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SOBOLEV TYPE EQUATION IN (n, p)-SECTORIAL CASE

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We consider a mathematical model of thermoelastic plate vibrations under certain assumptions. The model is based on the nonclassical high-order equation of the mathematical physics. In addition, this equation is unsolvable with respect to the time derivative of higher order. In appropriately chosen functional spaces, the considered mathematical model can be reduced to an abstract Sobolev type equation of the third order with relatively (n, p)-sectorial operator on the right-hand side. As is known, an equation of Sobolev type is not solvable for arbitrary initial values. Therefore, we construct a set of admissible initial values. The main research approach is the method to construct resolving groups.

Keywords: Sobolev type equation; relatively spectral-bounded operator; bundle of operators; mathematical model of thermoelastic plate vibrations.

1. Introduction

The mathematical model of thermoelastic plate vibrations [1] can be written as an initial-boundary problem for the equation of the form (1)

$$(\Delta - \lambda)u_{ttt} - k\Delta(\Delta - \lambda_1)u_{tt} - \gamma^2\Delta^2 u_t + k\Delta^3 u = 0. \tag{1}$$

Suppose that during heating a plate is not diffused, or the diffusion does not significantly effect on the physical properties and the vibration process. Then, parameter k can be set equal to zero. Therefore, the equation (1) is simplified to the equation of the form

$$(\Delta - \lambda)u_{ttt} = \gamma^2\Delta^2 u_t. \tag{2}$$

Let $D \subset \mathbb{R}^2$ is a bounded domain with a sufficiently smooth boundary ∂D. Supplement the equation (2) with the Benard boundary conditions

$$u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y, t) \in \partial D \times R, \tag{3}$$

and the initial conditions

$$u^{(m)}(x, y, 0) = u_m(x, y), \quad m = 0, 1, 2. \tag{4}$$

Integrate the equation (2) taking into account the initial conditions (4) and obtain

$$(\Delta - \lambda)u_{tt} = \gamma^2\Delta^2 u + f. \tag{5}$$

$$u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y, t) \in \partial D \times R, \tag{6}$$

$$u^{(m)}(x, y, 0) = u_m(x, y), \quad m = 0, 1. \tag{7}$$

Here $f = (\Delta - \lambda)u_2 - \gamma^2\Delta^2 u_0$. For certain values of the parameter λ, the operator of the highest time derivative is irreversible. Therefore (2) is a Sobolev type equation. Let us investigate the mathematical model (2)–(4) using the theory of degenerate semigroups of operators [2] and the theory of relatively p-sectorial operators [3, 4].

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, $\mathcal{L}(\mathfrak{U},\mathfrak{F})$ be a space of linear and bounded operators, $\mathcal{C}l(\mathfrak{U},\mathfrak{F})$ be a space of linear operators with a dense domain of definition. A map $U \in C(\mathfrak{U};\mathcal{L}(\mathfrak{U}))$ is called a semigroup of operators, if for all $s,t \in \mathbb{R}_+$ the following identity holds:

$$U^sU^t = U^{s+t}. \tag{8}$$

Usually, the semigroup of operators is identified with its graph $\{U^t : t \in \mathbb{R}_+\}$. A semigroup $\{U^t : t \in \mathbb{R}_+\}$ is said to be holomorphic, if it is analytically continued with preservation of the property (8) to some sector of the complex plane containing the semiaxis $\mathbb{R}_+$. A holomorphic semigroup is called degenerate, if its unit $P = s - \lim\limits_{t\to 0+} U^t$ is a projection in U.

The linear evolution equation of the Sobolev type

$$L\dot{u} = Mu, \tag{9}$$

where the operator $L \in \mathcal{L}(\mathfrak{U},\mathfrak{F})$, $\ker L \neq \{0\}$, is investigated and a holomorphic degenerate semigroup of operators is introduced in the papers [2, 3]. The complete theory of such semigroups is given in [4, ch. 3]. We note that the Cauchy problem for Sobolev type equations is solvable not for any initial value [4]. Therefore, in investigating the solvability of the Cauchy problem for such equations, it is necessary to find a set of initial values for which the solution exists and is unique. Such a set is called a phase space. Many works are devoted to the study of such spaces for various models (see, for example, [5, 6, 7]). Moreover, for nonlinear equations the stability of solutions [8].

