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THE BOUNDED SOLUTIONS ON A SEMIAXIS FOR THE LINEARIZED HOFF EQUATION IN QUASI-SOBOLEV SPACES

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In this paper we investigate the properties of the linearized Hoff equation in quasi-Sobolev spaces. Hoff equation, specified on the interval, describes the buckling of the H-beam. Due to the fact that for certain values of the parameters in the equation may be missing the derivative with respect to time, this equation refers in a frame of class of nonclassical equations of mathematical physics. The article by relatively spectral theorem describes the morphology of the phase space and the existence of invariant spaces of solutions. Using these results, we prove the existence of bounded on the semiaxis solutions for homogeneous evolution equations of Sobolev type in quasi-Sobolev spaces. Apart of the introduction and bibliography the article contains three parts. The first one shows the results on the solvability of the investigated class of equations. The second part shows the existence of bounded on the semiaxis solutions for the homogeneous equations of the research class. Finally, the third part presents the results of the existence of solutions bounded on the semiaxis for analog linearized Hoff equation in quasi-Sobolev spaces.

Keywords: Sobolev type equations; phase space; invariant subspaces of solutions; group of solving operators.

Introduction

Consider an analogue of the linearized Hoff equation

$$(\lambda + \Lambda)u_t = \alpha u, \qquad \lambda, \alpha \in \mathbb{R} \tag{1}$$

with Laplace quasi-operator $\Lambda : \ell_q^{r+2} \to \ell_q^r$ in quasi-Sobolev spaces [1]

$$\ell_q^r = \left\{ u = \{u_k\} \subset \mathbb{R} : \sum_{k=1}^{\infty} \left(\lambda_k^{\frac{r}{2}} |u_k| \right)^q < +\infty \right\},$$

where $r \in \mathbb{R}$ and $q \in (0,1)$, and sequence $\{\lambda_k\} \subset \mathbb{R}_+$ is such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. The prototype of the equation (1) is the boundary value problem for the linearized Hoff equation [2]

$$(\lambda - \Delta)u_t = \alpha u, \qquad \lambda, \alpha \in \mathbb{R} \tag{2}$$

in quasi-Sobolev spaces $W_2^r(\Omega)$. Here Ω is a bounded domain in $\mathbb{R}^d$ with infinitely smooth boundary $\partial\Omega$. Equation (2) describes the dynamics of H-beam construction. An operator in left side of (2) can degenerate and by this reason equation (2) is contained in wide class of nonclassical equations of mathematical physics [3].

In more general case we can consider equation (1) in frame of a class of dynamical equations
$$P_n(\Lambda)\dot{u} = Q_m(\Lambda)u, \tag{3}$$
where $P_n(x) = \sum\limits_{i=0}^{n} c_i x^i$, $c_i \in \mathbb{C}$, $c_n \neq 0$ and $Q_m(x) = \sum\limits_{j=0}^{m} d_j x^j$, $d_j \in \mathbb{C}$, $d_m \neq 0$ are polynomials such that $m \leq n$. The Sobolev type equation (3) is called dynamical if it's solutions exist on whole $\mathbb{R}$, and evolutionary if solutions of (3) exist only on $\mathbb{R}_+$. Note that dynamical Sobolev type equations was researched in space of "differentiable noises" [4]. Solvability of equation (3) was research in [5]. Main aim of this article is conditions for boundness of solutions for equations (3) and (1). To archive this aim we have to consider invariant spaces of solutions for Sobolev type equation [6]
$$L\dot{u} = Mu \tag{4}$$
with operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ in quasi-Sobolev spaces $\mathfrak{U}$ and $\mathfrak{F}$. Vector-function $u \in C^\infty(\mathbb{R}; \mathfrak{U})$ is called *a solution of equation* (4), if it satisfies this equation. The solution $u = u(t)$ of the equation (4) is called *solution of the Cauchy problem*
$$u(0) = u_0 \tag{5}$$
for equation (4) (shortly, the problem (4), (5)), if in addition it satisfies the Cauchy condition (5) at a some $u_0 \in \mathfrak{U}$.

