MSC 35Q93, 49J20

AN OPTIMAL CONTROL TO SOLUTIONS OF THE SHOWALTER – SIDOROV PROBLEM FOR THE HOFF MODEL ON THE GEOMETRICAL GRAPH

N.A. Manakova, South Ural State University, Chelyabinsk, Russian Federation, manakova@hotbox.ru

A lot of initial-boundary value problems for the equations and the systems of equations which are not resolved with respect to time derivative are considered in the framework of abstract Sobolev type equations that make up the vast field of non-classical equations of mathematical physics. We are interested in the optimal control problem to solutions of the Showalter – Sidorov problem for the semilinear Sobolev type equation. In this research we demonstrate the appliance of the abstract scheme to the solution of optimal control problem for the Hoff equations on a graph. The physical sense of the optimal control problem lies in the fact that the construction of I-beams should assume the desired shape with minimal costs. This scheme is based on the Galerkin method, allowing carrying out computational experiments. The sufficient conditions for the existence of optimal control to solutions of the Showalter – Sidorov problem for the Hoff equation on the geometrical graph are found.

Keywords: Sobolev type equation, optimal control, the Showalter – Sidorov problem, the Hoff equation.

Introduction

The Sobolev type equations, also known as degenerate or non-solvable with respect to the higher derivative equations, constitute an extensive field among the non-classical equations of mathematical physics [1 – 5]. For example, the Hoff equation

$$(\lambda + \Delta)x_t - \alpha x - \beta x^3 = u, \tag{0.1}$$

modelling the buckling of the I-beam [6], is the prototype of semilinear Sobolev type equation of the form:

$$L\dot{x} + M(x) = u, \ \ker L \neq \{0\}. \tag{0.2}$$

Consider the Showalter – Sidorov problem

$$L(x(0) - x_0) = 0 \tag{0.3}$$

for the semilinear Sobolev type equation (0.2). R.E. Showalter was the first who considered the problem (0.2), (0.3) in an explicit form, N.A. Sidorov came to the problem (0.2), (0.3) independently in another way. Condition (0.3) generalizes the classical Cauchy condition [7]. The solutions of both problems coincide in case of inversability of the operator L. The initial-boundary value problems for the equation (0.1) can be reduced to the Showalter – Sidorov problem (0.3) for the equation (0.2) in suitable functional spaces.

We are interested in the optimal control problem

$$J(x, u) \to \min \tag{0.4}$$

for solutions of (0.2), (0.3). Here $J(x, u)$ is some special penalty functional and $u \in \mathfrak{U}_{ad}$ is a control, where $\mathfrak{U}_{ad}$ is some closed convex set in the control space $\mathfrak{U}$. The optimal control of solutions for linear equation (0.2) satisfying the Cauchy condition was studied in [3]. The optimal control of solutions of the Showalter – Sidorov problem for the linear higher-order Sobolev type equation was studied in [8].

The paper is organized as follows. In the first part we find weak solutions of the problem (0.2), (0.3). This scheme is based on the Galerkin method, allowing carrying out computational experiments [9]. In the second part the optimal control problem (0.2) – (0.4) is studied and the sufficient conditions for the existence of its solutions are obtained. Many initial-boundary value problems for non-classical equations of mathematical physics can be solved by suggested abstract scheme. In addition to sufficient conditions of the optimal control existence we found some necessary (generalizing the Pontryagin maximum principle) conditions [10]. In [10] there is a representation of the abstract scheme application to the solution of the Cauchy – Dirichlet problem for the Hoff equation (0.1) in domain. In the third part we demonstrate the appliance of the abstract scheme to the solution of optimal control problem for the Hoff equations on a graph. The physical sense of the optimal control problem lies in the fact that the construction of I-beams should assume the desired shape with minimal costs. Numerical solution of the optimal measurement problem for Leontief type equations was investigated in [11].

