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SPECTRAL PROBLEMS FOR ONE MATHEMATICAL MODEL OF HYDRODYNAMICS

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This paper is devoted to the investigation of two spectral problems: the eigenvalue problem and the inverse spectral problem for one mathematical model of hydrodynamics, namely the mathematical model for the evolution of the free filtered-fluid surface. The Galerkin method is chosen as the main method for solving the eigenvalue problem. A theorem on the convergence of Galerkin's method applied to this problem was given. For the given spectral problem the algorithm was developed. A program that allows calculating the eigenvalues of the perturbed operator was produced in Maple. For the inverse spectral problem, the resolvent method was chosen as the main one. For this spectral problem, an algorithm is also developed. A program that allows one to approximately reconstruct the potential from the known spectrum of the perturbed operator was created in Maple. The theoretical results were illustrated by numerical experiments for a model problem. Numerous experiments carried out have shown a high computational efficiency of the developed algorithms.

Keywords: perturbed operator, discrete self-adjoint operator, eigenvalues of the inverse spectral problem, potential, Dzektser equation.

Introduction

The importance of the spectral theory lies in a wide range of applications in natural science and technology. Spectral problems appear in shell theory, hydrodynamics, quantum mechanics, etc. [1]–[5].

This paper describes spectral problems for equation

$$(\lambda - \Delta)u_t = \alpha\Delta u - \beta\Delta^2 u + f,$$

which simulates, as we know, the evolution of the free surface of a liquid filtration [6]. Parameter α is defined by formula

$$\alpha = \frac{\varepsilon_\alpha + k}{kh_0 a},$$

where α is the void ratio, ε_α – the power module of the flow through the free surface, k – coefficient of filtration, h_0 – the pressure on the free surface [6]. Parameters λ and β are determined by using the following formula

$$\lambda = \frac{2(\varepsilon_\alpha + k)}{k^2 H_0^2}, \quad \beta = \frac{h_0}{3a}.$$

Research of this problem we can find in the works of many authors. Nonstationary models of the free surface of a liquid filtration were considered [7]. A "quasi-Banach" analogue of the homogeneous Dirichlet problem in a limited area with smooth boundary for a linear Dzektser equation were considered [8]. The existence of of exponential dichotomies of solutions Dzektser evolution equation of the Sobolev type in the quasi-Sobolev spaces can be found in [9]. With regard to the spectral problem, the first attempt initiated in [10].

1. Eigenvalue problem

Let operators $T, L : \mathfrak{U} \longrightarrow \mathfrak{F}$ are determined by using the following formulas

$$T = \alpha\Delta - \beta\Delta^2, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad L = \lambda - \Delta, \tag{1}$$

whereas

$$\mathfrak{U} = \{u \in W_2^{k+2}(0,b) : u(0) = u(b)\},$$
$$\mathfrak{F} = \{u \in W_2^k\}, k \in 0\bigcup\mathbb{N},$$
$$domT = \{u \in W_2^{k+2}(0,b) : u''(0) = u''(b) = 0\}\bigcap\mathfrak{U}.$$

Let P — operator of multiplication by a function $p \in C^2(0,b)$.

Consider the operator $T+P$. We denote by $\{\nu_n\}_{n=1}^{\infty} = \sigma^L(T+P)$, where ν_n numbered in decreasing order of their real parts given algebraic multiplicity.

Using [11] we introduce the L–resolvent set of the operator T:

$$\rho^L(T) = \{\mu \in \mathbb{C} : (\mu L - T)^{-1} \in \mathcal{L}(F;U)\};$$

L-resolvent of the operator T $R_0(\mu) = (\mu L - T)^{-1}$; $R(\mu) = (\mu L - T - P)^{-1}$ — L-resolvent of operator $T+P$.

