta(\Delta - \lambda'').$$

For any $m \in \{0\} \cup \mathbb{N}$ operators $A, B_1, B_0 \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$. If $m > n/2 - 2$ then the operator $N(u)$ defined as $N(u) = \Delta f(u)$, is from the class C^∞ due to lemma (2).

Denote by $\{\lambda_k\}(= \sigma(\Delta))$ eigenvalues of the Dirichlet problem for the Laplace operator Δ numbered nonincreasingly taking into account their multiplicity. Denote by $\{\varphi_k\}$

corresponding orthonormal eigenfunctions in the sense of the scalar product in $L^2(\Omega)$. Since $\{\varphi_k\} \subset C^\infty(\Omega)$ then

$$\mu^2 A - \mu B_1 - B_0 = \sum_{k=1}^{\infty} [(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k)] < \varphi_k, \cdot > \varphi_k$$

where $< \cdot, \cdot >$ is the scalar product in $L^2(\Omega)$.

Lemma 4. *[3] Let one of the following conditions be fulfilled:*
(i) $\lambda \notin \sigma(\Delta)$;
(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$;
(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$.
Then the pencil $\vec{B}$ is polynomially A-bounded.

Let conditions (i) or (iii) of lemma 4 be fulfilled then condition (A) takes place. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$, i.e. the condition (ii) of lemma 4 is fulfilled, then (A) doesn't take place, therefore we eliminate it from further consideration. A-spectrum of pencil $\vec{B}$ consists of solutions $\mu_k^{1,2}$ of the equation

$$(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k) = 0, \ k \in \mathbb{N}.$$

Fix $m > n/2 - 2$ then according to regularity lemma the operator $N(u) : u \to \Delta f(u)$ belongs to $C^\infty(\mathfrak{U}; \mathfrak{F})$.

Thus we finished reduction. All condition of theorem 4 are fulfilled and the following statement takes place.

Theorem 5. *(i) For all $\lambda \notin \sigma(\Delta)$, $lm > n/2 - 2$, $u_0, u_1 \in \mathfrak{U}$ and $\tau > 0$ there exists a unique solution $u \in C^\infty\big((-\tau, \tau), \mathfrak{U}\big)$ of the problem* (4)–(6).
(ii) Let $(\lambda = \lambda' = \lambda_l \neq \lambda'')$, $m > n/2 - 2$, $(u_0, u_1) \in T\mathfrak{M}$ and condition (18) *be fulfilled. Then for $\tau > 0$ there exists a unique solution $u \in C^\infty\big((-\tau, \tau), \mathfrak{M}\big)$ of the problem* (4)–(6).

Now consider the Showalter – Sidorov conditions

$$P(u(0) - u_0) = 0, \quad P(\dot{u}(0) - u_1) = 0 \tag{19}$$

for the Sobolev-type equation (16). Note that initial data of a Showalter – Sidorov problem is projected on the image of operator at the highest time derivative. In particulaly case, when ∞ is a removable singularity of A-resolvent of pencil $\vec{B}$, the image coincides with the image of the projector P. Thus the initial data enters the phase space of given equation automatically as distinct from initial data of the Cauchy problem. The following statement takes place

Theorem 6. *Let the pencil $\vec{B}$ be polynomially A-bounded, ∞ be a removable singularity of A-resolvent of the pencil $\vec{B}$, the operator N be from the class C^∞ and condition* (18) *be fulfilled. For all u_0, u_1 there exists a unique local solution of problem* (16), (19).

Consider the equation (4) with boundary condition (5) and the Showalter – Sidorov conditions

$$(\lambda - \Delta)(u(x, 0) - u_0) = 0, \quad (\lambda - \Delta)(\dot{u}(x, 0) - u_1) = 0, \tag{20}$$

where α, β, λ, λ', $\lambda'' \in \mathbb{R}$, $f(u)$ is a function from the class C^{∞}.

The mathematical model (4), (5), (20) can be reduced to the abstract problem (16), (19) in the spaces

$$\mathfrak{U} = \{u \in W_2^{m+2}(\Omega) |\, u(x) = 0,\, (x) \in \partial\Omega\},$$

$$\mathfrak{F} = W_2^m(\Omega).$$

Thus, due to theorem (6), the following theorem true.

Theorem 7. *Let $m > n/2 - 2$, $\lambda \notin \sigma(\Delta)$ or $((\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda' \neq \lambda''))$ and (18) be fulfilled. Then for all $u_0, u_1 \in \mathfrak{U}$ and $\tau = \tau(u_0, u_1) > 0$ there exists a unique solution $u \in C^2((-\tau, \tau), \mathfrak{U})$ of the problem (4), (5), (20).*

3. Algorithm of the numerical solution

Now consider the algorithm of the numerical solution for mathematical model which can be reduced to the problem (16), (19). The flowchart of the method algorithm is shown on figure 1. In A.V. Keller works the algorithms of numerical solution of Showalter – Sidorov problem for Leontiev-type systems [11] are studied.