Besides the introduction and the references the article contains three parts. The first part provides preliminaries concepts, such that the relative resolvents in quasi-Sobolev, relatively (L, p)-bounded operators and solvability of problem (4),(5). The second one consider the bounded solutions on a semiaxis for homogeneous equations of the class (3). The third is considered an analog of the linearized equation Hoff (1) in quasi-Sobolev spaces. Existence the bounded solutions on a semiaxis for a homogeneous linearized analog Hoff equation.

1. Solvability for One Class of Dynamical Equations

Let $\mathfrak{U}$ and $\mathfrak{F}$ be a quasi-Sobolev spaces, operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Considerate an L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of operator M. By [7] the set $\rho^L(M)$ is always opened, therefore the $\sigma^L(M)$ is always closed.

Let $\rho^L(M) \neq \emptyset$ then the operator functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called respectively *L-resolvent*, the *right* and *left L-resolvent* of an operator M.

Let for the relative spectral of an operator $\sigma^L(M)$
$$\left.\begin{array}{c} \sigma^L(M) = \sigma^L_0(M) \bigcup \sigma^L_1(M),\ \sigma^L_1(M) \neq \emptyset, \\ \text{such that there exists a bounded domain } \Omega_1 \subset \mathbb{C} \\ \text{with } \partial\Omega_1 \text{ of class } C^1;\ \Omega_1 \supset \sigma^L_1(M) \text{ and } \overline{\Omega}_1 \cap \sigma^L_0(M) = \emptyset. \end{array}\right\}. \tag{6}$$

Let $\gamma_1 = \partial\Omega_1$, then we construct the operators
$$P_1 = \frac{1}{2\pi i}\int\limits_{\gamma_1} R^L_\mu(M)d\mu \quad \text{and} \quad Q_1 = \frac{1}{2\pi i}\int\limits_{\gamma_1} L^L_\mu(M)d\mu,$$

and the integrals are understood in the sense of Rieman [8]. By the construction of the operators $P_1 \in \mathcal{L}(\mathfrak{U})$ и $Q_1 \in \mathcal{L}(\mathfrak{F})$.

Theorem 1. [9] *Let $\mathfrak{U}$, $\mathfrak{F}$ be a quasi-Sobolev spaces, operators $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $\rho^L(M) \neq \emptyset$ and condition* (6) *is holds, then the operators $P_1 \in \mathcal{L}(\mathfrak{U})$ and $Q_1 \in \mathcal{L}(\mathfrak{F})$ are projectors.*

Definition 1. The operator M is called *a relatively spectral bounded* of the operator L (or shortly, (L,σ)*-bounded*), if $\exists a \in \mathbb{R}_+ \ \forall \mu \in \mathbb{C} \ (|\mu| > a) \ \Rightarrow \ \left(\mu \in \rho^L(M)\right)$.

Let $\gamma = \{\mu \in \mathbb{C} : \ |\mu| = h, \ h > a\}$, where constant a is from definition 1, then operators $P = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M) d\mu$, $Q = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M) d\mu$ and we have the following

Corollary 1. *Let an operator M (L,σ)-bounded, then the operators $P \in \mathcal{L}(\mathfrak{U})$ and $Q \in \mathcal{L}(\mathfrak{F})$ are projectors.*

Let $\mathfrak{U}^0$ $(\mathfrak{U}^1)$ $=$ ker$P$ (im$P$), $\mathfrak{F}^0$ $(\mathfrak{F}^1)$ $=$ ker$Q$ (im$Q$), then by L_k (M_k) be denote a restriction of an operator L (M) into $\mathfrak{U}^k$, $k = 0, 1$. From a Corollary 1 it follows, that the projectors P and Q are splitting the spaces $\mathfrak{U}$ and $\mathfrak{F}$ into direct sums $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$.

