SURVEY ARTICLES

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THE THEORY OF OPTIMAL MEASUREMENTS

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The mathematical model (MM) of the measuring transducer (MT) is discussed. The MM is intended for restoration of deterministic signals distorted by mechanical inertia of the MT, resonances in MT's circuits and stochastic perturbations. The MM is represented by the Leontieff type system of equations, reflecting the change in the state of MT under useful signal, deterministic and stochastic perturbations; algebraic system of equations modelling observations of distorted signal; and the Showalter – Sidorov initial condition. In addition the MM of the MT includes a cost functional. The minimum point of a cost functional is a required optimal measurement. Qualitative research of the MM of the MT is conducted by the methods of the degenerate operator group's theory. Namely, the existence of the unique optimal measurement is proved. This result corresponds to input signal without stochastic perturbation. To consider stochastic perturbations it is necessary to introduce so called Nelson – Gliklikh derivative for random process. In conclusion of article observations of "noises"(random perturbation, especially "white noise") are under consideration.

Keywords: mathematical model of the measuring transducer, the Leontieff type system, the Showalter – Sidorov condition, cost functional, the Nelson – Gliklikh derivative, "white noise".

Introduction

New approach to restoration of deterministic signals distorted by a mechanical inertia of the measuring transducer (MT) was proposed in [17]. In the basis of this approach is a mathematical model (MM) of MT one part of which is a *Leontieff type system of equations*

$$L\dot{x} = Mx + Du. \tag{1}$$

Here L, M and D are square matrices of order n modelling the construction of the MT. The vector functions $x = col(x_1, x_2, \dots, x_n)$ and $u = col(u_1, u_2, \dots, u_n)$ are responsible for the state of the MT and the input signal (hereinafter – *measurement*) accordingly. The system of algebraic equations

$$y = Cx \tag{2}$$

is another part of the MM, where $y = col(y_1, y_2, \dots, y_n)$ is the vector function corresponding to output signal (hereinafter *observation*). Square matrix C of order n models the output device (for example, oscillograph or recording device). Note that we can observe less parameters than we measure. For this purpose corresponding rows of the matrix C are replaced by zeros (i.e. the "corresponding recorder" is turned off). Another part of the mathematical model of the MT is represented by *the Showalter – Sidorov initial condition*

$$P(x(0) - x_0) = 0, \tag{3}$$

here P is the projector in the space $\mathbb{R}^n$, which is constructed using the matrices L and M. Let us note that we consider the Leontieff type system of equations as a finite-dimensional analogues of Sobolev type equations to be able to use the methods of the theory of degenerate operator semigroups (for example see [23], ch. 4). The initial condition (3) is more *natural* for the Sobolev type equations, than the Cauchy condition [25]. Moreover, the condition (3) is more convenient for the algorithms of numerical calculations [6].

In [18] the MM (1) – (3) of the MT has been extended to the case when the measurement is distorted by resonances in the circuits of the MT in addition to its mechanical inertia. Initially it was supposed that $detL \neq 0$, however the careful analysis [9] reveals the necessity of $detL = 0$ to take into account the resonances in the model. The condition $detL = 0$ finally pulls together the system (1) and the Leontieff's balance model (for example, see [8]).

Finally, *the cost functional*

$$J(u) = \alpha \sum_{k=0}^{1} \int_0^{\tau} \left|\left| y^{(k)}(t) - y_0^{(k)}(t) \right|\right|^2 dt + \beta \sum_{k=0}^{K} \int_0^{\tau} \langle N_k u^{(k)}(t), u^{(k)}(t) \rangle \, dt, \tag{4}$$

presents the last (and most important) part of the mathematical model of the MT. Where $y_0 = y_0(t)$ is the observation, obtained in natural experiment on the real MT, modelled by (1) – (4). The coefficients $\alpha \in (0, 1]$, $\beta \in \bar{\mathbb{R}}_+$, $\alpha + \beta = 1$; N_k are symmetric positively defined matrices of order n, $||\cdot||$ and $\langle \cdot, \cdot \rangle$ are the Euclidean norm and scalar product in $\mathbb{R}^n$ respectively. The restored signal is a minimum point v of functional (4),

$$J(v) = \min_{u \in \mathfrak{U}_{a\partial}} J(u), \tag{5}$$

on a closed convex set $\mathfrak{U}_{a\partial}$, which is understood as a *set of admissible measurements*. This set contains a priori information about the target value of *the optimal measurement* v. Metrologists consider that such information should always be because "the unknown is impossible to be measured" [15].

As a result of profound theoretical research of MM for MT (1) – (5) the numerical algorithms for finding the optimal measurement have been developed. The y based on the thesis [7] which represents a wide range of numerical algorithms for solving of the optimal control problems for the Leontieff type systems. The algorithm adapted to the situation (1) – (5) is presented in [5]. The results of computational experiments are given in [16].

Now turn to the restoration of the stochastic signals. The MM of MT is represented by (1) – (5), where u is a random process (for example, white noise) and x_0 is a random variable. Firstly make a digression about stochastic differential equations.

In the simplest case the system of linear stochastic equations is given by

$$d\eta = (S\eta + \psi)dt + Ad\omega. \tag{6}$$

Here S and A are some matrices, $\psi = \psi(t)$ is a deterministic external influence, $\omega = \omega(t)$ is a stochastic external influence, $\eta = \eta(t)$ is a target random process. Originally under $d\omega$ we understood the generalized differential of the Wiener process, which is traditionally treated as a white noise. The first equations of the type (6) were studyied by K. Ito, then R.L. Stratonovich and A.V. Skorohod joined this investigation. The Ito-Stratonovich-Skorokhod approach is still popular [3]. Moreover, it was successfully extended to the infinite-dimensional situation [1, 10] and even to the Sobolev type equations [10, 27]. Note also the Melnikova – Filinkov – Alshansky approach [12, 13] in which the equation (6) is considered in Schwartz spaces and where the generalized derivative of the Wiener process makes sense.