1. The Showalter – Sidorov Problem

Let $\mathcal{H} = (\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space identified with its dual. Let $(\mathfrak{A}, \mathfrak{A}^*)$ and $(\mathfrak{B}, \mathfrak{B}^*)$ be dual (with respect to the duality $\langle \cdot, \cdot \rangle$) pairs of reflexive Banach spaces, moreover, let the embeddings

$$\mathfrak{A} \hookrightarrow \mathfrak{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{B}^* \hookrightarrow \mathfrak{A}^* \tag{1.1}$$

be dense and continuous. Let $L \in \mathcal{L}(\mathfrak{A}; \mathfrak{A}^*)$ be a linear self-adjoint positive semidefinite Fredholm operator. Orthonormal (in the sense of $\mathcal{H}$) set $\{\varphi_k\}$ of eigenfunctions of L is a basis in the space $\mathfrak{A}$. Further, let $M \in C^r(\mathfrak{B}; \mathfrak{B}^*)$ be an s-monotone operator (i.e. $\left\langle M_y^{\prime} x, x \right\rangle > 0, \forall x, y \in \mathfrak{B} \backslash \{0\}$) and a p-coercive operator (i.e. $\langle M(x), x \rangle \geq C_M \|x\|^p$ and $\|M(x)\|_* \leq C^M \|x\|^{p-1}$ for some constants $C_M,\ C^M \in \mathbb{R}_+$ and $p \in [2, +\infty)$ and for any $x \in \mathfrak{B}$, where $\|\cdot\|$, $\|\cdot\|_*$ are the norms in the spaces $\mathfrak{B}$ and $\mathfrak{B}^*$ respectively). For smooth operator $M : \mathfrak{B} \to \mathfrak{B}^*$, strong monotonicity implies s-monotonicity, and s-monotonicity implies strict monotonicity [12].

Consider the Showalter – Sidorov problem

$$L(x(0) - x_0) = 0 \tag{1.2}$$

for the semilinear Sobolev type equation

$$L\dot{x} + M(x) = u. \tag{1.3}$$

Since L is a self-adjoint Fredholm operator, we have $\mathfrak{A} \supset \ker L \equiv \operatorname{coker} L \subset \mathfrak{A}^*$. Obviously, $\mathfrak{A}^* = \operatorname{coker} L \oplus \operatorname{im} L$, then $\mathfrak{B}^* = \operatorname{coker} L \oplus \operatorname{im} L \bigcap \mathfrak{B}^*$. By Q denote the projector along $\operatorname{coker} L$ onto $\operatorname{im} L \bigcap \mathfrak{B}^*$. Introduce the set $\operatorname{coim} L = \{x \in \mathfrak{A} : \langle x, \varphi \rangle = 0,\ \forall \varphi \in \ker L\}$. Obviously, $\operatorname{coim} L \oplus \ker L = \mathfrak{A}$. Let $\overline{\operatorname{coim} L}$ be closure of $\operatorname{coim} L$ in topology $\mathfrak{B}$. Then $\mathfrak{B} = \ker L \oplus \overline{\operatorname{coim} L}$.

If $x{=}x(t), t{\in}(0, T)$ is a solution of (1.3), then it necessarily belong to the set

$$\mathcal{P} = \begin{cases} \{x \in \mathfrak{B} : (I-Q)M(x) = (I-Q)u\}, & \text{if} \quad \ker L \neq \{0\}; \\ \mathfrak{B}, & \text{if} \quad \ker L = \{0\}. \end{cases}$$

The set $\{\varphi_k\}$ of eigenfunctions of operator L is total in $\mathfrak{B}$, therefore, we seek Galerkin approximation of the solution for (1.2), (1.3) in the form

$$x^m(t) = \sum_{k=1}^{m} a_k(t)\varphi_k, \; m > \dim\ker L.$$

Where the coefficients $a_k = a_k(t)$, $k = 1, ..., m$ are defined by the next problem

$$\langle L\dot{x}^m, \varphi_k\rangle + \langle M(x^m), \varphi_k\rangle = \langle u, \varphi_k\rangle, \tag{1.4}$$

$$\langle L(x^m(0) - x_0), \varphi_k\rangle = 0, \; k = 1, ..., m. \tag{1.5}$$

Equations (1.4) form a degenerate system of ordinary differential equations. Let $\mathfrak{B}^m = \text{span}\{\varphi_1, \varphi_2, ..., \varphi_m\}$, $T_m \in \mathbb{R}_+$, $T_m = T_m(x_0)$.