The Sobolev type equation

$$Lu^{(n)} = Mu$$

is considered and (L,n,p)-sectorial operator is introduced in the paper [9]. Then, on the basis of the abstract theory, the Benny-Luke equation is investigated and the semigroup of solving operators is constructed.

2. Relatively (n,p)-Sectorial Operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, $\mathcal{L}(\mathfrak{U},\mathfrak{F})$ be a space of linear and bounded operators, $\mathcal{C}l(\mathfrak{U},\mathfrak{F})$ be a space of linear operators with a dense domain of definition, and $L \in \mathcal{L}(\mathfrak{U},\mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U},\mathfrak{F})$. A set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1}\} \in \mathcal{L}(\mathfrak{U},\mathfrak{F})$ is called L-resolvent set of the operator M, $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ is called L-spectrum of the operator M. Denote $\sigma^L_n(M) = \{\mu^n : \mu \in \sigma^L(M)\}$, $\rho^L_n(M) = \mathbb{C} \setminus \sigma^L_n(M)$.

Definition 1. An operator M is called (n,p)*-sectorial with respect to the operator* L (or, briefly, (L,n,p)*-sectorial*), if there exist constants $K > 0$, $\theta \in (\frac{\pi}{2},\pi)$ such that the sector

$$S^L_{\theta,n}(M) = \{\mu \in \mathbb{C} : |\arg(\mu^n)| < \theta, \mu \neq 0\} \subset \rho^L_n(M)$$

and for all $\mu_k \in S^A_{\theta,n}(B), k = \overline{0,p}$, $\quad \max\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{F})}\} \leq \dfrac{K}{\prod\limits_{k=0}^{p} |\mu^n_k|}.$

Lemma 1. *Let M be a (L,n,p)-sectorial operator. Then the lengths of all chains of M–connected vectors are bounded by the number p.*

Let $\alpha \in \rho^L(M)$. Reduce the equation

$$Lu^{(n)} = Mu \tag{10}$$

to the following two equivalent equations defined on different spaces:

$$R^L_\alpha(M)u^{(n)} = (\alpha L - M)^{-1}Mu, \tag{11}$$

$$L^L_\alpha(M)f^{(n)} = M(\alpha L - M)^{-1}f. \tag{12}$$

The operators on the right-hand side can be identified with continuous operators defined on the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Therefore, it is convenient to consider these equations as specific interpretations of the equation

$$Av^{(n)} = Bv, \tag{13}$$

defined on some Banach space $\mathfrak{V}$, where the operators A and B are linear and continuous. A vector-function $v \in C^n(\mathbb{R}_+;\mathfrak{V})$ satisfying the equation (13) is said to be a solution of this equation.

Definition 2. An operator-function $V \in \mathbb{C}^\infty(\mathbb{R}_+;L(\mathfrak{V}))$ is called *a propagator of the equation* (13), if for any $v \in \mathfrak{V}$ the vector-function $v(t) = V^t v$ is a solution of this equation.

Lemma 2. [4] *Let M be a bundle of (L,n,p)-sectorial operators. Then integrals of the Dunford – Schwartz type*

$$U^t_m = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-m-1}(\mu^n L - M)^{-1}Le^{\mu t}d\mu, \qquad F^t_m = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-m-1}L(\mu^n L - M)^{-1}e^{\mu t}d\mu,$$

where $t \in \mathbb{R}_+$, $m = 0,1,\ldots,n-1$, and $\gamma \subset \rho^L_2(M)$ is a contour formed by the rays emerging from the origin at the angles θ and $-\theta$, determine the propagators of the homogeneous equations (11), (12), *respectively.*