Theorem 2. [7] *Let an operator M (L,σ)-bounded, then*
(i) *the operators $L_k, M_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;*
(ii) *there exists an operator $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.*

By Theorem 2 there exists the operators

$$H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0) \quad \text{and} \quad S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1).$$

Definition 2. (L,σ)-bounded an operator M is called
- $(L, 0)$*-bounded*, if $H \equiv \mathbb{O}$;
- (L, p)*-bounded*, if $H^p \neq \mathbb{O}$, and $H^{p+1} \equiv \mathbb{O}$ at $p \in \mathbb{N}$.

Corollary 2. [7] *Let an operator L is a continuously invertible, then holds $\sigma^L(M) = \sigma(S)$.*

The quasi-Sobolev spaces $\ell^r_q = \left\{ u = \{u_k\} \subset \mathbb{C} : \sum\limits_{k=1}^{\infty} \left(\lambda_k^{\frac{r}{2}} |u_k|\right)^q < +\infty \right\}$ with $r \in \mathbb{R}$, $q \in \mathbb{R}_+$ are not normed, but quasi-normed. The quasi-norm ${}_{\mathfrak{U}}\|\cdot\|$ differ from norm by "triagle axiom", which for quasi-norm has the following form

$${}_{\mathfrak{U}}\|u + v\| \leq C\left({}_{\mathfrak{U}}\|u\| + {}_{\mathfrak{U}}\|v\|\right) \qquad \forall u, v \in \mathfrak{U}$$

with $C \geq 1$. The spaces ℓ^r_q are quasi-Banach spaces for all $r \in \mathbb{R}$, $q \in \mathbb{R}_+$ with quasi-norm

$${}^r_q\|u\| = \left(\sum_{k=1}^{\infty} \left(\lambda_k^{\frac{r}{2}} |u_k|\right)^q \right)^{1/q},$$

and they are also Banach only if $q \in [1, +\infty)$. If $q \in (0, 1)$, then the constant $C = 2^{\frac{1-q}{q}}$. Note also, that if $r = 0$, then $\ell^0_q = \ell_q$, and the sequence $\{\lambda_k\} \subset \mathbb{R}_+$, such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$.

Remark 1. [1] The spaces ℓ_q^r $(r \in \mathbb{R})$ for $q \in (0,1)$ are metrizable.

Consider the solvability Cauchy problem (5) for dynamical equations (4) in quasi-Sobolev spaces. Note, that the solution of the Cauchy problem (5) for the equation (4) for an arbitrary initial condition is not always exist [6].

Definition 3. The set $\mathfrak{P} \subset \mathfrak{U}$ is called the *phase space* of the equation (4), if
1) for every $u_0 \in \mathfrak{P}$ there exists a unique solution of the Cauchy problem (4), (5),
2) any solution $u = u(t)$ of (4) lies in $\mathfrak{P}$ as the trajectory (ie $u(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$).

Theorem 3. [7] *Let the operator M (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the phase space of the equation (4) is the subspace $\mathfrak{U}^1$.*

Remark 2. If there exists an operator $L^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ then the phase space of the equation (4) is all the space $\mathfrak{U}$. In the case of the irreversibility of the operator L phase space of $\mathfrak{P}$ be a subspace of $\mathfrak{U}$.

Let us return to consideration of the equation (3). Consider the power of the Laplace quasi-operators $\Lambda^n u = \{\lambda_k^n u_k\}$ $(n \in \mathbb{N})$ [10]. It is easy to see, the operator $\Lambda^n : \ell_q^{r+2n} \to \ell_q^r$ is a toplinear isomorphism, $r \in \mathbb{R}$. Choose a space $\mathfrak{U} = \ell_q^{r+2n}$ and $\mathfrak{F} = \ell_q^r$. Operators $L = P_n(\Lambda)$ and $M = Q_m(\Lambda)$, where $P_n(x) = \sum\limits_{i=0}^{n} c_i x^i$ $(c_i \in \mathbb{C},\ c_n \neq 0)$ and $Q_m(x) = \sum\limits_{j=0}^{m} d_j x^j$ $(d_j \in \mathbb{C},\ d_m \neq 0)$ are polynomials, such that $m \leq n$. Then by [5] operators $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$.