Meanwhile there arose [19] and is actively developing [4, 20] a new approach in the study of equations of the form (6) where the "white noise" is understood as *the Nelson – Gliklikh derivative* of the Wiener process. (Note that this "white noise" is more appropriate to the Einstein – Smoluchowski theory of the Brownian motion than a traditional white noise [19, 20]). There was constructed the space of "noises" [4], which was developed to the infinite-dimensional space of "white noise" [21].

This article is specified as survey of last results of the optimal measurements theory. It's organized as follows. Section 1 contains well-known results of the theory of relatively p-regular matrices in the standpoint of the Weierstrass theorem about regular matrix pencils. Section 2 is devoted to establishment of existence and uniqueness theorem for optimal measurement. In section 3 we give very briefly an algorithm of numerical solution of optimal measurement problem. The main result here is the theorem of convergence of approximative solutions sequence to the strict solution which is obtained in section 2. All results of this section belong to Yury V. Khudyakov [9]. In section 4 we give some facts about the Nelson – Gliklih derivative and the Wiener process, and in final section 5 we consider observations of "noises", especially "white noise".

1. Relatively p-Regular Matrices and Degenerate Holomorphic Resolving Groups

Let L and M be square matrices of order n. Consider the *L-resolvent set* $\rho^L(M) = \{\mu \in \mathbb{C} : det(\mu L - M) \neq 0\}$ and the *L-spectrum* $\sigma^L(M) = \mathbb{C} \backslash \rho^L(M)$ of the matrix M. Obviously, the L-resolvent set $\rho^L(M) = \rho(L^{-1}M) = \rho(ML^{-1})$ if $detL \neq 0$. Further, the L-resolvent set $\rho^L(M) = \varnothing$ if $kerL \cap kerM \neq \{0\}$. Define the matrix M *to be regular with respect to the matrix* L (briefly, *L-regular*), if the L-spectrum of matrix M is bounded (in particular, *the set* $\sigma^L(M) = \varnothing$ for $M = \mathbb{I}_n$, and L being a nilpotent matrix). Note that the term "L-regular matrix M" is equivalent to the term "regular matrix pencil $\mu L - M$" in the sense of K. Weierstrass (cited by [2], Ch. 12). This term appeared similarly to the term "(L, σ)-bounded operator M"(see. for example, [23], Ch. 5).

Lemma 1. *Let the matrix M be L-regular. Then matrices*

$$P = \frac{1}{2\pi i} \int\limits_{\gamma} R^L_\mu(M) d\mu \qquad and \qquad Q = \frac{1}{2\pi i} \int\limits_{\gamma} L^L_\mu(M) d\mu \tag{7}$$

are idempotent.

Here $\gamma \subset \mathbb{C}$ is the (closed) contour bounding a domain containing the L-spectrum $\sigma^L(M)$ of matrix M; $R^L_\mu(M) = (\mu L - M)^{-1}L$ is the right, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the *left L-resolvent* of matrix M. The proof of lemma can be found in [23], Ch. 5.

Corollary 1. *Let the matrix M be L-regular, then $dimkerP = dimkerQ$ and $LP = QL$, $MP = QM$.*

Theorem 1. *(K. Weierstrass, [2], Ch.12). Let the matrix M be L-regular. Then there exist non-degenerate matrices A and B such that for any $\mu \in \mathbb{C}$ there is the representation*

$$B(\mu L - M)A = diag\{N_{n_1}, N_{n_2}, \ldots, N_{n_k}, \mu\mathbb{I}_l - S_l\}, \tag{8}$$

here on the right is a quasidiagonal matrix, $N_m = \mu H_m - \mathbb{I}_m$, H_m is a matrix of order m, wherein elements above the diagonal are equal to one, while the remaining elements are equal to zero.

From (8) we get the following

$$FLU = diag\{H_{n_1}, H_{n_2}, \ldots, H_{n_k}, \mathbb{I}_l\}, \; FMU = diag\{\mathbb{I}_{n-l}, S_l\}. \tag{9}$$

Let $p = \max\{n_1, n_2, \ldots, n_k\}$. Obviously, $p \in \mathbb{N}$ is an order of the pole at the point ∞ of L-*resolvent* $(\mu L - M)^{-1}$ of matrix M. Add here the case when $p = 0$ (i.e. $det\, L \neq 0$ or $det\, L = 0$ and $dimker L = dimker P$) and call the L-regular matrix M (L, p)-*regular*, $p \in \{0\} \cup \mathbb{N}$. Moreover, (8), (9) derive

$$A^{-1}PA = \frac{1}{2\pi i}\int\limits_{\gamma} A^{-1}(\mu L - M)^{-1}B^{-1}d\mu BLA = diag\{\mathbb{O}_{n-l}, \mathbb{I}_l\}, \tag{10}$$

$$BQB^{-1} = BLA\frac{1}{2\pi i}\int\limits_{\gamma} A^{-1}(\mu L - M)^{-1}B^{-1}d\mu = diag\{\mathbb{O}_{n-l}, \mathbb{I}_l\}. \tag{11}$$

Substituting

$$\Lambda = \frac{1}{2\pi i}\int\limits_{\gamma} (\mu L - M)^{-1}d\mu, \tag{12}$$

we get

$$BL\Lambda B^{-1} = A^{-1}\Lambda LA = diag\{\mathbb{O}_{n-l}, \mathbb{I}_l\}. \tag{13}$$

Corollary 2. *Let the matrix M be (L, p)-regular, $p \in \{0\} \cup \mathbb{N}$, and $det\, M \neq 0$. Then the matrix $H \equiv (\mathbb{I}_n - P)M^{-1}(\mathbb{I}_n - Q)L(\mathbb{I}_n - P)$ is nilpotent of degree p.*

Proof. Indeed, in view of (9) we get

$$A^{-1}HA = A^{-1}(\mathbb{I}_n - P)AA^{-1}M^{-1}B^{-1}B(\mathbb{I}_n - Q)B^{-1}BLAA^{-1}(\mathbb{I}_n - P)A =$$

$$= diag\{H_{n_1}, H_{n_2}, \ldots, H_{n_k}, \mathbb{O}_l\}.$$

□

Lemma 2. *Let the matrix M be (L, p)-regular, $p \in \{0\} \cup \mathbb{N}$. Then matrices*

$$P' = \lim_{\mu\to\infty} \left[\mu R^L_\mu(M)\right]^{p+1} \qquad and \qquad Q' = \lim_{\mu\to\infty} \left[\mu L^L_\mu(M)\right]^{p+1}$$

are idempotent.