Let us select the following subspaces in the spaces $\mathfrak{U}$ and $\mathfrak{F}$:

$$\mathfrak{U}^0 = \bigcap_{m=0}^{n-1} \ker U^t_m, \quad \mathfrak{F}^0 = \bigcap_{m=0}^{n-1} \ker F^t_m.$$

Let L_0 (M_0) be restriction of the operator L (M) to $\mathfrak{U}^0$ ($\mathfrak{U}^0 \cap \mathrm{dom} M$), respectively. Suppose

$$\mathfrak{U}^1 = \mathrm{im} U^t_0 = \{u \in \mathfrak{U} : \lim_{t\to 0+} U^t_0 u = u\}, \quad \mathfrak{F}^1 = \mathrm{im} F^t_0 = \{f \in \mathfrak{F} : \lim_{t\to 0+} F^t_0 f = f\}$$

and denote restriction of the operator L (M) to $\mathfrak{U}^1$ ($\mathfrak{U}^1 \cap \mathrm{dom} M$) by L_1 (M_1).

It is obvious that $\mathfrak{U}^0 \oplus \mathfrak{U}^1 \subset \mathfrak{F}$ and $\mathfrak{F}^0 \oplus \mathfrak{F}^1 \subset \mathfrak{F}$. Introduce the conditions:

$$\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{F} \quad (\mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}) \tag{14}$$

$$\exists\, L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1). \tag{15}$$

The condition (14) is satisfied, for example, if the spaces $\mathfrak{U}$ and $\mathfrak{F}$ are reflexive (the Yagi – Fedorov theorem). The condition (15) is satisfied in the case when (14) holds and $\mathrm{im} L_1 = \mathfrak{F}^1$ (the Banach theorem). Also, note that (14) provides existence of the projectors $P = s - \lim\limits_{t\to 0+} U_0^t$ and $Q = s - \lim\limits_{t\to 0+} F_0^t$ in the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively.

Corollary 1. [4] *Under the conditions of the previous lemma, the operators* $L_0 \in \mathcal{L}(\mathfrak{U}^0;\mathfrak{F}^0)$, $L_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{F}^1)$, $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{F}^0)$, $M_1 \in \mathcal{C}l(\mathfrak{U}^1;\mathfrak{F}^1)$, *and there exists an operator* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.

Consider the Cauchy problem

$$\lim_{t\to 0+} u^{(m)}(t) = u_m, \quad m = 0, 1, \ldots, n-1 \tag{16}$$

for the equation

$$Lu^{(n)} = Mu + f. \tag{17}$$

The equation (17) is reduced to the system

$$Hu^{0(n)} = u^0 + M_0^{-1} f^0, \tag{18}$$

for the equation

$$u^{1(n)} = Su^1 + L_1^{-1} f^0. \tag{19}$$

In the paper [10], it is shown that the operator $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is nilpotent of degree p and the following lemmas are proved.

Lemma 3. *Let M be (L,n,p)–sectorial operator and the conditions* (14), (15) *be satisfied. Then for any vector-function $f^0 \in C^{n(p+1)}([0,T];\mathfrak{F}^0)$ there exists the unique solution of the equation* (18), *which has the form*

$$u^0(t) = -\sum_{q=0}^{p} H^q M_0^{-1} f^{0(nq)}(t).$$

Lemma 4. *Under the conditions of the previous lemma, for any $u_m \in \mathfrak{U}^1, m = 0, ..., n-1$ and $f^1 \in C([0,T];\mathfrak{F}^1)$ there exists the unique solution of the Cauchy problem* (16) *for the equation* (18), *which has the form*

$$u^1(t) = \sum_{m=0}^{n-1} U_m^t u_m + \int_0^t U_{n-1}^{t-s} L_1^{-1} f^1(s) ds.$$

Let us construct a set of admissible initial values

$$M_f^m = \{u \in \mathfrak{U} : (I-P)u = -\sum_{q=0}^{p} H^q M_0^{-1} f^{0(nq+m)}(0), \ \ m = 0, ..., n-1\}.$$