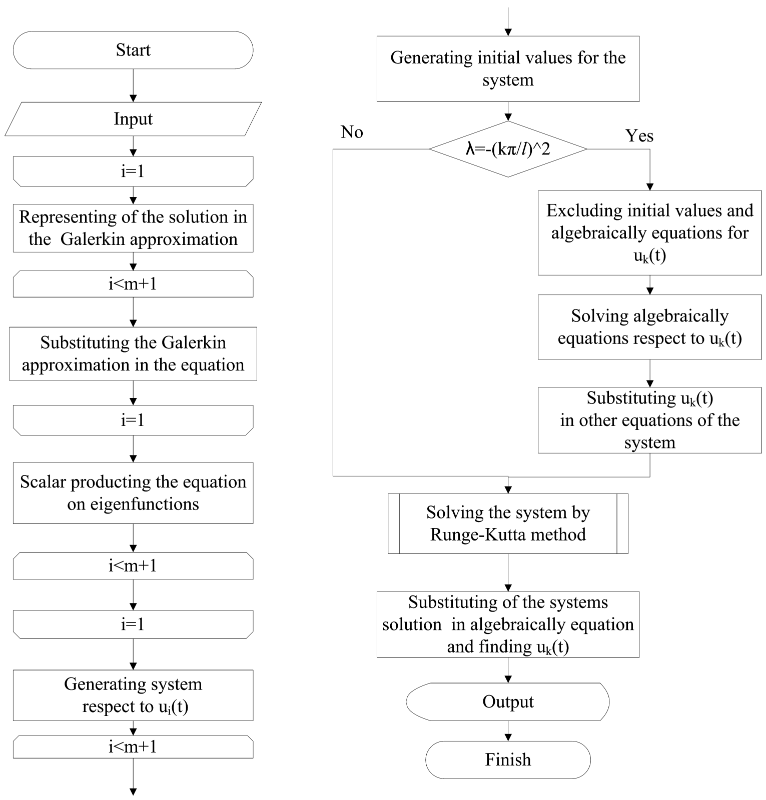

Fig. 1. The flowchart of the method algorithm solution of the problem (16), (19).

Describe the algorithm in details. Each step is responsible for one block.

1 step. After program starts, input data: m – number of Galerkin terms; λ, λ', λ'', α, β – parameters of equation; $f(u)$ – right side of equation; u_0, u_1 – initial data; Ω – domain; Δt, Δx – approximation steps of solution.

2 step. Generation of approximate solution $\tilde{u}$ in form of series $\sum\limits_{i=1}^{m} u_i(t)\varphi_i(x)$.

3 step. Substitution $\tilde{u}$ into equation.

4 step. Multiplication of the equation by $\varphi_i(x)$ in sense of $L_2(\Omega)$.

5 step. Combining of equations received on previous step in a system.

6 step. Expanding of the initial data to a Galerkin sum. Finding the initial data for system of equations from 5^{th} step.

7 step. Checking, if λ belongs to the spectrum of Laplace operator.

If 7 step is false:

8 step. System from the 5^{th} step with initial data from the 6^{th} step is solved by Runge – Kutta method.

If 7 step is true:

9 step. Exclude the algebraic equation №k form the system and corresponding the initial data.

10 step. Solve the algebraic equation with respect to $u_k(t)$.

11 step. Solution of the algebraic equation is substituted into the system.

12 step. System from the 11^{th} step with the initial data from the 9^{th} step is solved by Runge – Kutta method.

13 step. Combine solution and output it in the form of a graph and set of points.

Solve the following problem using that algorithm.

$$(-9-\Delta)\ddot{u} = (\Delta+9)\dot{u} + \Delta u + \Delta(u^3), \tag{21}$$

$$u(0,t) = u(\pi,t) = 0 \tag{22}$$

$$\begin{aligned} &(-9-\Delta)(u(x,0) - \sin(x) + 2\sin(2x) - 3\sin(3x)) = 0,\\ &(-9-\Delta)(\dot{u}(x,0) - 5\sin(x)) = 0. \end{aligned} \tag{23}$$

Equation (21) is degenerate. Galerkin sum for $m = 3$ has form

$$\tilde{u}(x,t) = \sqrt{\frac{2}{\pi}}\left(u_1(t)\sin(x) + u_2(t)\sin(2x) + u_3(t)\sin(3x)\right).$$

Algorithm was realized in Maple. The numerical solution of the problem (21)–(23) is shown on figure 2.

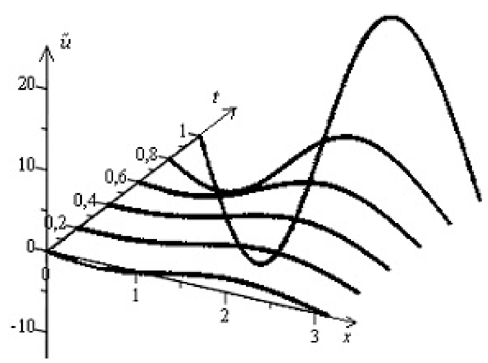

Fig. 2. Graph of the numerical solution of (21)–(23)

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