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CALCULATION OF EIGENVALUES OF DISCRETE SEMIBOUNDED DIFFERENTIAL OPERATORS

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We consider a problem of eigenvalues of an abstract discrete semibounded operator acting in a separable Hilbert space. The existence and uniqueness of the solution, as well as a convergence of the Galerkin method with reference to the problem, are proved. Simple formulas to calculate the eigenvalues are obtained. The formulas based on Galerkin method allow to calculate eigenvalues of discrete semibounded operators with high computational efficiency. In contrast to the classical methods, the formulas sharply reduce the number of calculations. Also, the formulas allow to find eigenvalues of the operator, regardless of whether the eigenvalues with smaller numbers are known or not. The formulas solve the problem on a calculation of all necessary spectrum points of the abstract discrete semibounded operators.

Keywords: eigenvalues, eigenfunctions, perturbation, discrete operator, Galerkin method, existence and uniqueness of the solution.

Introduction

Consider an abstract discrete semibounded operator L, acting in a separable Hilbert space H. Let $D_L \subset H$ be a domain of L. Suppose L is a differential operator. Then its eigenvalues μ are determined by finding nontrivial solutions of equation

$$Lu = \mu u, \tag{1}$$

satisfying homogeneous boundary conditions

$$Gu|_{\Gamma} = 0, \tag{2}$$

where Γ is a boundary of domain D_L.

Consider a spectral problem such that L can be represented as $L = T + P$, where T is a discrete semibounded operator and P is a bounded operator. One of the methods applied to the research of such spectral problems is the Regularized traces method. In 1956 г., in their paper [1] V.A. Sadovnichy and V.E. Podolsky for the first time theoretically justified a calculation of the first eigenvalues of the Sturm-Liouville operator, based on a system of nonlinear equations composed of regularized traces of operator T. In future, S.I. Kadchenko obtained linear formulas, which allow to remove existing restrictions on the perturbing operator and essentially expand the range of problems to which the regularized traces method can be applied (see [2]–[10]). S.I. Kadchenko and S.N. Kakushkin developed a non-iterative method in [11]. The method allows to find the values of perturbed discrete operator eigenfunctions beginning with any number, if spectral characteristics of the unperturbed operator and eigenvalues of the perturbed operator are known. The method also showed high computational efficiency. The problem was investigated on the basis of the modified Galerkin method in [8].

To calculate eigenvalues of the spectral problem (1), (2), we use the Galerkin method. Consider a sequence $\{H_n\}_{n=1}^{\infty}$ of finite-dimensional spaces $H_n \subseteq H$, which is complete in H. Suppose an orthonormal basis of the space H_n is known and consists of functions $\{\varphi_k\}_{k=1}^{n}$. In addition, the functions φ_k satisfy the boundary conditions (2). According to the Galerkin method, an approximate solution of the spectral problem (1), (2) can be found in the form:

$$u_n = \sum_{k=1}^{n} a_k(n)\varphi_k.$$

We prove the existence and uniqueness of the solution, as well as a convergence of the Galerkin method with reference to the problem. Based on the previously obtained results, we obtain simple formulas to calculate the eigenvalues of the operator L with high computational efficiency.

1. Support information

Let us define the terms used in the paper and give some known facts (see. [13], [14]).

We call an operator L, acting in a separable Hilbert space H, *a discrete operator*, if there exists a complex number λ_0 such that $R_{\lambda_0} = (L - \lambda_0 E)^{-1}$ is completely continuous in H (see [13], ch. V, §4).

A spectrum of the discrete operator L consists of isolated points having no limit points, except for infinity.

Definition 1. (see [13], ch. V, §4) If there exists a real number c such that the condition $(Lf, f) \geq c(f, f)$ holds for all $f \in D_L$, then the operator L is called semibounded below. (If the opposite inequality holds, then the operator is called semibounded above.)