Using the theorem 1 we can obtain

$$A^{-1}\left[\mu R^L_\mu(M)\right]^{p+1}A = \mu^{p+1}\left[A^{-1}(\mu L - M)^{-1}BB^{-1}LA\right]^{p+1} =$$

$$= \mu^{p+1}\left[diag(\mathbb{O}_{n-l},(\mu\mathbb{I}_l - S_l)^{-1}\right]^{p+1} = diag\left(\mathbb{O}_{n-l},(\mathbb{I}_l - \mu^{-1}S)^{-p-1}\right).$$

It means that

$$A^{-1}P'A = diag(\mathbb{O}_{n-l},\mathbb{I}_l), \tag{14}$$

and also

$$BQ'B^{-1} = diag(\mathbb{O}_{n-l},\mathbb{I}_l). \tag{15}$$

Corollary 3. *Under assumptions of lemma 1 (or lemma 2)*

$$P' = P \text{ and } Q' = Q.$$

Substituting

$$\Lambda' = \lim_{\mu\to\infty}\mu^{p+2}\left[R^L_\mu(M)\right]^{p+1}(\mu L - M) \tag{16}$$

we can obtain from (13) by analogy with (10) → (14) and (11) → (15) the following result.

Corollary 4. *Under assumptions of corollary 3* $\Lambda' = \Lambda$.

Consider now the Leontieff type system of equations

$$L\dot{x} = Mx. \tag{17}$$

The vector function $x \in C^\infty(\mathbb{R};\mathbb{R}^n)$ satisfying (17) is called *a classical solution* of this system. The classical solution $x = x(t)$ is called *a classical solution of the Cauchy problem*

$$x(0) = x_0 \tag{18}$$

for (17) (in short, *a classical solution of the problem* (17), (18)) if it satisfies in addition (18) for some $x_0 \in \mathbb{R}^n$. The matrix function $U^\bullet \in C^\infty(\mathbb{R};\mathbb{R}^{2n})$ is called *the group* (and is indicated by its graph $\{U^t : t \in \mathbb{R}\}$) if

$$U^sU^t = U^{s+t} \tag{19}$$

for all $s,t \in \mathbb{R}$. The group $\{U^t : t \in \mathbb{R}\}$ is called *holomorphic*, if it is analytically continued to the whole complex plane with conservation of the property (19); it is called *a resolving group* if $x(t) = U^tx_0$ is a classical solution of (17) for any $x_0 \in \mathbb{R}^n$; and it is called *a degenerate group* if its identify is a projector P (i.e. $P = U^0$).

Theorem 2. *Let the matrix M be (L,p)-regular, $p \in \{0\}\cup\mathbb{N}$. There exists a unique degenerate holomorphic resolving group of (17).*

It is easy to show (see for example, [23], Ch.5), that the searched group is represented by the integral

$$U^t = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)e^{\mu t}d\mu,\ t \in \mathbb{R},$$

here the contour $\gamma \subset \mathbb{C}$ is the same as in Lemma 1. Moreover, $U^tA = Adiag\{\mathbb{O}_{n-l}, e^{tS_l}\}$, where $e^{tS_l} = \sum\limits_{k=0}^{\infty}\frac{S_l^k}{k!}t^k$. Clearly, the solution $x(t) = U^tx_0$ of (17) is a solution of (17), (18) if

$Px_0 = x_0$. Show that the sufficient condition $x_0 \in imP$ is a necessary condition. Introduce *the phase space* of (17), which is understood as the set $\mathfrak{P} \subset \mathbb{R}^n$ such that, firstly, any solution of system (17) lies in $\mathfrak{P}$, i.e. the $x(t) \in \mathfrak{P}$ for all $t \in \mathbb{R}$. Secondly, there exists a unique solution of the problem (17), (18) for any $x_0 \in \mathfrak{P}$.

Theorem 3. *Let the matrix M be (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, $det\, M \neq 0$. The subspace $im\, P$ is a phase space of (17).*

Proof. Indeed, reduce (17) to the equivalent form

$$H\dot{x}^0 = x^0,\ \dot{x}^1 = P\Lambda QMx^1,\ x^1 = Px,\ x^0 = x - x^1. \tag{20}$$

Differentiating the first equation in (20) with respect to t and consistently multiplying by H on the left, due to the corollary 2, we get

$$0 = H^{p+1}x^{0(p+1)} = H^p x^{0(p)} = \cdots = H\dot{x}^0 = x^0.$$

The solution of the second equation in (20) has the form $x^1(t) = Px(t) = PU^t x_0 = U^t x_0$, i.e. it belongs to $im\, P$. Existence of solutions and the uniqueness are obvious. □