It is clear that L–eigenvalues of the operator T

$$\mu_n = \lambda_n \frac{\beta\lambda_n - \alpha}{\lambda_n - a^2}, \tag{2}$$

where $\{\lambda_n\}_{n=1}^{\infty} = \sigma(\Delta)$ — the eigenvalues of the Laplace operator generated by the Dirichlet boundary value problem:

$$\Delta u = a^2 u, \quad u(0) = u(b) = 0,$$
$$\lambda_n = -n^2, n \in N.$$

Consider the direct spectral problem: we know the eigenvalues T and perturbation p. We need to find the L– eigenvalues of the perturbed operator $T+P$. For the operator L we have

$$L\varphi_s = (a^2 - \Delta)\varphi_s = (a^2 - \lambda_s)\varphi_s = \begin{cases} (a^2 - \lambda_s)\varphi_s, & a^2 \neq \lambda_s,\ \lambda_s = -n^2, \\ 0, & a^2 = \lambda_s,\ a^2 \neq -n^2, \end{cases}$$

therefore:

$$R_0^L(\mu) = \begin{cases} \dfrac{1}{\mu - \mu_s}, & a^2 \neq \lambda_s, \\ 0, & a^2 = \lambda_s, \end{cases} \quad R_0^L(\mu)\varphi_s = \begin{cases} \dfrac{\varphi_s}{\mu - \mu_s}, & a^2 \neq \lambda_s, \\ 0, & a^2 = \lambda_s, \end{cases}$$

$$R_0(\mu) = \begin{cases} \dfrac{1}{(\mu - \mu_s)(a^2 - \lambda_s)}, & a^2 \neq \lambda_s, \\ \dfrac{1}{\beta a^4 - \alpha a^2}, & a^2 = \lambda_s, \end{cases} \quad R_0(\mu)\varphi_s = \begin{cases} \dfrac{\varphi_s}{(\mu - \mu_s)(a^2 - \lambda_s)}, & a^2 \neq \lambda_s, \\ \dfrac{\varphi_s}{\beta a^4 - \alpha a^2}, & a^2 = \lambda_s. \end{cases}$$

Operators $R_0(\mu), R_0^L(\mu), \mu \in \rho(T)$, are nuclear, since the series of eigenvalues of these operators are converge. Let $\gamma_n = \{\mu \in \mathbb{C} : |\mu - \mu_n| = r_n, \quad r_n = n\beta - \frac{\beta}{2}\}$, where $\beta \in R_+$.

1.1. Approximate calculation of the eigenvalues

The relative eigenvalues ν_k of the operator $T+P$ are defined by determining nontrivial solutions of the equation

$$(T+P)u = \nu L u \tag{3}$$

that satisfies certain homogeneous boundary conditions.

To find the eigenvalues of the operator $T+P$ we use the method of Galerkin. We introduce the sequence $\{H_n\}_{n=1}^{\infty}$ of finite-dimensional spaces $H_n \subseteq H$, which will be full in H. Let an orthonormal basis in the space H_n is known and it consists of functions $\{\varphi_k\}_{k=1}^{\infty}$. The functions φ_k must satisfy the boundary conditions of problem. According to the Galerkin method, we seek an approximate solution of the spectral problem (3) in the form of

$$u_n = \sum_{k=1}^{n} a_k(n)\varphi_k, k \in N. \tag{4}$$

Operator $T+P$ is a discrete, consequently, its resolvent is completely continuous.

Theorem 1. [12] *If the operator $T = A_0^{-1}K$ is completely continuous in the H_0, then the Galerkin process is convergent in the problem of determining eigenvalues.*

Using ideas from the works [12], [13] we can prove the next theorem.

Theorem 2. *Let operators $T, L : U \to F$ are defined by formulas*

$$T = \alpha\Delta - \beta\Delta^2, \quad \Delta = \frac{d^2}{dx^2}, \quad L = a^2 - \Delta, \tag{5}$$

and besides

$$U = \{u \in W_2^{k+2}(0,b) : u(x) = 0, x \in (0,b)\},$$

$$F = W_2^k(0,b), k \in \{0\} \cup \mathbb{N}, u = W_2^{k+2}(0,b),$$

$$domT = \big\{u \in W_2^{k+2}(0,b) : u''(0) = u''(b) = 0\big\} \cap U.$$

Let operator P be the operator of multiplication on the function $p \in C^2(0,b)$.

If the system of the coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ is the basis H, the Galerkin method, which is applied to the spectral problem (3) of determining the eigenvalues is convergent.