Theorem 1. ([14], ch. I, §6) If $\|L_n - T\| \to 0$, where L and L_n are completely continuous operators, then the characteristic numbers of the equation

$$u - \lambda L u = 0$$

are obtained by proceeding to limit as $n \to \infty$ from the characteristic numbers of the equation

$$u_n - \lambda L_n u_n = 0.$$

As is well known (e.g., see [13], ch. V, §4), every semibounded operator can be expressed using some positive operator by means of one of the formulas $L = S + cE$, $L = S - cE$. Therefore, in future we will consider only positive operators. Since $(Lf, f) \geq 0$, then for the eigenfunctions u_n the inequalities

$$(Lu_n, u_n) = \mu_n(u_n, u_n) \geq 0$$

hold, i.e., the eigenvalues of a positive discrete operator are nonnegative and can accumulate only to $+\infty$.

2. Main results

Theorem 2. *Let L be a discrete semibounded operator acting in a separable Hilbert space H. If a system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in H, then the Galerkin method, which is applied to the problem of finding eigenvalues of the spectral problem (1) and constructed on this system of functions, converges.*

Proof. We write equation (1) in the form

$$(L-\lambda_0 E)u=(\mu-\lambda_0)u. \tag{3}$$

Since the operator L is discrete, then for it there exists a resolvent operator $R_{\lambda_0}L=(L-\lambda_0 E)^{-1}$, which is completely continuous in H. Acting on the left side of both sides of equation (3) by the operator $R_{\lambda_0}L$, we obtain

$$u=(\mu-\lambda_0)Mu, \quad Mu\equiv R_{\lambda_0}Lu. \tag{4}$$

Let us repeat the reasoning of Theorem 1 in ([14], ch.XI,§94) and show that application of the Galerkin method to the problem of finding the eigenvalues of equation (2) is equivalent to finding the eigenvalues of equation

$$u_n=(\mu-\lambda_0)M_n u_n. \tag{5}$$

Suppose the system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis of the space H. Decompose $Mu=R_{\lambda_0}Lu$ into an orthogonal series

$$Mu=\sum_{k=1}^{\infty}(Mu,\varphi_k)\varphi_k$$

and assume

$$M_n u=\sum_{n=1}^{n}(Mu,\varphi_k)\varphi_k.$$

The operator M is completely continuous in H and therefore it can be decomposed into a sum $M=M'+M''$, where M' is a degenerate operator, and $\| M'' \|<\dfrac{\varepsilon}{2}$, $\varepsilon>0$. Then

$$(M-M_n)u=\sum_{k=n+1}^{\infty}(Mu,\varphi_k)\varphi_k=\sum_{k=n+1}^{\infty}(M'u,\varphi_k)\varphi_k+\sum_{k=n+1}^{\infty}(M''u,\varphi_k)\varphi_k. \tag{6}$$

First we estimate the second sum in (6):

$$\| \sum_{k=n+1}^{\infty}(M''u,\varphi_k)\varphi_k \|^2=\Big(\sum_{k=n+1}^{\infty}(M''u,\varphi_k)\varphi_k,\sum_{k=n+1}^{\infty}(M''u,\varphi_k)\varphi_k\Big)=$$

$$=\sum_{k=n+1}^{\infty}(M''u,\varphi_k)\sum_{m=n+1}^{\infty}(M''u,\varphi_m)(\varphi_k,\varphi_m)=\sum_{k=n+1}^{\infty}\mid (M''u,\varphi_k)\mid^2.$$

Use the Bessel inequality and obtain

$$\sum_{k=n+1}^{\infty} \mid (M''u, \varphi_k) \mid^2 \leq \parallel M''u \parallel^2 \leq \parallel M'' \parallel^2 \parallel u \parallel^2 < \frac{\varepsilon^2}{4} \parallel u \parallel^2 .$$

Hence,

$$\parallel \sum_{k=n+1}^{\infty} (M''u, \varphi_k)\varphi_k \parallel < \frac{\varepsilon}{2} \parallel u \parallel . \tag{7}$$