Remark 1. The degenerate holomorphic resolving group is also given by the formula

$$U^t = \lim_{k\to\infty} \left[k(kL - tM)^{-1}L\right]^{k(p+1)}.$$

2. The Optimal Measurement Problem: Strict Solution

Consider now the nonhomogeneous Leontieff type system

$$L\dot{x} = Mx + f, \tag{21}$$

where the vector function $f : [0,\tau) \to \mathbb{R}^n, \tau \in \mathbb{R}_+$ will be determined later. The vector function $x \in C([0,\tau);\mathbb{R}^n) \cap C^1((0,\tau);\mathbb{R}^n)$ is called *a classical solution of* (21) if its satisfies (21) on the $(0,\tau)$. The solution $x = x(t)$ of (21) is called a *classical solution of the Showalter – Sidorov problem* (3) (briefly, a *classical solution of the problem* (21), (3)) if it satisfies in addition (3). Note that the condition (3) occurs only in the case of the (L,p)-regularity of the matrix M, $p \in \{0\} \cup \mathbb{N}$. In this case, the condition (3) is equivalent to the condition

$$[R^L_\alpha(M)]^{p+1}(x(0) - x_0) = 0 \tag{22}$$

for any $\alpha \in \rho^L(M)$ [25]. And if $detL \neq 0$ then (22) is equivalent to (18).

Theorem 4. *Let the matrix M be (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, and $det\, M \neq 0$. For all $\tau \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, $f^0 = (\mathbb{I}_n - Q)f \in C^{p+1}([0,\tau);\mathbb{R}^n)$, and $f^1 = Qf \in C([0,\tau);\mathbb{R}^n)$ there exists a unique classical solution $x = x(t)$ of the problem (22), (3) given by the formula*

$$x(t) = -\sum_{k=0}^{p} H^k M^{-1} f^{0(k)}(t) + U^t x_0 + \int_0^t U^{t-s}\Lambda f^1(s)ds.$$

Here U^t is a degenerate holomorphic resolving group (see section 1), and the matrix Λ is given by 12 or 16 (corollary 4).

Let us turn to the system (1). Denote by $\mathfrak{X} = \{x \in L_2((0,\tau);\mathbb{R}^n) : \dot{x} \in L_2((0,\tau);\mathbb{R}^n)\}$ *a state space of the MT*, and by $\mathfrak{U} = \{u \in L_2((0,\tau);\mathbb{R}^n) : u^{(p+1)} \in L_2((0,\tau);\mathbb{R}^n)\}$ denote a *measurement space*, $\tau \in \mathbb{R}_+$ is some fixed number. The vector function $x \in \mathfrak{X}$ is called a *strong solution of* (1) if it satisfies (1) for some $u \in \mathfrak{U}$ and almost all $t \in (0,\tau)$. The strong solution $x = x(t)$ of (1) is called *a strong solution of the problem* (1), (3), if it satisfies in addition (3) for some $x_0 \in \mathbb{R}^n$. Note that in this case the condition (3) is correct due to the embedding $\mathfrak{X} \hookrightarrow C([0,\tau] : \mathbb{R}^n)$.

Corollary 5. *Let all assumptions of Theorem 4 be fulfilled. Then for every $\tau \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ and $u \in \mathfrak{U}$ there exists a unique strong solution to the problem (1),(3) given by the formula*

$$x(t) = -\sum_{k=0}^{p} H^k M^{-1}(\mathbb{I}_n - Q)Du^{(k)}(t) + U^t x_0 + \int_0^t U^{t-s}\Lambda Q u(s)ds. \qquad (23)$$

Introduce the space of observations $\mathfrak{Y} = C(\mathfrak{X})$ and new cost functional

$$J(u) = \alpha \sum_{k=0}^{1} \int_0^{\tau} \left|\left| y^{(k)}(t) + \tilde{y}_0^{(k)}(t) - y_0^{(k)}(t) \right|\right|^2 dt + \beta \sum_{k=0}^{K} \int_0^{\tau} \left\langle N_k u^{(k)}(t), u^{(k)}(t) \right\rangle dt, \qquad (24)$$

which differs from (4) by summand $\tilde{y}_0^{(k)}(t)$. This vector function responds to the observation obtained on real MT without useful input signal. In other words, $\tilde{y}_0^{(k)}$ is responsible for noise caused by resonances in chains of the MT. The necessity of such modernization of the functional (4) was substantiated in [9]. Note also that originally [18] it was supposed that $K = p+1$, however careful analysis [11] reveals that $K \in \{0,1,...,p+1\}$. Finally isolate a *set of admissible measurements* $\mathfrak{U}_{a\partial}$, i.e.closed and convex subset of $\mathfrak{U}$. A minimum point $v \in \mathfrak{U}_{a\partial}$ of the functional (24) is called an *optimal measurement*.

Theorem 5. *Let the matrix M be (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, $det\, M \neq 0$. Then for all $\tau \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, $K \in \{0,1,...,p+1\}$ there exists a unique optimal measurement.*

Remark 2. Theorem 5 will be valid if we replace the functional (24) by (4). The proof of this theorem you can find in [23], Ch.7.

3. The Optimal Measurement Problem: Approximate Solutions

In this section we present a numerical algorithm for solution of the optimal measurement problem (1)-(3),(5) where the functional J has the form (24). In *the first step* of this algorithm we note that the space $\mathfrak{U}$ is separable by construction. It means that there exists a sequence of the finite-dimensional ($dim\mathfrak{U}^\ell = \ell$) subspaces $\mathfrak{U}^\ell \subset \mathfrak{U}$ monotonously exhausting the space $\mathfrak{U}$ (i.e. $\mathfrak{U}^\ell \subset \mathfrak{U}^{\ell+1}$ and $\bigcup_{\ell=1}^{\infty} \mathfrak{U}^\ell$ is densely embedded in $\mathfrak{U}$). *An approximation $u^\ell \in \mathfrak{U}^\ell$* of the measurement u is represented in the form

$$u^\ell = col\left(\sum_{j=1}^{\ell} a_{1j}\varphi_j, \sum_{j=1}^{\ell} a_{2j}\varphi_j, \ldots, \sum_{j=1}^{\ell} a_{nj}\varphi_t\right), \qquad (25)$$

