SHORT NOTES

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THE SPECTRAL IDENTITY FOR THE OPERATOR WITH NON-NUCLEAR RESOLVENT

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Direct spectral problems play an important role in many branches of science and technology. In a hight number of mathematical and physical problems is required to find the spectrum of various operators. The inverse spectral problems also have a wide range of applications. To solve them, we often find a solution to the direct problem. The method of regularized traces effectively al lows us to find the eigenvalues of the perturbed operator. This method is not feasible to the operator with a non-nuclear resolution. This is related to the selection of a special function that transforms the eigenvalues of the operator. Currently, there is an active search for methods that makes it possible to calculate the eigenvalues of a perturbed operator with a non-nuclear resolvent. In this paper, we consider a direct spectral problem for an operator with a non-nuclear resolvent perturbed by a bounded one.The method of regularized traces is chosen as the main method for solving this problem. Broadly speaking, this method can not be applied to this problem. It is impossible to take advantage of Lidsky's theorem because the operator has a non-nuclear resolvent. We proposed to introduce the relative resolvent of the operator. In this case, the operator L was chosen so that the relative resolvent of the operator is a nuclear operator. As a result of applying the resolvent method to the relative spectrum of the perturbed operator, we obtain the relative eigenvalues of the perturbed operator with the non-nuclear resolvent.

Keywords: perturbed operator, discrete self-adjoint operator, direct spectral problem, relative resolvent.

Introduction

The work is devoted to the study of the direct spectral problem for operator, whose resolvent is not a nuclear operator. Given problems play a huge role in various areas of mathematics, physics and engineering, as well as used for the formulation and solution of inverse spectral problems, which in turn also have an extensive area of applications. To solve this problem the resolvent method developed by V.A. Sadovnichy and V.V. Dubrovsky [1], is applied to relative or L-resolvent of operator [2]. This approach was used in [3] to the operator with nuclear resolvent. Note that the nuclear operators play inportant role in construction of space of "differentiable noises" (see [4], [5]).

The origins of the theory of regularized traces, in which developed the method used, refer to [6]. In [7] formulas for regularized traces of abstract operators with the nuclear resolvent and limited perturbation were obtained. Analogues of these formulas were obtained for unbounded perturbations in [8]. The conditions for limited perturbation had been replaced by condition on it's subordination with regard to perturbed operator. The formulas obtained in this article are correct for the operator with non-nuclear, but compact resolvent and unlimited perturbation. The main difference from other works devoted to spectral problems is the fact that there are no conditions on the resolvent of operator, but on the relative resolvent. Perturbing operator relies limited.

1. Necessary statements

Let operators M and L act in the separable Hilbert space $\mathfrak{H}$. M – discrete, self-adjoint and positive operator, L – linear, closed, continuously invertible operator. Let the operator P is bounded operator acting in the same space $\mathfrak{H}$. $R_0(\mu) = (\mu L - M)^{-1}$ is the L-resolvent of operator M, $R(\mu) = (\mu L - M - P)$ is the L-resolvent of perturbed operator $M + P$. $\{\lambda_n\}_{n=1}^{\infty}$ denotes the eigenvalues of operator M, $\{\mu_n\}_{n=1}^{\infty} = \sigma^L(M)$ – L-spectrum of operator M, $\{\nu_n\}_{n=1}^{\infty} = \sigma^L(M+P)$ – L-spectrum operator $M + P$.

Lemma 1. [9] *If* $\|P\|_{\mathfrak{H}} < r/2$, *where* $0 < r \le r_0$, *then operator* $M + P$ *discrete, and*
(i) if $R_0(\lambda) \in \mathfrak{S}_q$, *then* $R(\lambda) \in \mathfrak{S}_q$, $1 \le q < \infty$,
(ii) if $\lambda_t \in \mathbb{C} \setminus \Omega_{r_t}$, *then* $\mu_t^s \in \mathbb{C} \setminus \Omega_{r_t}$, $s=\overline{1,\nu_t}$, ν_t *– the multiplicity of the eigenvalue* λ_t.