Estimate the first sum in (6). By definition, the degenerate operator $M'u$ has the form ([14], ch.XI,§94)

$$M'u = \sum_{j=1}^{s} (u, \psi_j)\omega_j,$$

where s is a finite number, and ψ_j and ω_j are elements of the space H. Thus,

$$\parallel \sum_{k=n+1}^{\infty} (M'u, \varphi_k)\varphi_k \parallel = \parallel \sum_{k=n+1}^{\infty} \Big(\sum_{j=1}^{s} (u, \psi_j)\omega_j, \varphi_k \Big)\varphi_k \parallel =$$

$$= \parallel \sum_{k=n+1}^{\infty} \sum_{j=1}^{s} \Big((u, \psi_j)\omega_j, \varphi_k \Big)\varphi_k \parallel = \parallel \sum_{k=n+1}^{\infty} \sum_{j=1}^{s} (u, \psi_j)(\omega_j, \varphi_k)\varphi_k \parallel =$$

$$= \parallel \sum_{j=1}^{s} (u, \psi_j) \sum_{k=n+1}^{\infty} (\omega_j, \varphi_k)\varphi_k \parallel \leq \sum_{j=1}^{s} \mid (u, \psi_j) \mid \cdot \parallel \sum_{k=n+1}^{\infty} (\omega_j, \varphi_k)\varphi_k \parallel \leq$$

$$\leq \parallel u \parallel \sum_{j=1}^{s} \parallel \psi_j \parallel \sqrt{\sum_{k=n+1}^{\infty} \mid (\omega_j, \varphi_k) \mid^2}.$$

The number series $\sum_{k=n+1}^{\infty} \mid (\omega_j, \varphi_k) \mid^2$ converges. Then there always exists a number N_ε such that for all $n \geq N_\varepsilon$ a coefficient of $\parallel u \parallel$ is less than $\frac{\varepsilon}{2}$

$$\parallel \sum_{k=n+1}^{\infty} (M'u, \varphi_k)\varphi_k \parallel < \frac{\varepsilon}{2} \parallel u \parallel . \tag{8}$$

Then it follows from (6)-(8) that for $n \geq N_\varepsilon$ the inequality $\parallel (M - M_n)u \parallel < \varepsilon$ or $\parallel M - M_n \parallel < \varepsilon$ holds.

On the basis of Theorem 1, if $\parallel M - M_n \parallel \to 0$ as $n \to \infty$ and M, M_n are completely continuous operators, then the eigenvalues of equation (4) are obtained from the eigenvalues of equation (5) by proceeding to limit as $n \to \infty$.

□

N.I. Polsky introduced the so-called (A)-condition, or the Polsky condition, of the projection methods convergence (e.g., see [17]). Also, N.I. Polsky noted that for a positive operator this condition in the Galerkin method automatically holds. Thus, under the conditions of Theorem 3 and on the basis of the results obtained in papers [15]–[17], the eigenvalues of equation (1) can be obtained as the limits of approximate eigenvalues. Hence, we have

Theorem 3. *If L is a discrete semibounded operator acting in a separable Hilbert space H, then there exists a unique solution of the problem of finding the eigenvalues and eigenfunctions of the operator L. Approximate values of the eigenvalues can be found by the Galerkin method.*

Let $\widetilde{\mu}_k(n)$ be n-th approximate eigenvalues of k-th eigenvalue μ_k of the operator L found by the Galerkin method. Then on the basis of Theorems 1 and 2 we have

$$\lim_{n\to\infty} \widetilde{\mu}_k(n) = \mu_k, \;\; k \in N. \tag{9}$$

The following theorem was proved in the paper [12].