Lemma 1. [11] *For arbitrary $x_0 \in \mathfrak{B}$ and $m > \dim\ker L$ there is a unique solution $x^m \in C^r(0, T_m; \mathfrak{B}^m)$ of (1.4), (1.5).*

We scalar multiply (1.3) by $\varphi \in L_p(0, \tau; \mathfrak{B})$ and integrate the resulting equation over $(0, T)$. Then

$$\int_0^T < L\dot{x}(t) + M(x(t)), \varphi(t) > dt = \int_0^T < u(t), \varphi(t) > dt. \tag{1.6}$$

Definition 1. *The vector function $x \in L_\infty(0, \tau; \text{coim}L) \bigcap L_p(0, \tau; \mathfrak{B})$ is called a solution of equation (1.3) if it satisfies (1.6) for some $T \in \mathbb{R}_+$ ($t \in (0, T)$) and $u \in L_q(0, T; \mathfrak{B}^*)$, $p^{-1} + q^{-1} = 1$.*

Theorem 1. *For arbitrary $x_0 \in \mathfrak{B}$, $T \in \mathbb{R}_+$, $u \in L_q(0, T; \mathfrak{B}^*)$ there exists a unique solution $x \in L_\infty(0, T; \text{coim}\, L) \bigcap L_p(0, T; \mathfrak{B})$ of (1.2), (1.3).*

Proof.

Uniqueness. Let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ be two solutions of (1.2), (1.3). Then for the difference $w = x_1 - x_2$, we have

$$\partial_t \langle Lw, w\rangle + 2\langle M(x_1) - M(x_2), w\rangle = 0.$$

By integrating this relation over the interval $(0, t)$, we obtain

$$\langle Lw, w\rangle + 2\int_0^t \langle M(x_1) - M(x_2), w\rangle \, d\tau = 0. \tag{1.7}$$

The first term in (1.7) is nonnegative, since L is a positive semidefinite operator, and the second term is positive, since M is an s-monotone operator. Therefore, relation (1.7) is valid only with $w \equiv 0$.

Existence. In coimL we introduce the norm $|x|^2 = \langle Lx, x\rangle$. By the Courant principe, this norm is equivalent to the norm induced from the superspace $\mathcal{H}$. Multiply the k-th equation in (1.4) by a_k, take a sum of the resulting relations over $k = 1, ..., m$ and integrate the resulting equation over $(0, t)$. Since M is an p-coercive operator, we have

$$|x^m(t)|^2 + C_1 \int_0^t \|x^m(\tau)\|^p d\tau \leq C;\ C > 0.$$

Hence it follows that all T_m as provided by Lemma 1 can be set equal to each other: $T_m = T$. In addition, since the Banach spaces $L_p(0,T;\mathfrak{B})$ and $L_q(0,T;\mathfrak{B}^*)$ are reflexive, it follows from the Banach – Alaoglu theorem that there are weak limits

$$x^m \to x\ *-\text{weakly in}\, L_\infty(0,T;\text{coim}L),$$

$$x^m \to x\ -\text{ weakly in } L_p(0,T;\mathfrak{B}),$$

$$x^m(T) \to \xi\ *-\text{weakly in }\ L_\infty(0,T;\text{coim}L),$$

$$M(x^m) \to \mu\ -\text{ weakly in } L_q(0,T;\mathfrak{B}^*),$$

where $x^m : m \in \mathbb{N}$, is some subsequence of the sequence of Galerkin approximations provided by Lemma 1.

Let us show that the sequence $\{x^m(t)\}$ of Galerkin approximations converges to the solution $x \in L_\infty(0,T;\text{coim}\, L) \bigcap L_p(0,T;\mathfrak{B})$ of (1.2), (1.3). It can be shown that the weak limit provides the solution of the problem (1.2) for the equation $L\dot{x}+\mu = u$ and $\mu = M(x)$. These assertions can be justified by the usual argument [13] with some modifications.

□

2. Optimal Control Problem

Let $T \in \mathbb{R}_+$. Consider the space $\mathfrak{U} = L_q(0,T;\mathfrak{B}^*), p^{-1} + q^{-1} = 1$, and define a closed convex set $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Consider the optimal control problem (1.2), (1.3), (0.4), where the penalty function is given by the relation

$$J(x,u) = \frac{1}{p}\int_0^T \|x(t) - z_d(t)\|^p_{\mathfrak{B}}\, dt + \frac{N}{q}\int_0^T \|u(t)\|^q_{\mathfrak{B}^*} dt,$$

$$J(x,u) \to \min, \tag{2.1}$$

$z_d = z_d(t)$ is the desired state.