Lemma 2. *If* $\mu \in \gamma_n$ $\|PR_0(\mu)\|_{\mathfrak{H}} = q < 1$, *then equality is true*

$$LR(\mu) = LR_0(\mu) + \sum_{k=1}^{\infty}[R_0(\mu)P]^k LR_0(\mu) \tag{1}$$

Proof.
Consider the identity $\mu L - T - P = (\mathbb{I} - pR_0(\mu))(\mu L - T)$. Because $\|PR_0(\mu)\| < 1$, then here exists a linear bounded operator

$$R(\mu) = (\mu L - T - P)^{-1} = R_0(\mu)(\mathbb{I} - pR_0(\mu))^{-1}.$$

It follows $R(\mu) = R_0(\mu)B(\mu)$, where $B(\mu)$ is a bounded operator. Because the T is discrete, then $R_0(\mu)$ it is completely continuous, therefore, $R(\mu)$ is also a completely continuous operator, that is, the operator $T + P$ it is a discrete. From this relation it also follows that $R(\mu)$ – nuclear operator, for which we have the expansion in the norm convergent series:

$$R(\mu) = R_0(\mu) + \sum_{k=1}^{\infty}[R_0(\mu)P]^k R_0(\mu),$$

hence, by multiplying the previous identity in the L to the left, we get

$$LR(\mu) = LR_0(\mu) + \sum_{k=1}^{\infty}[R_0(\mu)P]^k LR_0(\mu).$$

□

Lemma 3. *Let* $\mu \in \rho^L(T)$, *then the following estimate:*

$$\|LR_0(\mu)\|_{\mathfrak{H}} \le \frac{1}{\rho(\mu, \sigma^L(T))}, \tag{2}$$

here $\rho(\mu, \sigma^L(T))$ *it means the distance from the point* μ *before* L*–spectrum of operator* T.

Proof.

Let $\lambda \in \rho^L(T)$. From [2] it is known the representation of T operator's L–resolvent in the form of a Neumann series $R_0(\lambda) = R_0(\mu)\sum_{k=0}^{\infty}(\mu-\lambda)^k(LR_0(\mu))$.

Obviously, the number of the right side is absolutely convergent, at least for those λ, which satisfy the condition $|\lambda-\mu| < \frac{1}{\|LR_0(\mu)\|}$.

Hence we obtain

$$\|LR_0(\mu)\| \leq \frac{1}{\rho(\mu, \sigma^L(T))}.$$

Consider the difference in norm of Riesz projectors

$$\Big\|\frac{1}{2\pi i}\int_{\gamma_n}(R(\mu)-R_0(\mu))d\mu\Big\| \leq \int_{\gamma_n}\|R_0(\mu)P\|\cdot\|R(\mu)\||d\mu| < \int_{\gamma_n}\|R_0(\mu)L\|\cdot\|R(\mu)\||d\mu| < 1,$$

therefore all root subspaces of $T+P$ have the same dimension as the operator T, therefore, the spectrum of $T+P$ will be a single.

□

Theorem 1. *Let M – discrete, self-adjoint positive operator, L – linear, closed, is continuously invertible operator, and L^{-1} bounded operator. $R_0(\mu)$ – nuclear operator and $P \leq \frac{r_0}{2}$, $\gamma_n = \{\mu : |\mu-\mu_n = r_n\}$, $r_n = \frac{1}{2}\min_n\{\mu_{n+1}-\mu_n; \mu_n-\mu_{n-1}\}, r_0 = \inf_n r_n$, then it is a spectral identity:*

$$\nu_n = \lambda_n + (L^{-1}P\varphi_n, \varphi_n) + \alpha_n, \tag{3}$$

where

$$\alpha_n = \sum_{k=2}^{\infty}\frac{(-1)^k}{2\pi i}\int_{\gamma_n}\mu Sp[R_0(\mu)P]^k LR_0(\mu)d\mu.$$

Proof.