Theorem 4. [5] *Let the numbers λ_k, being the roots of the polynomial $P_n(x)$, are not roots of $Q_m(x)$. Then the operator M $(L,0)$-bounded.*

The L-spectrum of operator M have the form

$$\sigma^L(M) = \left\{ \mu \in \mathbb{C} : \ \mu_k = \frac{Q_m(\lambda_k)}{P_n(\lambda_k)}, \ \text{at } k : \ P_n(\lambda_k) \neq 0 \right\}. \tag{7}$$

Further, let $\{U^t : t \in \mathbb{R}\}$ is a holomorphic degenerate group of operators, and U^0 is an its identity. Consider the image of $\mathrm{im}U^\bullet = \mathrm{im}U^0$ and the kernel $\ker U^\bullet = \ker U^0$ of the group. Let us call the group $\{U^t : t \in \mathbb{R}\}$ a resolution group of the equation (4), if in the first, the vector function $u(t) = U^t u_0$ is a solution of equation (4) for every $u_0 \in \mathfrak{U}$, a and secondly, the image of $\mathrm{im}U^\bullet$ coincides with phase space of equation (4).

Theorem 5. [7] *Let the operator M (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then there is a unique group of resolving the equation (4), which also has the form*

$$U^t = \frac{1}{2\pi i} \int\limits_{\gamma} R_\mu^L(M) e^{\mu t} d\mu, \quad t \in \mathbb{R},$$

where the contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > a\}$.

By Theorems 4 and 5 it is easy to show a group of a holomorphic resolution of equation (3) have the form

$$U^t \cdot = \begin{cases} \sum\limits_{k=1}^{\infty} e^{\mu_k t} \langle \cdot, e_k \rangle e_k, & \text{if } P_n(\lambda_k) \neq 0, \ k \in \mathbb{N}; \\ \sum\limits_{k \neq l} e^{\mu_k t} \langle \cdot, e_k \rangle e_k, & \text{if there exist } l \in \mathbb{N} : \ P_n(\lambda_l) = 0, \end{cases}$$

where $\mu \in \sigma^L(M)$ from (7). Here the vectors $e_k = (0, 0, ..., 0, 1, 0, ...)$, where the identity is worth in the k-th place. By Theorems 3 and 4 the phase space of (4) is the set

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U}, & \text{if } P_n(\lambda_k) \neq 0, \ k \in \mathbb{N}; \\ \{u \in \mathfrak{U} : \ u_l = 0, \ P_n(\lambda_l) = 0\}. \end{cases}$$

Definition 4. Let $\mathfrak{P} \subset \mathfrak{U}$ is a phase space of equation (4). Set $\mathfrak{J} \subset \mathfrak{P}$ is called *an invariant subspace,* of this equation, if any $u_0 \in \mathfrak{J}$ there exists a unique solution $u = u(t)$ Cauchy problem (5) for the equation (4), and $u(t) \in \mathfrak{J}$ for all $t \in \mathbb{R}$.

Theorem 6. [11] *Let operator M be an (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$ and condition (6) is holds. Then the image of group*

$$U_1^t = \frac{1}{2\pi i} \int\limits_{\gamma_1} R_\mu^L(M) e^{\mu t} d\mu, \quad t \in \mathbb{R}, \tag{8}$$

be an invariant space for equation (4). Here $\gamma_1 = \partial\Omega_1$ and Ω_1 from condition (6).

Remark 3. Condition analogously (6) is important for wide class of multipoint initial-final problems [12].

2. Boundness on a Semiaxis of Solutions for a Class of Dynamical Equations

Let $\mathfrak{U}$ and $\mathfrak{F}$ are quasi-Sobolev spaces, operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. Let us find the necessary and sufficient conditions for existing the bounded solution of the Cauchy problem (4), (5) on a semiaxis in terms of the L-spectrum of the operator M.