Definition 2. *A pair $(\tilde{x},\tilde{u}) \in [L_\infty(0,T;\text{coim}\, L) \bigcap L_p(0,T;\mathfrak{B})] \times \mathfrak{U}_{ad}$ is called a solution of (1.2), (1.3), (2.1), if $J(\tilde{x},\tilde{u}) = \inf\limits_{(x,u)} J(x,u)$ and every (x,u) satisfies (1.2), (1.3). The vector $\tilde{u}$ is called an optimal control in problem (1.2), (1.3), (2.1).*

Theorem 2. *For arbitrary $x_0 \in \mathfrak{B}, T \in \mathbb{R}_+$ there exists a solution of (1.2), (1.3), (2.1).*

Proof.

It follows from Theorem 1 that the operator

$$(L\frac{d}{dt} + M) : L_\infty(0,T;\text{coim}L) \bigcap L_p(0,T;\mathfrak{B}) \to L_q(0,T;\mathfrak{B}^*)$$

is a homeomorphism. Since M is an s-monotone operator and L is a positive semidefinite operator, it follows from the implicit function theorem that the operator $(L\frac{d}{dt} + M)^{-1}$ is a C^r-diffeomorphism. Therefore, the penalty functional (2.1) can be representated in the form

$$J(x,u) = J(u) = \frac{1}{p}\|x(u) - z_d\|^p_{L_p(0,T;\mathfrak{B})} + \frac{N}{q}\|u\|^q_{L_q(0,T;\mathfrak{B}^*)}. \quad (2.2)$$

Let $\{u_m\} \subset L_q(0,T;\mathfrak{B}^*)$ be a minimizing sequence, relation (2.2) implies that

$$\|u^m\|_{L_q(0,T;\mathfrak{B}^*)} \leq \text{const} \quad (2.3)$$

for all $m \in \mathbb{N}$. By taking a subsequence if necessary, from (2.3) we get a weakly convergent sequence $u_m \to \tilde{u}$. By the Mazur theorem $\tilde{u} \in \mathfrak{U}_{ad}$. Set $x_m = x(u_m)$. By Theorem 1 $M(x) \in L_q(0,T;\mathfrak{B}^*)$ and $u \in L_q(0,T;\mathfrak{B}^*)$; therefore, by (1.3) $L\dot{x} \in L_q(0,T;\mathfrak{B}^*)$. Then $L\dot{x}_m \in L_q(0,T;\mathfrak{B}^*)$ remains in a bounded set in $L_q(0,T;\mathfrak{B}^*)$, $x_m \in L_\infty(0,T;\text{coim}L) \bigcap L_p(0,T;\mathfrak{B})$ remains in a bounded set in $L_\infty(0,T;\text{coim}L) \bigcap L_p(0,T;\mathfrak{B})$; therefore, we can single out a subsequence (denote again by x_m) such that

$$x^m \to x \ * -\text{weakly in } L_\infty(0,T;\text{coim}L),$$

$$x^m \to x \ - \text{weakly in } L_p(0,T;\mathfrak{B}),$$

$$\frac{d}{dt}Lx^m \to \frac{d}{dt}Lx \ - \text{weakly in } L_q(0,T;\mathfrak{B}^*),$$

$$M(x^m) \to \mu \ - \text{weakly in } L_q(0,T;\mathfrak{B}^*).$$

Thus, to prove the existence of an optimal control, it suffices to show that $\mu = M(x(\tilde{u}))$. This fact follows from the s-monotonicity of operator M. The complete proof of this fact is contained in [14]. Therefore, passing to the limit in the state equation

$$L\dot{x}_m + M(x_m) = u_m,$$

we obtain

$$L\dot{x} + M(x) = \tilde{u}.$$

Consequently, $x = x(\tilde{u})$ and $\liminf J(u_m) \geq J(\tilde{u})$. Thus, $\tilde{u}$ is the optimal control.