Consider the series (5)

$$LR(\mu) = LR_0(\mu) + \sum_{k=1}^{\infty}[R_0(\mu)P]^k LR_0(\mu).$$

Multiply the right and left sides of this equation by $\frac{\mu}{2\pi i}$ and integrate the resulting equation along the contour γ_n. We get

$$\frac{1}{2\pi i}\int_{\gamma_n}\mu LR(\mu)d\mu = \frac{1}{2\pi i}\int_{\gamma_n}\mu LR_0(\mu)d\mu + \sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{\gamma_n}\mu[R_0(\mu)P]^k LR_0(\mu)d\mu.$$

We find the matrix trace of both sides of the resulting equality, with the use of nuclear operators T и $T+P$.

$$Sp\frac{1}{2\pi i}\int_{\gamma_n}\mu LR_0(\mu)d\mu = Sp\frac{1}{2\pi i}\int_{\gamma_n}\mu\sum_{k=1}^{\infty}\frac{P_k}{\mu-\mu_k}d\mu = \mu_n SpP_n = \mu_n\sum_{s=1}^{\infty}(P_n\varphi_s, \varphi_s) =$$

$$= \mu_n\sum_{s=1}^{\infty}\Big((\varphi_s, \varphi_n)\varphi_n, \varphi_s\Big) = \mu_n\sum_{s=1}^{\infty}\Big((\varphi_s, \varphi_n)(\varphi_n, \varphi_s)\Big) = \mu_n.$$

Similarly

$$Sp\frac{1}{2\pi i}\int_{\gamma_n}\mu LR(\mu)d\mu=\nu_n.$$

We calculate the first correction of perturbation theory:

$$\frac{1}{2\pi i}\int_{\gamma_n}Sp\mu R_0(\mu)PLR_0(\mu)=\frac{1}{2\pi i}\int_{\gamma_n}\mu\sum_{s=1}^{\infty}(R_0(\mu)PLR_0(\mu)\varphi_s,\varphi_s)d\mu=$$

$$=\frac{1}{2\pi i}\int_{\gamma_n}\mu\sum_{s=1}^{\infty}(\frac{1}{\mu-\mu_s}L^{-1}PL\frac{1}{\mu-\mu_s}L^{-1}\varphi_s,\varphi_s)d\mu=$$

$$=\frac{1}{2\pi i}\int_{\gamma_n}\mu\sum_{s=1}^{\infty}(\frac{1}{(\mu-\mu_s)^2}L^{-1}PLL^{-1}\varphi_s,\varphi_s)d\mu=\frac{1}{2\pi i}\sum_{s=1}^{\infty}\int_{\gamma_n}\mu\frac{1}{(\mu-\mu_s)^2}(L^{-1}P\varphi_s,\varphi_s)d\mu=$$

$$=\frac{1}{2\pi i}\sum_{s=1}^{\infty}(L^{-1}P\varphi_s,\varphi_s)\int_{\gamma_n}\frac{\mu}{(\mu-\mu_s)^2}d\mu=(L^{-1}P\varphi_n,\varphi_n).$$

We obtain the following spectral identity:

$$\nu_n=\mu+(L^{-1}P\varphi_n,\varphi_n)+\frac{1}{2\pi i}\sum_{k=2}^{\infty}\int_{\gamma_n}\mu\left(R_0(\mu)P\right)^k LR_0(\mu).$$

We show that $\alpha_n\to 0$ при $n\to\infty$. Consider

$$\alpha_n=\sum_{k=2}^{\infty}\frac{(-1)^k}{2\pi i}\int_{\gamma_n}\mu\mathrm{Sp}[R_0(\mu)P]^kLR_0(\mu)d\mu=\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{2\pi ik}\int_{\gamma_n}\mu\mathrm{Sp}[R_0(\mu)P]^kLR_0(\mu)d\mu.$$

We estimate the number of members at $k\geq 2$. $d(\mu)$ – distance from the point $\mu\in\mathbb{C}$ to the relative spectrum T.

$$\int_{\gamma_n}|\mathrm{Sp}[PR_0(\mu)]^kLR_0(\mu)||d\mu|\leq\int_{\gamma_n}\|PR_0(\mu)\|^{k-1}\|P\|\|R_0(\mu)\|_1\|LR_0(\mu)\|d\mu||\leq$$

$$\leq\|P\|^k\|R_0(\mu)\|_1\int_{\gamma_n}\frac{|d\mu|}{d^{k-1}(\mu)}\|LR_0(\mu)\|.$$