Lemma 1. *Let the operator M (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$, and any non-trivial solution of the equation (4) is bounded on $\overline{\mathbb{R}_+}$. Then L is the spectrum of the operator M lies in the closed left half-plane.*

Proof. By virtue of Theorem 5 the solution of problem (4), (5) has the form $u(t) = U^t u_0$, where U^t is a resolution group and $u_0 \in \mathfrak{U}^1$. Since by hypothesis any solution of equation (4) is bounded on $\overline{\mathbb{R}_+}$, then

$$_{\mathfrak{U}}\|u(t)\| = {_{\mathfrak{U}}\|U^t u_0\|} \leq K {_{\mathfrak{U}}\|u_0\|} \quad \text{at} \quad t \geq 0, \quad K = \frac{K_1}{_{\mathfrak{U}}\|u_0\|}$$

for all $K_1 > 0$ and $t \geq 0$ such that condition $_{\mathfrak{U}}\|u(t)\| \leq K_1$ is satisfied. By virtue of consequence group of $U^t\Big|_{\mathfrak{U}^1} = e^{St}$, where $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U})$.

Let us fix $t > 0$. A mapping of the spectrum has the form $\sigma(e^{St}) = e^{t\sigma(S)}$, and means the set of the form $t\sigma(S)$ lies in a halfplane $\{\mathrm{Re}\lambda \leq \ln K\}$. Consequently, the spectrum $\sigma(S)$ lies in a halfplane

$$\{\mathrm{Re}\mu \leq \ln K/t\}.$$

Since the $t > 0$ is arbitrary and, by Corollary 2, $\sigma(S) = \sigma^L(M)$, then we obtain the required result.

□

Let $\mathfrak{U} = \ell_q^{r+2n}$ and $\mathfrak{F} = \ell_q^r$ are quasi-Sobolev spaces, where $r \in \mathbb{R}$ and $q \in (0,1)$, and the sequence $\{\lambda_k\} \subset \mathbb{R}_+$, such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. The operators $P_n(\Lambda), Q_m(\Lambda) \in \mathcal{L}(\ell_q^{r+2n}; \ell_q^r)$, $m \leq n$ and the number λ_k, the root of polynomial $P_n(x)$, is not root of $Q_m(x)$. Consider bounded solutions of the class of equations of the form

$$P_n(\Lambda)\dot{u} = Q_m(\Lambda)u. \tag{9}$$

Theorem 7. *Let $m \leq n$, the operators $L = P_n(\Lambda), M = Q_m(\Lambda)$ and the numbers λ_k, the roots of the polynomial $P_n(x)$, are not roots of $Q_m(x)$, and satisfy the condition*

$$\sigma^L(M) = \sigma_1^L(M) \cup \sigma_2^L(M), \tag{10}$$

where $\sigma_1^L(M) = \{\mu\in\sigma^L(M) : \mathrm{Re}\mu<0\}$, $\sigma_2^L(M) = \{\mu\in\sigma^L(M) : \mathrm{Re}\mu\geq 0\}$.

Then the solution of problem (5), (9) *is bounded on $\overline{\mathbb{R}_+}$ if and only if a $u_0 \in \mathfrak{U}^{11}$.*