□

3. The Hoff Model on a Graph

The Hoff equation (0.1) models the buckling of the I-beam, under a constant load. The function $x = x(s,t)$, $(s,t) \in (a,b) \times \mathbb{R}_+$ describes the deviation from the vertical beam, the parameters $\lambda \in \mathbb{R}_+$, $\alpha, \beta \in \mathbb{R}_+$ characterize the load and material properties of the beam, respectively.

Let $\mathbf{G} = \mathbf{G}(\mathrm{V};\mathrm{E})$ be a finite connected oriented graph, where $\mathrm{V} = \{V_i\}$ is the vertex set and $\mathrm{E} = \{E_i\}$ is a set of edges. Every edge of the graph has a length $l_j \in \mathbb{R}_+$ and cross-sectional area $d_j \in \mathbb{R}_+$. Consider the Hoff equations on a graph $\mathbf{G}$

$$j: \ -\lambda_j x_{jt} - x_{jtss} + \alpha_j x_j + \beta_j x_j^3 = u_j, \ \alpha_j, \beta_j \in \mathbb{R}_+, \ \lambda_j \in \mathbb{R}_+. \quad (3.1)$$

They model the dynamics of buckling construction of I-beams.

We are interested in the solution of (3.1) that satisfies the Showalter – Sidorov condition

$$j: \ (\lambda_j + \frac{\partial^2}{\partial s^2})(x_j(s,0) - x_{j0}(s)) = 0, \ s \in (0,l_j) \quad (3.2)$$

and the conditions

$$\sum_{j:E_j\in E^{\alpha}(V_i)} d_j x_{js}(0,t) - \sum_{k:E_k\in E^{\omega}(V_i)} d_k x_{ks}(l_k,t) = 0, \tag{3.3}$$

$$x_j(0,t) = x_k(0,t) = x_m(l_m,t) = x_n(l_n,t), \tag{3.4}$$

where $E_j, E_k \in E^{\alpha}(V_i)$, $E_m, E_n \in E^{\omega}(V_i)$ ($E^{\alpha(\omega)}(V_i)$ is the set of edges of the graph with the beginning (end) in the edge V_i. Consider a Hilbert space $\mathcal{H} = L_2(\mathbf{G}) = \{x = (x_1, x_2, ..., x_j, ...) : x_j \in L_2(0, l_j)\}$ with scalar product

$$\langle x, y\rangle = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} x_j y_j ds,$$

and Banach spaces $\mathfrak{A} = \{x = (x_1, x_2, ..., x_j, ...) : x_j \in W_2^1(0, l_j)$ and (3.4) is fullilled$\}$ with norm

$$||x||^2_{\mathfrak{A}} = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (x^2_{js} + x^2_j) ds,$$

$B = L_4(\mathbf{G}) = \{x = (x_1, x_2, ..., x_j, ...) : x_j \in L_4(0, l_j)\}$ with norm

$$||x||^4_{\mathfrak{B}} = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} |x_j|^4 ds.$$

By the Sobolev theorem functions from $W_2^1(0, l_j)$ are absolutely continuous, therefore the space $\mathfrak{A}$ is correctly defined.

Let $\mathfrak{A}^*$ be an adjoint space to $\mathfrak{A}$ on duality $\langle\cdot,\cdot\rangle$ and the relation

$$\langle Cx, y\rangle = -\sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} x_{js} y_{js} ds, \quad x, y \in \mathfrak{A}$$

defines the operator $C \in Z(\mathfrak{A}; \mathfrak{A}^*)$ [14]. Its spectrum $\sigma(C)$ is negative, discrete, with finite multiplicity and concentrates only to $-\infty$. By $\{\lambda_k\}$ denote the sequence of eigenvalues of the operator numbered in nonincreasing order with regard to multiplicities. Then the set of eigenfunctions $\{\varphi_k\}$ of operator C, whose orthonormal (in the sense of $\mathfrak{A}$), is a basis in the space $\mathfrak{A}$ since embedding $\mathfrak{A}$ in $\mathcal{H}$ is dense and continuous. Note that, by the Sobolev embedding theorem, the embedding (1.1) is continuous and dense.