We estimate the integral $\int_{\gamma_n}\frac{|d\mu|}{d^{k-1}(\mu)}$, lying in the first quarter of a circle of radius $\frac{r_n}{2}$ centered at μ_n (similarly in the other quarters). Let $\mu=re^{i\beta}$, тогда $d(\mu)\geq r\sin\beta$ and circuit $\gamma_n d(\mu)=\frac{r_n}{2}$.

$$\int_{\gamma_n}\frac{|d\mu|}{d^{k-1}(\mu)}=\int_0^{\theta}\frac{r|e^{i\beta}|d\beta}{d^{k-1}(re^i\beta)}\leq\int_0^{\theta}\frac{rd\beta}{r\sin\beta d^{k-2}(re^{i\beta})}=\int_0^{\theta}\frac{d\beta}{\sin\beta d^{k-2}(re^{i\beta})}=$$

$$=\int_0^{\theta}\frac{d\beta}{\sin\beta(\frac{r_n}{2})^{k-2}})\leq\frac{const}{(\frac{r_n}{2})^{k-2}},\quad\theta=\frac{r_n}{2\sqrt{\frac{r_n^2}{4}+\mu_n^2}}.$$

We find that

$$\left| \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{2\pi ik} \int_{\gamma_n} \mu \mathrm{Sp}[R_0(\mu)P]^k LR_0(\mu)d\mu \right| \leq$$

$$\leq const\|R_0(r_n)\|_1 \sum_{k=2}^{\infty} \frac{\|P\|^k}{(\frac{r_n}{2})^{k-2}} \|LR_0(\mu) \leq const\|R_0(r_n)\|_1\|LR_0(r_n)\|.$$

The theorem is proved.

□

In the future, it is planned to study perturbation theory corrections α_n. Namely, under what conditions do they tend to 0, and from which number. A numerical method will also be developed to calculate the relative eigenvalues of the perturbed operator with a non-nuclear resolution.

References

1. Sadovnichy V.A., Dubrovskii V.V. An Abstract Theorem of Perturbation Theory, Regularized Trace Formulas and Zeta function Operator. *Differ. Uravn.*, 1977, vol. 13, № 7, pp. 1264–1271 (in Russian)

2. Sviridyuk G.A., Fedorov V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators.* Utrecht, Boston, VSP, 2003.

3. Zakirova G.A, Kirillov E.V. L-regularized Trace of One of the Perturbed Operator. *Bulletin of the Odessa National University. Mathematics and Mechanics*, 2013, vol. 18, issue 2(18), pp. 7–13. (in Russian)

4. Sviridyuk G.A., Manakova M.A., Zakirova G.A. The Asymptotics of Eigenvalueso of a Differential Operator in the Stochastic Models with "White Noise". *Applied Mathematical Sciences*, 2014, vol. 8, no. 173-176, pp. 8747–8754. doi: 10.12988/ams.2014.49756

5. Sviridyuk G.A., Manakova M.A. The Dynamical Models of Sobolev Type with Showalter – Sidorov Condition and Additive "Noise". *Bulletin of the South Ural State University. Series Mathematical Modeling, Programming and Computer Sofrware*, 2014, vol. 7, no. 1, pp. 90–103. doi: 10.14529/mmp140108 (in Russian)

6. Gelfand I.M., Levitan B.M. On an Identity for the Eigenvalues of a Differential Operator of the Second Order. *Doklady Akademii nauk SSSR*, 1953, vol. 88, № 4, pp. 593–596. (in Russian)

7. Sadovnichii V. A., Konyagin S. V., Podol'skii V. E. Regularized Trace of a Resolution of the Nuclear Perturbed by Limited. *Doklady Akademii nauk*, 2000, vol. 373, № 1, pp. 26–28. (in Russian)

8. Sadovnichy V.A., Podolsky V.E. Traces of Operators with Relatively Compact Perturbations. *Mat. Sb.*, 2002, vol. 193, № 2, pp. 129–152. doi: 10.4213/sm630 (in Russian)