Proof. The problem (4), (5) can be represented in the form of an equivalent system of problems

$$L_{11}\dot{u}^{11} = M_{11}u^{11}, \qquad u^{11}(0) = u_0^{11}, \tag{11}$$

$$L_{21}\dot{u}^{21} = M_{21}u^{21}, \qquad u^{21}(0) = u_0^{21}. \tag{12}$$

By the condition (10) there exists a contour γ_1', lying in the left half-plane, and a contour γ_2', which bound the domains containing $\sigma_1^L(M)$ and $\sigma_2^L(M)$ respectively. It's clearly that from (10) follows that (6) holds. By Theorem 5, we construct the groups U_1^t

$$U_1^t = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^L(M)e^{\mu t}d\mu, \quad t \in \mathbb{R},$$

and also U_2^t, where the contour γ first is replaced by γ_1', and then by γ_2'. We denote the invariant spaces of equation (4) $\mathfrak{U}^{11} = \mathrm{im}U_1^\bullet$ and $\mathfrak{U}^{21} = \mathrm{im}U_2^\bullet$, which exists by Theorem 5. By virtue of the fact that $U_1^t + U_2^t = U^t$ the equality $\mathfrak{U}^{11} \oplus \mathfrak{U}^{21} = \mathfrak{U}^1 = \mathrm{im}U^\bullet$ is holds.

Now let $u_0 \in \mathfrak{U}^{11}$, then following the estimate holds

$$_{\mathfrak{U}}\|U_1^t u_0\|^\alpha = {}^{r+2n}_{\quad q}\left\|\sum_{k:\ \mu_k\in\sigma_1^L(M)} e^{\mu_k t}\langle u_0, e_k\rangle e_k\right\|^q = \sum_{k:\ \mu_k\in\sigma_1^L(M)} \left(e^{\mu_k t}\lambda_k^{\frac{r+2n}{2}}|u_{0k}|\right)^q \leq$$

$$\leq 2C^q e^{qa_1 t}\sum_{k:\ \mu_k\in\sigma_1^L(M)}\left(\lambda_k^{\frac{r+2n}{2}}|u_{0k}|\right)^q = C_1^\alpha e^{a_1 t\alpha}{}_{\mathfrak{U}}\|u_0\|^\alpha \leq C_1^\alpha{}_{\mathfrak{U}}\|u_0\|,$$

where $a_1 = \max\limits_{\mu\in\gamma_1'}\mathrm{Re}\mu$.

If the solution $u(t) = U^t u_0$ of the problems (4), (5) is bounded on $\overline{\mathbb{R}_+}$, then, by Lemma 1, L-spectrum of the operator M lies in the left half-plane, and hence $\sigma_2^L(M) = \emptyset$. Hence we get that the problem (12) has a solution, trivial, only for $\mathfrak{U}^{21} = \{0\}$. Therefore, $u_0 = u_0^{11} \in \mathfrak{U}^{11}$.

□

Corollary 3. *If in the Theorem 7 change the condition of the spectrum to condition* $\sigma^L(M) = \sigma_1^L(M) \cup \sigma_2^L(M)$*, where* $\sigma_1^L(M) = \{\mu \in \sigma^L(M) : \operatorname{Re}\mu \leq 0\}$*,* $\sigma_2^L(M) = \{\mu \in \sigma^L(M) : \operatorname{Re}\mu > 0\}$*. Then the solution of problems* (4), (5) *is bounded on* $\overline{\mathbb{R}_-}$ *if and only if* $u_0 \in \mathfrak{U}^{21}$*.*

By Theorems 4 and 7 we have the following

Theorem 8. *Let* $n > m$*, the numbers* λ_k*, which are the roots of the polynomial* $P_n(x)$*, are not roots of* $Q_m(x)$*, for all* $k \in \mathbb{N}$ *and points of the form* $\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}$ *with condition* $\operatorname{Re}\left(\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) < 0$ *finite number. Then the solution of problem* (5), (9) *is bounded on* $\overline{\mathbb{R}_+}$ *if and only if* $\langle u_0, e_k\rangle = 0$ *for* $k \in \mathbb{N} :\ \operatorname{Re}\left(\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) \geq 0.$

By Corollary 3 we have

Corollary 4. *If in Theorem 8 we require that points with condition* $\operatorname{Re}\left(\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) > 0$ *has a finite number. Then the solution of problem* (5), (9) *is bounded on* $\overline{\mathbb{R}_-}$ *if and only if*

$$\langle u_0, e_k\rangle = 0 \quad \textit{for} \quad k \in \mathbb{N} :\ \operatorname{Re}\left(\frac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) \leq 0.$$