where $\{\varphi_j\}_{j=1}^{\ell}$ is an orthonormal basis of the subspace U^ℓ, and the coefficients $a_{11},...,a_{1\ell},a_{21},...,a_{2\ell},...,a_{n\ell}$ are unknown. It is natural to assume that the resonances arising in chains of the MT are the perturbations of the measurements u^ℓ, i.e. instead of u^ℓ consider

$$\tilde{u}^\ell = col\left(u_1^\ell + A_1\sin\omega_1 t, u_2^\ell + A_2\sin\omega_2 t, ..., u_n^\ell + A_n\sin\omega_n t\right), \tag{26}$$

where the resonance frequencies $\omega_1,\omega_2,...,\omega_n$ are assumed to be known, and the amplitudes $A_1,A_2,...,A_n$ are not. Construct an *approximate solution* $x_k^\ell = x_k^\ell(t)$ based on (23). This solution has the form

$$x_k^\ell(t) = -\sum_{j=0}^{p} H_k^j M^{-1}(\mathbb{I}_n - Q_k)D\tilde{u}^{\ell(j)}(t) + U_k^t x_0 + \tag{27}$$

$$\sum_{j=0}^{J}\left(\left(\left(L - \frac{t-s_j}{k}M\right)^{-1}L\right)^{k(p+1)-1}\left(L-\frac{t-s_j}{k}M\right)^{-1}Q_k D\tilde{u}^\ell(s_j)\right)\Delta c_j,$$

where s_j and c_j are nodes and weights of the Gauss quadrature formula. Note that the choice of k should be bounded below [6]. By substituting x_k^ℓ in (2) instead of x we find *an approximate observation* $y_k^\ell = y_k^\ell(t)$.

Remark 3. Note that the function $\varphi_j = \varphi_j(t)$ (25) has the form $\varphi_j(t) = \sin jt$ [9].

In **the second step** of the algorithm substitute the data $y_0^{(j)}$ and $\tilde{y}_0^{(j)}$, $j = 0,1$, the approximation y_k^ℓ instead of y and u^ℓ instead of u in the cost functional (24). Note that the second summand in (24) acts as a filter that decreases the high amplitudes of the resonances. After the calculations in (24) we obtain a functional $J^\ell = J^\ell(\mathbf{a})$, where the vector $\bar{\mathbf{a}} = col(a_{11},...,a_{1\ell},a_{21},...,a_{2\ell},...,a_{n\ell},A_1,A_2,...,A_n)$ belongs to the space $\mathbb{R}^\ell\times\mathbb{R}^n$. Where the subspace $\mathbb{R}^n$ is called a *space of resonances amplitudes.*

Refer to the set of admissible measurements $\mathfrak{U}_{a\partial}$. Typically, in applications it is not only a closed and convex, but in addition it is bounded. Let the set $\mathfrak{U}_{a\partial}$ be closed, convex and bounded then there exists a sequence of convex compacts $\{\mathfrak{U}_{a\partial}^\ell\}$, $\mathfrak{U}_{a\partial}^\ell \subset \mathfrak{U}^\ell$ monotonically exhausting the set $\mathfrak{U}^\ell$. In our considerations we can construct a convex compact set in the space $\mathbb{R}^{\ell n}$ isomorphic to $\mathfrak{U}_{a\partial}^\ell$. Further this compact set will be denoted by the same symbol $\mathfrak{U}_{a\partial}^\ell$. In the space of resonances amplitudes $\mathbb{R}^n$ choose a convex compact set $\mathfrak{U}_{a\partial}^n$ accumulating a priori information about the MT resonance's amplitudes. Find the minimum of the functional J^ℓ on the set $\mathfrak{U}_\partial^\ell\times\mathfrak{U}_\partial^n$ that exists (and is unique) due to the Mazur Theorem.

Substituting the values $\tilde{a}_{11},...,\tilde{a}_{1\ell},\tilde{a}_{21},...,\tilde{a}_{n\ell}$ of the minimum point

$$\tilde{\mathbf{a}} = col(\tilde{a}_{11},...,\tilde{a}_{1\ell},\tilde{a}_{21},...,\tilde{a}_{2\ell},...,\tilde{a}_{n\ell},\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n)$$

of the functional J^ℓ on the set $\mathfrak{U}_{a\partial}^\ell\times\mathfrak{U}_{a\partial}^n$ into (25) we get the vector function

$$u_k^\ell = col\left(\sum_{j=1}^{\ell}\tilde{a}_{1j}\varphi_j, \sum_{j=1}^{\ell}\tilde{a}_{2j}\varphi_j, \ldots, \sum_{j=1}^{\ell}\tilde{a}_{nj}\varphi_j\right), \tag{28}$$

which is called *an approximate optimal measurement.* The superscript of u_k^ℓ defines the dependence on "approximate space" $\mathfrak{U}^\ell$, and the subscript defines the dependence on approximation (27). Note that we have simultaneously found the resonance amplitudes $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ that we are not interested in as an approximate state of MT. The state of MT can be found from (27) by substitution of the vector function

$$\tilde{u}_k^\ell = col\left(u_{k1}^\ell + \tilde{A}_1 sin\omega_1 t, u_{k2}^\ell + \tilde{A}_2 sin\omega_2 t, ..., u_{kn}^\ell + \tilde{A}_n sin\omega_n t\right)$$

instead of $\tilde{u}^\ell$. Note also that, for simplicity, the time t in the algorithm ranges within $(0, \pi)$ (i.e. in (24) we assume $\tau = \pi$). To consider the other intervals it is necessary to use the correction coefficients for t in (25), (28). The following theorem completes the algorithm

Theorem 6. *Let the conditions of the theorem 5 be fulfilled. Then* $\lim\limits_{\ell\to\infty} \lim\limits_{k\to\infty} u_k^\ell = u.$

Proof can be found in [9].