Theorem 4. *Let L be a discrete semibounded operator acting in a separable Hilbert space H. If a system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in H and satisfies the homogeneous boundary conditions (2), then approximate eigenvalues $\widetilde{\mu}_k$ of the operator L are found by the formulas [12]*

$$\widetilde{\mu}_k(n) = (L\varphi_k, \varphi_k) + \delta_n, \;\; k = \overline{1, n}, \tag{10}$$

where $\delta_n = \sum\limits_{k=1}^{n-1} [\widetilde{\mu}_k(n-1) - \widetilde{\mu}_k(n)]$.

Note that for any $k \in N$ $\lim\limits_{n\to\infty} (\widetilde{\mu}_k(n-1) - \widetilde{\mu}_k(n)) = 0$, therefore $\lim\limits_{n\to\infty} \delta_n = 0$. Then from (9) and (10) we obtain

$$\mu_k = (L\varphi_k, \varphi_k), \;\; \forall k \in N. \tag{11}$$

Thus, the following theorem is proved.

Theorem 5. *Let L be a discrete semibounded operator acting in a separable Hilbert space H. If a system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in H and satisfies the homogeneous boundary conditions of spectral problem (1), (2), then eigenvalues μ_k of the operator L are found by the formulas (11).*

The formulas (11) obtained by the Galerkin method allow to calculate eigenvalues of discrete semibounded operators with high computational efficiency. In contrast to the classical methods, the formulas (11) sharply reduce the number of calculations. Also, the formulas allow to find eigenvalues of the operator L, regardless of whether the eigenvalues with smaller numbers are known or not. The formulas solve the problem on a calculation of all necessary spectrum points of the abstract discrete semibounded operators.

3. Computational Experiments

In order to verify the obtained formulas (11), which allow to calculate eigenvalues of discrete semibounded operators, we consider the spectral problem

$$\begin{cases} -u'' + p_1(x)u' + p_2(x)u = \mu u, \;\; a < x < b, \\ \cos\alpha \; u'(a) + \sin\alpha \; u(a) = 0, \\ \cos\beta \; u'(b) + \sin\beta \; u(b) = 0, \quad \alpha, \beta \; \in R. \end{cases} \tag{12}$$

The functions $p_1(x)$, $p_2(x)$ are continuous on the interval $[a, b]$. In order to construct a system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ in the Galerkin method, which satisfies the boundary conditions (12), we find eigenvalues and eigenfunctions of the auxiliary spectral problem

$$\begin{cases} -\varphi'' = \lambda\varphi, \ a < x < b, \\ \cos\alpha \ \varphi'(a) + \sin\alpha \ \varphi(a) = 0, \\ \cos\beta \ \varphi'(b) + \sin\beta \ \varphi(b) = 0, \quad \alpha, \beta \ \in R. \end{cases} \tag{13}$$

The eigenvalues λ_k of the problem (13) are roots of a transcendental equation

$$[\sin\alpha\sin(\sqrt{\lambda}a) + \sqrt{\lambda}\cos\alpha\cos(\sqrt{\lambda}a)] \times [\sin\beta\cos(\sqrt{\lambda}b) - \sqrt{\lambda}\cos\beta\sin(\sqrt{\lambda}b)] +$$

$$+[\sqrt{\lambda}\cos\alpha\sin(\sqrt{\lambda}a) - \sin\alpha\cos(\sqrt{\lambda}a)] \times [\sin\beta\sin(\sqrt{\lambda}b) + \sqrt{\lambda}\cos\beta\cos(\sqrt{\lambda}b)] = 0,$$

and the eigenfunctions have the form:

$$\varphi_k(x) = C_k\{[\sin\alpha\sin(\sqrt{\lambda_k}a) + \sqrt{\lambda_k}\cos\alpha\cos(\sqrt{\lambda_k}a)]\cos(\sqrt{\lambda_k}x) +$$

$$+[\sqrt{\lambda_k}\cos\alpha\sin(\sqrt{\lambda_k}a) - \sin\alpha\cos(\sqrt{\lambda_k}a)]\sin(\sqrt{\lambda_k}x)\}, \quad k = \overline{1, \infty}.$$

The constants C_k are found from the normalization condition.