Theorem 9. *Let* $n = m$*, the numbers* λ_k*, which are the roots of the polynomial* $P_n(x)$*, are not roots of* $Q_m(x)$*. Then, if the points of the form* $\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}$ *with the condition* $\operatorname{Re}\left(\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) < 0$ *a finite number, then the solution of problem* (5), (9) *bounded on* $\overline{\mathbb{R}_+}$ *if and only if* $\langle u_0, e_k\rangle = 0$ *for* $k \in \mathbb{N} :\ \operatorname{Re}\left(\dfrac{Q_m(\lambda_k)}{P_n(\lambda_k)}\right) \geq 0.$

Remark 4. If we change the conditions in Theorem 9 as in Corollary 4, then we obtain the conditions for existence the bounded solution of the problem (5), (9) on $\overline{\mathbb{R}_-}$.

Remark 5. Results of Theorem 9 and Remark 4 can be applied to research properies of solution for Barenblatt–Zheltov–Kochina equation [3], linearized Oskolkov equation [13] and other [3].

Remark 6. All result of this part in Banach spaces investigate in [14].

3. Boundness on a Semiaxis of Solutions for the Hoff Equation in Quasi-Sobolev Spaces

Consider the analog of the linearized Hoff equation

$$(\lambda + \Lambda)u_t = \alpha u, \qquad \lambda, \alpha \in \mathbb{R}, \tag{13}$$

in quasi-Sobolev spaces $\mathfrak{U} = \ell_q^{r+2}$ and $\mathfrak{F} = \ell_q^r$ at $r \in \mathbb{R}$ and $q \in \mathbb{R}_+$. But the operators $L = P_1(\Lambda) = \lambda + \Lambda$ and $M = Q_0(\Lambda) = \alpha\mathbb{I}$, Then the operators $L, M \in \mathcal{L}(\ell_q^{r+2}; \ell_q^r)$.

Lemma 2. *Let $\mathfrak{U} = \ell_q^{r+2}$ and $\mathfrak{F} = \ell_q^r$ at $r \in \mathbb{R}$ and $q \in \mathbb{R}_+$. Then for any $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ the operator M is $(L, 0)$-bounded.*

Proof. The L-spectrum of the operator M has the form

$$\sigma^L(M) = \left\{ \mu \in \mathbb{C} : \mu_k = \frac{\alpha}{\lambda + \lambda_k}, \text{ at } k : \lambda_k \neq -\lambda \right\}. \tag{14}$$

Since $\lambda_k \to +\infty$, then the points of the relative spectrum $\sigma^L(M)$ converge to zero. Hence, the set $\sigma^L(M)$ is bounded.

Space

$$\mathfrak{U}^0 = \begin{cases} \{0\}, & \text{if } \lambda_k \neq -\lambda \text{ for all } k \in \mathbb{N}; \\ \{u \in \mathfrak{U} : u_k = 0, \ k \in \mathbb{N} \setminus \{l : \lambda_l = -\lambda\}\}; \end{cases}$$

Therefore the operator $H = M_0^{-1} L_0 = \mathbb{O}$. Hence, the operator M is $(L, 0)$-bounded.

□

By Theorem 5 there exists a resolving group of equation (13), which has the form

$$U^t = \begin{cases} \sum_{k=1}^{\infty} e^{\frac{\alpha}{\lambda + \lambda_k} t} \langle \cdot, e_k \rangle e_k, & \text{if } \lambda_k \neq -\lambda, \ k \in \mathbb{N}; \\ \sum_{k \neq l} e^{\frac{\alpha}{\lambda + \lambda_k} t} \langle \cdot, e_k \rangle e_k, & \text{if there exist } l \in \mathbb{N} : \ \lambda_l = -\lambda. \end{cases}$$