4. The Nelson – Gliklikh Derivative

The mean derivative of the random process was introduced by E. Nelson [14]. The theory of such derivatives was developed by Y.E. Gliklikh [3]. In [19] the symmetric mean derivative of the random process $\eta = \eta(t)$ was suggested to be called the *Nelson – Gliklikh derivative* and to be denoted $\overset{\circ}{\eta} = \overset{\circ}{\eta}(t)$. Such derivatives are widely used in the study of the Leontieff type systems (1) with additive "white noise" [4, 20].

Consider (one-dimensional) Wiener process $\beta = \beta(t)$ which models the Brownian motion on the line in Einstein – Smolukhovsky theory. It possesses the following properties:

(W1) almost surely (a.s.) $\beta(0) = 0$, a.s. all trajectories $\beta(t)$ are continuous and for all $t \in \overline{\mathbb{R}}_+ (= \{0\} \cup \mathbb{R}_+)$ $\beta(t)$ is a Gaussian random variable;

(W2) mathematical expectation $\mathbf{E}(\beta(t)) = 0$ and the autocorrelation function $\mathbf{E}((\beta(t) - \beta(s))^2) = |t - s|$ for all $s, t \in \overline{(\mathbb{R})}_+$;

(W3) trajectories of $\beta(t)$ are not differentiable at any point $t \in \overline{\mathbb{R}}_+$ and at any arbitrarily small interval have unbounded variation.

Theorem 7. *With probability equal to one there exists a unique random process β satisfying the properties $(W1)$ – $(W2)$ and it can be given by*

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t,$$

where ξ_k are independent Gaussian random variables, $\mathbf{E}\xi_k = 0,\ \mathbf{D}\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$.

Further the random process β satisfying the properties $(W1)$ – $(W3)$ will be called a *Brownian motion*. This random process belongs to the space $\mathbf{C}(\overline{\mathbb{R}}_+; \mathbb{R})$ of random processes which trajectories are continuous a.s. on $\overline{\mathbb{R}}_+$. The *spaces of differentiable "noises"* $\mathbf{C}^m((\xi, \tau); \mathbb{R})$ as the spaces of random processes with trajectories of Nelson – Gliklikh derivatives to order $m \in \mathbb{N}$ being continuous a.s. on the $(\xi, \tau) \subset \mathbb{R}$ were introduced in [24]. By $\overset{\circ}{\beta}{}^{(m)}$ denote the Nelson – Gliklich derivative of order $m \in \mathbb{N}$ of the Brownian motion β.

Theorem 8. *(Yu.E. Gliklich)* $\overset{\circ}{\beta}{}^{(m)}(t) = (-1)^{m+1}(2t)^{-m}\beta(t)$ *for all* $t \in \mathbb{R}_+$ *and* $m \in \mathbb{N}$.

Due to the same theorem the Brownian motion $\beta \in \mathbf{C}^\infty(\mathbb{R}_+; \mathbb{R})$, where $\mathbf{C}^\infty(\mathbb{R}_+; \mathbb{R})$ is the space of stochastic processes with trajectories of Nelson – Gliklich derivatives being continuous a.s. on $\mathbb{R}_+$ to any order. Moreover the Nelson – Gliklich derivative $\overset{\circ}{\beta}$, called *one-dimensional "white noise"*, is also an element of the space $\mathbf{C}^\infty(\mathbb{R}_+;\mathbb{R})$. Now fix the interval (ε,τ), the number $n\in\mathbb{N}$ and take n independent random processes $\{\eta_1,\eta_2,\ldots,\eta_n\}\subset\mathbf{C}((\varepsilon,\tau);\mathbb{R})$. By formula

$$\eta(t)=\sum_{j=1}^{n}\eta_j(t)e_j,$$

where e_j are orts of the space $\mathbb{R}^n$, $j=\overline{1,n}$, define the *n-dimensional stochastic process* (briefly, *n-stochastic process*). By analogy with the previous introduce the space of continuous $\mathbf{C}((\varepsilon,\tau);\mathbb{R}^n)$, continuously Nelson – Gliklih differentiable to order $m\in\mathbb{N}$ $\mathbf{C}^m((\varepsilon,\tau);\mathbb{R}^n)$ and infinitely Nelson – Gliklih differentiable $\mathbf{C}^\infty((\varepsilon,\tau);\mathbb{R})$ n-dimensional "noises". As an example, consider the n-dimensional Wiener process (*n-Wiener process*)

$$W_n(t)=\sum_{j=1}^{n}\beta_j(t)e_j, t\in\overline{\mathbb{R}}_+, \tag{29}$$

where β_j, $j=\overline{1,n}$ are independent Brownian motions. Due to Theorem 8, the following statement takes place.

Corollary 6. $\overset{\circ}{W}{}_n^{(m)}(t)=(-1)^{m+1}(2t)^{-m}W_n(t)$ *for all* $t\in\mathbb{R}_+$, $m,n\in\mathbb{N}$.

It follows from [2] that the n-Wiener process W_n satisfies conditions (W1) – (W3) if we substitute β for W_n. Considering that this substitution was done we get

Theorem 9. *With probability equal to one there exists a unique n-Wiener process* W_n *for any* $n\in\mathbb{N}$ *that satisfies conditions (W1) – (W3), and it can be given by (29).*

Remark 4. Emphasize that under "noises" we understand random noises only.