Let $\{\widetilde{\mu}_k\}_{k=1}^{n}$ and $\{\widehat{\mu}_k\}_{k=1}^{n}$ be eigenvalues of the Sturm-Liouville spectral problem found by the formulas (11) and the Galerkin method, respectively. We compare the results of the eigenvalues calculation. An example of calculation of the boundary problem (12) eigenvalues is given in Table for $a = 0$, $b = 1$, $\alpha = Pi/3$, $\beta = Pi/5$, $p_1(x) = -2\sin(5x) + 3x$, $p_0(x) = -3\cos(2x) + x$. As the number of basic functions used in the Galerkin method increases, the values of approximate eigenvalues $\{\widehat{\mu}_k\}_{k=1}^{n}$ found by the Galerkin method approach to the eigenvalues $\{\widetilde{\mu}_k\}_{k=1}^{n}$ found by the formulas (11).

Table

k	$\widetilde{\mu}_k$	$\widehat{\mu}_k$	$\lvert\widetilde{\mu}_k - \widehat{\mu}_k\rvert$
1	10, 541004	10, 628951	0, 087947
2	39, 303090	41, 233238	1, 930148
3	88, 515916	90, 163626	1, 861448
4	157, 556701	159, 191315	1, 634614
5	246, 361792	247, 960088	1, 598295
6	354, 915883	356, 506057	1, 590174
7	483, 213785	484, 793380	1, 579595
8	631, 253343	632, 828726	1, 575384
9	799, 033529	800, 603920	1, 570392
10	985, 553804	988, 121318	1, 587514
11	1193, 813863	1195, 378096	1, 564233
12	1420, 813523	1422, 375281	1, 561758
13	1667, 552667	1669, 111637	1, 558971
14	1934, 031219	1935, 587668	1, 556449
15	2220, 249128	2221, 802734	1, 553606
16	2526, 206359	2527, 757093	1, 550734
17	2851, 902885	2853, 45O361	1, 547476
18	3197, 338688	3198, 882655	1, 543967
19	3562, 513753	3564, 053703	1, 539950
20	3947, 428071	3948, 963524	1, 535453
21	4352, 081633	4353, 611879	1, 530246

Conclusion

The theorem on the existence and uniqueness of solution of the problem on finding eigenvalues and eigenfunctions of discrete semibounded operators is proved. The theorem on the convergence of the Galerkin method in application to the problem on finding eigenvalues is proved. Simple formulas to calculate the eigenvalues are obtained. In contrast to the classical methods, the formulas sharply reduce the number of calculations. Also, the formulas allow to find eigenvalues of the operator, regardless of whether the eigenvalues with smaller numbers are known or not. The formulas solve the problem on a calculation of all necessary spectrum points of the abstract discrete semibounded operators.

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ВЫЧИСЛЕНИЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ ДИСКРЕТНЫХ ПОЛУОГРАНИЧЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ

С. И. Кадченко, Г. А. Закирова

В работе рассматривается задача на собственные значения для абстрактного дискретного полуограниченного оператора, действующего в сепарабельном гильбертовом пространстве. Доказываются теоремы о существовании и единственности решения данной спектральной задачи, а так же доказана сходимость метода Галеркина применительно к этой задаче. На основе метода Галеркина получены формулы для вычисления собственных значений абстрактного дискретного полуограниченного оператора. Данные формулы позволяют проводить расчет собственных значений дискретных полуограниченных операторов с высокой вычислительной эффективностью. В отличие от классических методов, данные формулы резко сокращают количество вычислений. Кроме того, собственные значения оператора можно вычислять, независимо от того, известны или нет все предыдущие собственные значения. В работе представлены результаты вычислительных экспериментов.

Ключевые слова: собственные числа, собственные функции, возмущение, дискретный оператор, метод Галеркина, существование и единственность решения.

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