If we solve the problem by the Galerkin method, then solution is presented in the form of

$$\varphi(x) = \sum_{k=1}^{n} a_k \sin\frac{\pi k x}{b}. \tag{6}$$

Further we substitute (6) to (3) and construct a discrepancy:

$$N(x) = L\left[\sum_{k=1}^{n} a_k\varphi_k(x)\right],$$

$$N(x) = \alpha u_{xx} - \beta(u_{xx})^2 + Pu - \nu a^2 u + \nu u_{xx}.$$

Next, if we will demand that the discrepancy of the basis functions be orthogonal, we obtain:

$$\int_a^b N(x)\varphi_k(x)dx = 0,$$

so $$\int_a^b a_k(\alpha u_{xx} - \beta(u_{xx})^2 + Pu - \nu a^2 u + \nu u_{xx})\sin(\frac{\pi}{b}kx)dx = 0.$$

As a result, when our homogeneous system of equations for the coefficients in the expansion will be solved, we'll have approximate eigenvalues.

1.2. Computational experiment

The program in Maple 16 was developed. With the help of this program we can determine the approximate L–eigenvalues of the operator $T + P$ using the eigenvalues of the operator T and potential p, that we know. Numerical experiments confirm the good computational accuracy. In some experiments, the outrage has been put equal to zero. In this case, was shown high correlation with the results obtained analytically

The results of computational experiments are presented in tables 1 and 2.

Table 1

The Numerical Solution of the Problem (3),
when $p = 0, \alpha = 0, \beta = 1, a = 0, b = \frac{\pi}{4}, a^2 = 2.$

№	ν_k
1	–14.22222222
2	–62.06060606
3	–142.0273973
4	–254.0155039
5	–398.0099502
6	–574.0069204
7	–782.0050891
8	–1022.003899
9	–1294.003082
10	–1598.002497
11	–1934.002064
12	–2302.001735
13	–2702.001478
14	–3134.001275
15	–3598.001110
16	–4094.000976
17	–4622.000865
18	–5182.000771
19	–5774.000692
20	–6398.000625

Table 2

The Numerical Solution of the Problem (3),
for $p = \sin(4x), \alpha = 0, \beta = 1, a = 0, b = \frac{\pi}{4}, \gamma = 1.$

№	ν_k
1	–899.0011099
2	–728.0013699
3	–575.0017331
4	–440.0022624
5	–323.0030769
6	–224.0044248
7	–143.0068966
8	–80.01219512
9	–35.02702703
10	–8.100000000

2. Inverse spectral problem

We will consider the inverse spectral problem: we know the eigenvalues of the operator T and the perturbation p. We need to find the eigenvalues of the perturbed operator $T+P$. One dimensional case was considered in [10]. Further we will consider two-dimensional case, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

It can be shown that the equation

$$p = \alpha_0 - \alpha(p),$$

where

$$\alpha_0 = \sqrt{\frac{4}{ab}} \sum_{t=1}^{\infty} \sum_{k=1}^{\nu_t} (\xi_t^k - \lambda_t)\varphi_t^k,$$

$$\alpha(p) = (-1)^N \sqrt{2^N V} \sum_{t=1}^{\infty} \sum_{k=1}^{\nu_t} \frac{\alpha_t(p)}{\nu_t} \varphi_t^k,$$

$$\alpha_t(p) = \frac{1}{2\pi i} \int_{\gamma_{r_t}} \lambda Sp[R(\lambda)(PR_0(\lambda))^2] d\lambda = \frac{1}{2\pi i} \int_{\gamma_{r_t}} Sp[\sum_{k=2}^{\infty} R_0(\lambda)(PR_0(\lambda))^k] d\lambda,$$

has a unique solution p.

This solution can be found by using the method of successive approximations

$$\widetilde{\alpha}(p) = \sum_{t=1}^{\infty} \widetilde{\alpha_t}(p)\varphi_t, \quad \widetilde{\alpha_t}(p) = \frac{1}{2\pi i} \int_{\gamma_{r_t}} \lambda Sp[R_0(\lambda)(PR_0(\lambda))^2] d\lambda.$$

We figure out $\widetilde{\alpha_t}(\alpha_0)$.

$$\widetilde{\alpha_t}(\alpha_0) = \frac{1}{2\pi i} \int_{\gamma_{r_t}} \lambda Sp[R_0(\lambda)(P_1 R_0(\lambda))^2] d\lambda =$$

$$= \sum_{j \neq t} \sum_{k=1}^{\nu_t} \frac{(\alpha_0 \varphi_t^k, \varphi_j^k) \cdot (\alpha_0 \varphi_j^k, \varphi_t^k)}{\lambda_t^k - \lambda_j^k} = \sum_{j \neq t} \sum_{k=1}^{\nu_t} \frac{(\alpha_0 \varphi_t^k, \varphi_j^k)^2}{(\lambda_t^k - \lambda_j^k)}.$$