5. Observations of "noises"

By analogy with the spaces of differentiable "noises" introduce *the space of integrable "noises"*. Fix the interval (ε,τ) and by $\mathbf{L_2}((\varepsilon,\tau);\mathbb{R})$ denote the space of stochastic processes with any trajectory a.s. lying in $L_2((\varepsilon,\tau);\mathbb{R})$. The space $\mathbf{L_2}((\varepsilon,\tau);\mathbb{R})$ is a Hilbert space with inner product $[\eta,\xi]=\int\limits_\varepsilon^\tau\mathbf{E}\eta(t)\xi(t)dt$. Similarly construct the Hilbert space $\mathbf{L_2}((\varepsilon,\tau);\mathbb{R}^n)$ with inner product $[\eta,\xi]_n=\int\limits_\varepsilon^\tau\mathbf{E}<\eta(t),\xi(t)>dt$ where $<\cdot,\cdot>$ is a Euclidean scalar product in $\mathbb{R}^n$.

Further, by analogy with (1), (6) consider the stochastic Leontieff type equations

$$L\overset{\circ}{\xi}=M\xi+D\varphi. \tag{30}$$

It models the random changes of the MT states $\xi=\xi(t)$ under the influence of inertia and resonances (the matrices L, M and D are the same as in (1)), $\overset{\circ}{\xi}$ denotes the Nelson – Gliklikh derivative of random process ξ. The random process $\varphi=\varphi(t)$ describes the additive stochastic disturbance (i.e. "noise").

Assuming that the matrix M is (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$ supply the system (30) with the Showalter – Sidorov initial condition

$$P(\xi(0)-\xi_0)=0. \tag{31}$$

Fix the interval $(0,\tau) \subset \mathbb{R}_+$ and construct a stochastic MT state space $\Xi = \{\xi \in \mathbf{L}_2((0,\tau);\mathbb{R}^n) : \overset{\circ}{\xi} \in \mathbf{L}_2((0,\tau);\mathbb{R}^n)\}$ and *a stochastic measurements space* $\Phi = \{\varphi \in \mathbf{L}_2((0,\tau);\mathbb{R}^n) : \overset{\circ}{\varphi}^{(p+1)} \in \mathbf{L}_2((0,\tau);\mathbb{R}^n\}$. Note that if any trajectory of a random process $\overset{\circ}{\psi}^{k+1} = \overset{\circ}{\psi}^{k+1}(t)$, $t \in (0,\tau)$, $k \in \{0\} \cup \mathbb{N}$ lies in $L_2((0,\tau);\mathbb{R}^n)$ then the same trajectory of the random process $\overset{\circ}{\psi}^{k+1}$ is absolutely continuous on $[0,\tau]$ by the Sobolev imbedding theorems. Therefore, the condition (31) and stochastic spaces Ξ, Φ are defined correctly. Fix $\varphi \in \Phi$. The random process $\xi \in \Xi$ is called *a strong solution* of (30), if for any trajectory of φ there exists a trajectory ξ almost everywhere (a.e.) on the $(0,\tau)$ satisfying (30). It is called *a strong solution of the problem* (30), (31) if it satisfies (31) for some $\xi_0 \in \mathbf{L}_2$.

Theorem 10. *Let the matrix M be (L,p)-regular, $p \in \{0\}\cup\mathbb{N}$, $\det M \neq 0$. For all $\tau \in \mathbb{R}_+$, $\varphi \in \Phi$, $\xi_0 \in \mathbf{L_2}$ independent from φ there exists a unique strong solution $\xi = \xi(t)$ of the problem (30), (31) and a.s. all its trajectories are given by*

$$\xi(t) = -\sum_{k=0}^{p} H^k M^{-1}(\mathbb{I}_n - Q)D\overset{\circ}{\varphi}^{(k)}(t) + U^t\xi_0 + \int_0^t U^{t-s}\Lambda Q\varphi(s)ds. \tag{32}$$

Supplementing (30) by equations

$$\eta = C\xi \tag{33}$$

we obtain *the observation* η of the "noise" φ where $\xi = \xi(t)$ is given by (32). To get an observation of the "white noise" we need to replace φ in (30) by $\overset{\circ}{W}_n$ and besides that to replace condition (31) by *the wakened* (in the sense of S.G. Krein) *Showalter – Sidorov condition*

$$\lim_{t\to 0+} P(\xi(t)-\xi_0)=0. \tag{34}$$

Corollary 7. *Let all assumptions of the Theorem 10 be fulfilled. Then for every $\xi_0 \in \mathbf{L}_2^n$ independent from $\overset{\circ}{W}_n$ there exists a unique solution $\xi \in \Xi$ of the Leontieff type system*

$$L\overset{\circ}{\xi} = M\xi + D\overset{\circ}{W}_n \tag{35}$$

which satisfies the initial condition (34). All trajectories of this solution a.s. are given by the formula

$$\xi(t) = -\sum_{k=0}^{p} H^k M^{-1}(\mathbb{I}_n - Q)D\overset{\circ}{W}{}_n^{(k)}(t) + U^t\xi_0 + \int_0^t U^{t-s}\Lambda QD\overset{\circ}{W}_n(s)ds. \tag{36}$$

for every $t \in \mathbb{R}_+$.

Since

$$P\left[\sum_{k=0}^{p} H^k M^{-1}(\mathbb{I}_n - Q)D\overset{\circ}{W}{}_n^{(k)}(t)\right] = 0$$

for every $t \in \mathbb{R}_+$ we are to show that the integral in the right hand side of (36) exists. Taking $\varepsilon \in (0,t)$, $t \in \mathbb{R}_+$, and integrating this integral by parts we obtain

$$\int_0^t U^{t-s}\Lambda Q\overset{\circ}{W}_n(s)ds = \Lambda QW_n(t) - U^{t-\varepsilon}\Lambda QW_n(\varepsilon) - \Lambda M\int_\varepsilon^t U^{t-s}\Lambda QW_n(s)ds. \qquad (37)$$

Since

$$\lim_{\varepsilon\to 0+}\int_\varepsilon^t U^{t-s}\Lambda QW_n(s)ds = \int_0^t U^{t-s}\Lambda QW_n(s)ds,$$

$$\lim_{\varepsilon\to 0+} U^{t-\varepsilon}\Lambda QW_n(\varepsilon)ds = \mathbb{O}_n,$$

then from (37) we get required. Finally from (33), where $\xi = \xi(t)$ is given by (36), we obtain *the observation of "white noise"* $\overset{\circ}{W}_n$.