Define operators L and M by the relations

$$\langle Lx, y\rangle = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (x_{js}y_{js} - \lambda_j x_j y_j) ds, \quad x, y \in \mathfrak{A},$$

$$\langle M(x), y\rangle = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (\alpha_j x_j y_j + \beta_j x_j^3 y_j) ds, \quad x, y \in \mathfrak{B}.$$

Lemma 2. (i) $L \in \mathcal{L}(\mathfrak{A}; \mathfrak{A}^*)$ *is a self-adjoin positive semidefinite Fredholm operator for all* $\lambda_j \leq -\lambda_1$; *moreover, the orthonormal family* $\{\varphi_k\}$ *of its eigenfunctions is total in space* $\mathfrak{A}$.

(ii) $M \in C^\infty(\mathfrak{B};\mathfrak{B}^*)$ *is an s-monotone, 4-coercive operator for all* $\alpha_j, \beta_j \in \mathbb{R}_+$.

Proof.

Assertion (i) is well know [15]. Let us prove assertion (ii). The Frechet derivative of M at point $x \in \mathfrak{B}$ satisfies the relation

$$\langle M_x^{'}y, w\rangle = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (\alpha_j y_j w_j + 3\beta_j x_j^2 y_j w_j)ds.$$

$$|\,\langle M(x), x\rangle\,| \leq \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (\alpha_j x_j^2 + \beta_j x_j^4)ds \leq$$

$$\leq \max_j\{|\alpha_j|, |\beta_j|\}(\|x\|^2_{L_2(\mathbf{G})} + \|x\|^4_{L_4(\mathbf{G})}) \leq \mathrm{const}\|x\|^4_{L_4(\mathbf{G})},$$

$$|\langle M_x^{'}y, y\rangle| \leq \mathrm{const}\|x\|^2_{L_4(\mathbf{G})}\|y\|^2_{L_4(\mathbf{G})},$$

$$|\langle M_x^{''}(y,w), z\rangle| \leq \mathrm{const}\|x\|_{L_4(\mathbf{G})}\|y\|_{L_4(\mathbf{G})}\|w\|_{L_4(\mathbf{G})}\|z\|_{L_4(\mathbf{G})},$$

$$|\langle M_x^{'''}(v,w,y), z\rangle| \leq \mathrm{const}\|v\|_{L_4(\mathbf{G})}\|w\|_{L_4(\mathbf{G})}\|y\|_{L_4(\mathbf{G})}\|z\|_{L_4(\mathbf{G})},$$

where symbol const indicates different constants, but they all do not depend on vectors $x, v, w, y, z \in L_4(\mathbf{G})$ and by $M_x^{'}, M_x^{''}, M_x^{'''}$ the Frechet derivatives of the operator M at point x are designated respectively. Note also that the Frechet derivative $M_x^{(4)} \equiv 0$. Hence it follows that $M \in C^\infty(\mathfrak{B};\mathfrak{B}^*)$ and it is an s-monotone operator

$$\langle M_x^{'}y, y\rangle = \sum_{E_j\in \mathrm{E}} d_j \int_0^{l_j} (\alpha_j y_j^2 + 3\beta_j x_j^2 y_j^2)ds > 0,\ \forall y \in \mathfrak{B}\backslash\{0\}.$$

Prove that the operator M is 4-coercive.

$$\langle M(x), x\rangle \geq \min_j\{|\alpha_j|\}\|x\|^2_{L_2(\mathbf{G})} + \min_j\{|\beta_j|\}\|x\|^4_{L_4(\mathbf{G})} \geq \mathrm{const}\|x\|^4_{\mathfrak{B}},$$

$$|\langle M(x), y\rangle| \leq \max_j\{|\alpha_j|\}\|x\|_{L_2(\mathbf{G})}\|y\|_{L_2(\mathbf{G})} + \max_j\{|\beta_j|\}\|x\|^3_{L_4(\mathbf{G}}\|y\|_{L_4(\mathbf{G})} \leq \mathrm{const}\|x\|^3_{\mathfrak{B}}\|y\|_{\mathfrak{B}}.$$

□

Theorem 3. *Let* $\lambda_j \leq -\lambda_1$, $\alpha_j, \beta_j, T \in \mathbb{R}_+$. *Then for arbitrary* $x_0 \in \mathfrak{B}$ *and* $u \in L_{\frac{4}{3}}(0,T;\mathfrak{B}^*)$ *there exists a unique solution* $x \in L_\infty(0,T;\mathrm{coim}\, L)\bigcap L_4(0,T;\mathfrak{B})$ *of (3.1)–(3.4).*

Proof.