9. Zakirova G.A., Sedov A.I. An Inverse Spectral Problem for Laplace Operator and It's Approximate Solution. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2008, no. 27(127), issue 2, pp. 19–27. (in Russian)

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СПЕКТРАЛЬНОЕ ТОЖДЕСТВО ДЛЯ ОПЕРАТОРА С НЕЯДЕРНОЙ РЕЗОЛЬВЕНТОЙ

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Прямые спектральные задачи играют важную роль во многих отраслях науки и техники. В огромном количестве математических и физических задачах требуется находить спектр различных операторов. Так же широкую область применения имеют обратные спектральные задачи. Для их решения часто нужно находить решение прямой задачи. Эффективно позволяет находить собственные числа возмущенного оператора метод регуляризованных следов. Но для операторов с неядерной резольвентой данный метод применяется тяжело. Это связанно с подбором специальной функции, преобразующей собственные числа оператора. В настоящее время ведется активный поиск метода, позволяющего вычислять собственные числа возмущенного оператора с неядерной резольвентой. В данной статье рассматривается прямая спектральная задача для оператора с нядерной резольвентой возмущенного ограниченным. В качестве метода решения используется метод регуляризованных следов. На прямую этот метод применить к данной задаче не удается. Нельзя воспользоваться в ходе решения теоремой Лидского, т.к. оператор имеет неядерную резольвенту. Предлагается ввести в рассмотрение относительную резольвенту оператора. При этом оператор L выбран таким образом, что относительная резольвента оператора является ядерным оператором. В результате применения резольвентного метода к относительному спектру возмущенного оператора получаем относительные собственные числа возмущенного оператора с нядерной резольвентой.

Ключевые слова: возмущенный оператор, дискретный самосопряженный оператор, прямая спектральная задача, относительная резольвента.

References

1. Садовничий В.А. Об одной абстрактной теореме теории возмущений, о формулах регуляризованных следов и о дзета-функции операторов / В.А. Садовничий, В.В. Дубровский // Дифференц. уравнения. – 1977. – Т. 13, № 7. – С. 1264–1271.

2. Sviridyuk G.A. Linear sobolev type equations and degenerate semigroups of operators / G.A. Sviridyuk, V.E. Fedorov. – Utrecht, Boston: VSP, 2003.

3. Закирова Г.А. Регуляризованный L-след одного возмущенного оператора / Г.А. Закирова, Е.В. Кириллов // Вестник Одесского национального университета. Математика и механика. – 2013. – Т. 18, вып. 2(18). – С. 7–13.

4. Sviridyuk G.A. The Asymptotics of Eigenvalueso of a Differential Operator in the Stochastic Models with "White Noise" / G.A. Sviridyuk, N.A. Manakova, G.A. Zakirova // Applied Mathematical Sciences. – 2014. – V. 8, № 173-176. – P. 8747–8754.

5. Свиридюк, Г.А. Динамические модели соболевского типа с условием Шоуолтера – Сидорова и аддитивными шумами / Г.А. Свиридюк, М.А. Манакова // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2014. – Т. 7, № 1. – С. 90–103.

6. Гельфанд И.М. Об одном простом тождестве для собственных значений дифференциального оператора второго порядка / И.М. Гельфанд, Б.М. Левитан // ДАН СССР. – 1953. – Т. 88, № 4. – С. 593–596.

7. Садовничий В.А. Регуляризованный след оператора с ядерной резольвентой, возмущенного ограниченным / В.А. Садовничий, С.В. Конягин, В.Е. Подольский // Доклады Академии наук. – 2000. – Т. 373, № 1. – С. 26–28.

8. Садовничий В.А. Следы операторов с относительно компактным возмущением / В.А. Садовничий, В.Е. Подольский // Матем. сб. – 2002. – Т. 193, № 2. – С. 129–152.

9. Закирова Г.А. Обратная спектральная задача для оператора Лапласа и ее приближенное решение / Г.А. Закирова, А.И. Седов // Вестник Южно-Уральского государственного университета. Серия: Математическое моделирование и программирование. – 2008. – № 27(127), вып. 2. – С. 19–27.

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