References

1. Da Prato G., Zabczyk J. *Stochastic Equations in Infinite Dimensions.* Cambridge, Cambridge University Press, 1992.

2. Gantmacher F.R. *The Theory of Matrices.* AMS Chelsea Publishing, Reprinted by American Mathematical Society, 2000.

3. Gliklikh Yu.E. *Global and Stochastic Analysis with Applications to Mathematical Physics.* London, Springer, 2011. DOI: 10.1007/978-0-85729-163-9

4. Gliklikh Yu.E. Study of the Leontieff Type Equations with White Noise by the Methods of Mean Derivatives of Stochastic Processes. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2012, no. 27 (286), issue 13, pp. 24–34. (in Russian)

5. Keller A.V. Numerical Solution of the Optimal Control Problem for Degenerate Linear System of Equations with Showalter–Sidorov Initial Conditions. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2008, no. 27 (127), issue 2, pp. 50–56. (in Russian)

6. Keller A.V. The Leontieff type systems: classes of problems with the Showalter-Sidorov intial condition and numerical solving *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2010, vol. 3, no. 2, pp. 30–43. (in Russian)

7. Keller A.V. [*Numerical Study of Optimal Control Problems for Leontieff Type Models. Doctoral Thesis of Physicomathematical Science*]. Chelyabinsk, South Ural State University, 2011. (in Russian)

8. Keller A.V., Nazarova E.I. The Regularization Property and the Computational Solution of the Dynamic Measure Problem. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2010, no. 16 (195), issue 5, pp. 32–38. (in Russian)

9. Khudyakov Yu.V. The Numerical Algorithm to Study the Shestakov-Sviridyuk Model of the Measuring Device with Inertia and Resonances. *Matematicheskiye Zametki YAGU* [Mathematical Note of the YASU], 2013, vol. 20, issue 2, pp. 225–236. (in Russian)

10. Kovács M., Larsson S. Introduction to stochastic partial differential equations. *Processing of "New Directions in the Mathematical and Computer Sciences National Universities Commission. October 8-12. 2007.* Abuja. Nigeria. Publications of the ICMCS, 2008, no. 4, pp. 159–232.

11. Manakova N.A. On a Hypothesis of G.A. Sviridyuk *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2011, vol. 4, no. 4, pp. 87–93. (in Russian)

12. Melnikova I.V., Filinkov A.I., Alshansky M.A. Abstract Stochastic Equations II. Solutions in Spaces of Abstract Stochastic Distribotions. *Journal of Mathematical Sciences.* 2003, vol. 116, no. 5, pp. 3620–3656.

13. Melnikova I.V., Filinkov A.I. Generalized solutions to abstract stochastic problems. *J. Integ. Transf. and Special Funct.*, 2009, vol. 20, no. 3–4, pp. 199–206.

14. Nelson E. *Dynamical Theories of Brownian Motion.* Princeton, Princeton University Press, 1967.

15. Shestakov A.L. Dynamic Accuracy of the Transmitter With a Correction Device as a Sensor Model. *Metrology.* 1987, no. 2, pp. 26. (in Russian)

16. Shestakov A.L., Keller A.V., Nazarova E.I. Numerical solution of the optimal measurement problem. *Automation and Remote Control.* 2012, no. 1, pp. 107–115.

17. Shestakov A.L., Sviridyuk G.A. A New Approach to Measurement of Dynamically Perturbed Signal. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2010, no. 16 (192), issue 5, pp. 116–120. (in Russian)

18. Shestakov A.L., Sviridyuk G.A. Optimal Measurement of Dynamically Distorted Signals. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2011, no. 17 (234), issue 8, pp. 70–75.

19. Shestakov A.L., Sviridyuk G.A. A New Concept of White Noise. *Obozrenie prikladnoy i promyshlennoy matematiki* [Survey of Applied and Industrial Mathematics]. 2012, issue 19, no. 2, pp. 287–288.

20. Shestakov A.L., Sviridyuk G.A. On the Measurement of the "White Noise". *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2012, no. 27 (286), issue 13, pp. 99–108.

21. Shestakov A.L., Sviridyuk G.A., Khudyakov Yu.V. Dynamic Measurements in Spaces of "Noise". *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2013, no. 2, issue 13, pp. 4–11. (in Russian)

22. Shestakov A.L., Sviridyuk G.A., Sagadeeva M.A. Reconstruction of a Dynamically Distorted Signal with Respect to the Measure Tranducer Degradation. *Applied Mathematical Sciences*. 2014, vol. 8, no. 41–44, pp. 2125–2130.

23. Sviridyuk G.A., Fedorov V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*. Utrecht, Boston, Köln, Tokyo, VSP, 2003.

24. Sviridyuk G.A., Manakova N.A. The Dynamical Models of Sobolev Type with Showalter Ц Sidorov Condition and Additive "Noise". *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2014, no. 7, issue 1, pp. 90–103. (in Russian)

25. Sviridyuk G.A., Zagrebina S.A. The Showalter – Sidorov Problem as Phenomena of the Sobolev-Type Equations. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2010, vol. 3, no. 1, pp. 51–72. (in Russian)

26. Zagrebina, S.A., Soldatova, E.A. The linear Sobolev-type Equations With Relatively p-bounded Operators and Additive White Noise. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2013, no. 1, pp. 20–34. (in Russian)

27. Zamyshlyaeva A.A. Stochastic Incomplete Linear Sobolev Type High-Ordered Equations with Additive White Noise. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2012, no. 40, issue 14, pp. 73–82. (in Russian)

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