From Theorem 1 and Lemma 2 we have the assertion.

□

Let us proceed to the analysis of the optimal control problem for the Hoff model on a graph. Take a closed convex set $U_{ad} \subset L_{\frac{4}{3}}(0,T;L_{\frac{4}{3}}(\mathbf{G}))$.

Theorem 4. *Let* $\lambda_j \leq -\lambda_1, \alpha_j, \beta_j, T \in \mathbb{R}_+$. *Then there exists an optimal control for the problem (3.1)–(3.4), (2.1).*

Proof.

By Theorm 2 and Lemma 2 we have the assertion.

□

The author is grateful to G.A. Sviridyuk for useful discussion of the results.

References

1. Al'shin A.B., Korpusov M.O., Sveshnikov A.G. *Blow-up in Nonlinear Sobolev-Type Equations.* Berlin, N.-Y., Walter de Gruyter GmbH & Co. KG, 2011.
2. Demidenko G.V., Uspenskii G.V. *Partial Differential Equations and Systems not Solvable with Respect to the Highest-Order Derivative.* N.-Y., Basel, Hong Kong, Marcel Dekker, Inc., 2003.
3. Sviridyuk G.A., Fedorov V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators.* Utrecht, Boston, Köln, Tokyo, VSP, 2003.
4. Sviridyuk G.A., Zamyshlyaeva A.A. The Phase Spaces of a Class of Linear Higher-Order Sobolev Type Equations. *Differential Equations*, 2006, vol. 42, no. 2, pp. 269–278.
5. Sviridyuk G.A., Zagrebina S.A. Verigin's Problem for Linear Equations of the Sobolev Type with Relatively p-Sectorial Operators. *Differential Equations*, 2002, vol. 38, no. 12, pp. 1745–1752.
6. Hoff N.J. Creep Buckling. *The Aeronautical Quarterly*, 1956, vol. 7, no. 1, pp. 1–20.
7. Sviridyuk G.A., Zagrebina S.A. The Showalter – Sidorov Problem as Phenomena of the Sobolev-Type Equations. *The Bulletin of Irkutsk State University. Series "Mathematics"*, 2010, vol. 3, no. 1, pp. 51–72. (in Russian)
8. Zamyshlyaeva, A.A., Tsyplenkova, O.N. Optimal Control of Solutions of the Showalter – Sidorov – Dirichlet Problem for the Boussinesq – Love Equation. *Differential Equations*, 2013, vol. 49, no. 11, pp. 1356–1365.
9. Manakova N.A. On a Model of Optimal Control of the Oskolkov Equation. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2008, no. 27 (127), issue 2, pp. 63–70. (in Russian)
10. Sviridyuk G.A., Manakova N.A. An Optimal Control Problem for the Hoff Equation. *Journal of Applied and Industrial Mathematics*, 2007, vol. 1, no. 2, pp. 247–253.
11. Shestakov A.L., Keller A.V., Nazarova E.I. Numerical Solution of the Optimal Measurement Problem. *Automation and Remote Control*, 2012, vol. 73, no. 1, pp. 97–104.
12. Sviridyuk G.A. One Problem for the Generalized Boussinesq Filtration. *Russian Mathematics (Izvestiya VUZ. Matematika)*, 1989, vol. 33, no. 2, pp. 62–73.
13. Lyons J.-L. *Quelques methodes de résolution des problémes aux limites nonlineaires.* Paris, Dunod de Gauthiers–Villars, 1969.
14. Manakova N.A. Optimal Control Problem for the Oskolkov Nonlinear Filtration Equation. *Differential Equations*, 2007, vol. 43, no. 9, pp. 1213–1221.
15. Bayazitova A.A. Computational Investigation of Processes in Hoff Models. *Bulletin of the South Ural State University. Series "Mathematical, Modelling, Programming & Computer Software"*, 2011, no. 4 (221), issue 7, pp. 4–9. (in Russian)

Received April 26, 2014