The initial value $\{u_{0k}\} = u_0 \in \ell_q^{r+2}$, the vector $e_k = (0, ..., 0, 1, 0, ...)$, where a unit stands for k-th place By Theorem 3, the image of im$U^\bullet$ coincides with the phase space of equation (13). And this phase space has the form

$$\mathfrak{U}^1 = \begin{cases} \ell_q^{r+2}, & \text{if } \lambda_k \neq -\lambda \text{ for all } k \in \mathbb{N}; \\ \{u \in \ell_q^{r+2} : u_k = 0, \ \lambda_k = -\lambda\}. \end{cases}$$

Now let consider the properties of the solutions of equation (13). First we consider invariant subspaces of solutions (13). The relative spectrum has the form (14), and hence it is discrete and obviously is satisfied, the condition (6) And so, we have the

Lemma 3. *Suppose that the conditions of Lemma 2 are satisfied and $\lambda < 0$ such that there exist $\tilde{k} \in \mathbb{N}$, for which $\lambda_{\tilde{k}} < -\lambda$. Then there exist at least two invariant subspaces of solutions of equation (13).*

Moreover, the invariant subspace of equation (13) of the form span$\{e_{\tilde{k}} : \lambda_{\tilde{k}} < -\lambda\}$ is finite-dimensional. Finally, we consider solutions of the homogeneous analog of the Hoff equation that are bounded on the semiaxis. By the Theorem 7, the following

Theorem 10. *Let $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ and there exist $\tilde{k} \in \mathbb{N}$ such that $\lambda_{\tilde{k}} < -\lambda$; either $\alpha < 0$ and there exist $\tilde{k} \in \mathbb{N}$ such that $\lambda_{\tilde{k}} > -\lambda$. Then the solution of problem (5), (13) is bounded on $\overline{\mathbb{R}_+}$ if and only if*

$$\langle u_0, e_k \rangle = 0 \quad \textit{for} \quad k : \lambda_k \geq -\lambda.$$

Similarly the Corollary 4, we formulate the following

Corollary 5. *Let $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}_-$ and there exists $\tilde{k} \in \mathbb{N}$ such that $\lambda < -\lambda_{\tilde{k}}$; either $\alpha > 0$ and there exists $\tilde{k} \in \mathbb{N}$ such that $\lambda > -\lambda_{\tilde{k}}$. Then the solution of problem* (5), (13) *is bounded on $\overline{\mathbb{R}_-}$ if and only if*

$$\langle u_0, e_k \rangle = 0 \quad \textit{for} \quad k : \lambda_k \geq -\lambda.$$

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ОГРАНИЧЕННОСТЬ НА ПОЛУОСИ РЕШЕНИЙ ЛИНЕАРИЗОВАННОГО УРАВНЕНИЯ ХОФФА В КВАЗИСОБОЛЕВЫХ ПРОСТРАНСТВАХ

Ф.Л. Хасан

В данной работе исследованы свойства решений линеаризованного уравнения Хоффа в квазисоболевых пространствах. Уравнения Хоффа заданное на отрезке, описывает выпучивание двутавровой балки. В силу того, что при определенных значениях параметров в уравнении может отсутствовать производная по времени, то это уравнение относится к обширному классу неклассических уравнений математической физики. В статье с помощью относительно спектральной теоремы описана морфология фазового пространства и показано существование инвариантных пространств уравнения. С использованием этих результатов доказано существование ограниченных на полуоси решений однородных эволюционных уравнений соболевского типа в квазисоболевых пространствах. Статья кроме введения и списка литературы содержит три части. В первой из них приведены результаты о разрешимости исследуемого класса уравнений. Во второй части показывается существование ограниченных на полуоси решения для однородных уравнений исследуемого класса. Наконец, в третьей части приведены результаты о существовании ограниченных на полуоси решений для аналога линеризованного уравнения Хоффа в квазисоболевых пространствах.

Ключевые слова: уравнения соболевского типа; фазовое пространство; инвариантные подпространства решений; разрешающая группа операторов.

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