Theorem 1. *Let M be (L,n,p)-sectorial operator and the conditions* (14), (15) *be satisfied. Then for any $u_k \in M_f^m, m = 0, ..., n-1$ and a vector-function $f = f(t), t \in [0,T]$, satisfying the conditions of Lemmas* 3, 4, *there exists the unique solution of the problem* (10), (16), *which has the form $u(t) = u^0(t) + u^1(t)$.*

3. Mathematical Model of Thermoelastic Plate Vibrations

In appropriately chosen spaces, the mathematical model of thermoelastic plate vibrations (5)–(7) can be represented as an initial problem for an operator-differential equation. Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂D. Introduce the spaces $\mathfrak{U} = \{u \in W_2^{k+2}(D) : u(x,y,t) = 0 \; \forall \, (x,y) \in \partial D\}$, $\mathfrak{F} = W_2^k(D)$ and define the operators

$$L = \Delta - \lambda \mathbb{I}, \quad M = \gamma^2 \Delta^2.$$

Then $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M \in \mathcal{C}\ell(\mathfrak{U};\mathfrak{F})$, and dom $M = \{u \in W_2^{k+4}(D) : u(x,y,t) = \Delta u(x,y,t) = 0 \; \forall \; (x,y) \in \partial D\}$. Therefore, the mathematical model (5)–(7) takes the form

$$\dot{u}(0) = u_1, \quad u(0) = u_0, \tag{20}$$

$$L\ddot{u} = Mu + f. \tag{21}$$

Lemma 5. *For any values $\lambda, \gamma \in \mathbb{R}$ $(\gamma \neq 0)$, the operator M is $(L, 2, 0)$-sectorial.*

Let us determine L-spectrum of the operator M. Denote by λ_q eigenvalues of the homogeneous Dirichlet problem for the Laplace operator Δ enumerated by nonincreasing, taking into account their multiplicity. It is known that λ_q have finite multiplicity and condense to the point $-\infty$. So $\sigma^L(M) = \left\{\mu_k = \dfrac{\gamma^2 \lambda_q^2}{\lambda_q - \lambda}; q \in \mathbb{N} \setminus \{q : \lambda_q = \lambda\}\right\}$. Obviously, $\mu_q \sim -q$. Therefore, there exists an angle θ such that

$$S^L_{\theta,2}(M) = \{\mu \in \mathbb{C} : \| \arg(\mu^2) \| < \theta, \mu \neq 0\} \subset \rho^L(M),$$

and we have the estimate

$$\max\{||R^L_{\mu^2}(M)||_{\mathcal{L}(\mathfrak{U})}, ||L^L_{\mu^2}(M)||_{\mathcal{L}(\mathfrak{U})}\} \leq \text{const}|\mu|^{-2}, \; \mu \in S^L_{\theta,2}(M).$$

Since the condition of Theorem 1 is satisfied, we have

Theorem 2. *Let $u_2^0 \in kerL$, $\lambda, \gamma \in \mathbb{R}$ $(\gamma \neq 0)$ and $u_k \in M_f^m$. Then there exists the unique solution of the problem* (5) – (7).

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УРАВНЕНИЕ СОБОЛЕВСКОГО ТИПА В (n, p)-СЕКТОРИАЛЬНОМ СЛУЧАЕ

Е.В. Бычков, К.Ю. Котлованов

В данной статье рассматривается математическая модель колебаний термоупругой пластины при некоторых допущениях. В основе модели лежит неклассическое уравнение математической физики высокого порядка. Кроме того данное уравнение является неразрешимым относительно старшей производной по времени. Исследуемая математическая модель в подходящим образом выбранных функциональных пространствах может быть редуцирована к абстрактному уравнению соболевского типа третьего порядка с относительно (n, p)-секториальным оператором в правой части. Как известно уравнения соболевского типа не является разрешимым при произвольных начальных значениях. Поэтому в статье строится множество допустимых начальных значений. Основным подходом к исследованию является метод построения разрешающих групп.

Ключевые слова: уравнение соболевского типа; относительно спектрально ограниченный оператор; пучок операторов; модель колебания термоупругой пластины.

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