As a result, we obtain a more convenient formula for constructing an algorithm

$$\widetilde{p} = \alpha_0 - \sqrt{4ab}(\sum_{j\neq t}\sum_{k=1}^{\nu_t}\frac{(\alpha_0\varphi_t,\varphi_j)^2}{(\lambda_t-\lambda_j)})\varphi_t^k.$$

2.1. Computational experiment

Alas, there is no single numerical solution for the problem of determining the potential. We have developed a program in Maple 16. Use this program you can you can determine the approximate potential explicitly, so that the spectrum of the perturbed operator will coincide with this sequence.

We give an example illustrating the work of the program.

$$(\lambda-\Delta)u_t = \alpha\Delta u - \beta\Delta^2 u + f,$$

Operators $T, L : \mathfrak{U} \longrightarrow \mathfrak{F}$ are determined by formulas

$$T = \alpha\Delta - \beta\Delta^2, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad L = \lambda - \Delta, \tag{7}$$

where $\alpha = \frac{8}{10}, \lambda = 2.1, \beta = \frac{1}{\pi^2}$. Suppose also, $\xi_n = \mu_n + 0.0001, n \leq 10$. There is the potential $p \in L_2(0;\pi)$ such that for any $n \in \mathbb{N}$

$$\xi_n = \nu_n \tag{8}$$

$\{\nu_{mn}\} = \sigma^L(T+P)$. An approximate potential which we have reconstituted using the program by the first ten members of the sequence $\{\xi_n\}$ shown in Figure.

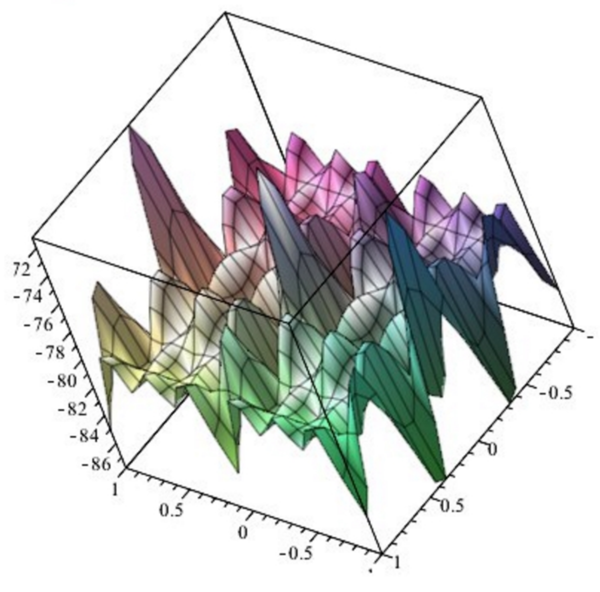

Fig. Reconstituted potential

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СПЕКТРАЛЬНЫЕ ЗАДАЧИ ДЛЯ ОДНОЙ МОДЕЛИ ГИДРОДИНАМИКИ

И. С. Стрепетова, Л. М. Фаткуллина, Г. А. Закирова

Работа посвящена исследованию двух спектральных задач: задаче на собственные значения и обратной спектральной задаче для одной математической модели гидродинамики, а именно математической модели эволюции свободной поверхности фильтрующейся жидкости. Основным методом решения задачи на собственные значения выбран метод Галеркина. Приведена теорема о сходимости метода Галеркина применительно к данной задаче. Для данной спектральной задачи разработан алгоритм и на его основе в среде Maple написана программа, позволяющая вычислять собственные числа возмущенного оператора. Для обратной спектральной задачи в качестве основного выбран резольвентный метод. Для данной спектральной задачи также разработан алгоритм и на его основе в среде Maple написана программа, позволяющая приближенно восстановить потенциал по известному спектру возмущенного оператора. Теоретические результаты проиллюстрированы с помощью вычислительных экспериментов для модельных задач. Многочисленные проведенные эксперименты показали высокую вычислительную эффективность разработанных алгоритмов.

Ключевые слова: возмущенный оператор, дискретный самосопряженный оператор, собственные значения, уравнение Дзекцера.

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