MSC 60H30, 60H10

ON THE STOCHASTIC SYSTEMS OF DIFFERENTIAL-ALGEBRAIC TYPE

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Under the stochastic system of differential-algebraic type we understand the special class of stochastic differential equations in the Ito form, in which in the left- and right-hand sides there are time-dependent continuous rectangular real matrices of the same size, and, in the case of a square matrix, the matrix in the left-hand side is degenerated. In addition, in the right-hand side there is a term that depends only on time. This class of equations is a natural generalization of the class of ordinary differential-algebraic equations. It is assumed that the initial conditions for this class of equations are solutions of some systems of linear algebraic equations, matrices in which are constant and have the same size as in the stochastic system. For the study of this class of equations, we use the machinery that is a generalization of the methods suggested for the study of ordinary differential-algebraic equations in the works by Yu. E. Boyarintsev, V. F. Chistyakov and others. Note that for investigation of these equations we do not use derivative of the right-hand side. We give the necessary information from the theory of pseudo-inverse matrix, and then thransform the system to a form more convenient for study. The result of the article is the statemants, in which sufficient conditions for the existence of solutions are obtained and formulae for finding the solutions are given.

Keywords: differential-algebraic system, Wiener process.

Statement of the problem

Under differential-algebraic system [1, 2] we understand a system of the form

$$\frac{dL(t)x(t)}{dt} = M(t)x(t) + f(t), \ \ 0 \le t \le T,$$

with continuous matrix coefficients of $L(t), M(t) \in R^{m\times n}$ where $f(t) \in R^m$ is a continuous vector-function and $x(t) \in R^n$ is a solution. In the case of a square matrix, we suppose $L(t)$ to be degenerate. In papers [1, 2] such systems have been studied quite extensively. However, the question remains about the solvability of equations in the case where in the right-hand side there is a summand of white noise type. In this article we study this case, i.e., we investigate the stochastic system of the form

$$dL(t)\xi(t) = M(t)\xi(t)dt + f(t)dt + N(t)dw(t), 0 \le t \le T, \tag{1}$$

and the vector $\xi(t)$ satisfies the condition

$$S\xi(0) = a, \tag{2}$$

(see [3, 4]) where $\xi(t) \in R^n$ is a stochastic process, $L(t)$, $M(t)$ and $N(t)$ are real continuous $m \times n$-matrices, S is a constant $m \times n$-matrix, $f(t) \in R^m$ is an integrable vector-function and $w(t) \in R^n$ is a Wiener process given on full filtered probability space $\{\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, P\}$ started from zero and subordinated to $(\mathfrak{F}_t)_{t\in[0,T]}$. Its derivative $\acute{w}(t)$ is the white noise and $a \in R^m$ is a constant vector.

Definition 1. *A solution of* (1), (2) *is a stochastic process* $\xi(t) \in R^n$*, non-anticipating with respect to family* $(\mathfrak{F}_t)_{t\in[0,T]}$ *of complete* σ*-algebras, that satisfies* (1) *and* (2) *with probability one.*

1. Pseudo-inverse matrices

Everywhere below, by $C^i[0,T]$ we denote the space of C^i-smooth matrix function of corresponding dimension on the interval $[0,T] \subset R$. By E we denote the unit matrix, by 0 – the zero matrix, by imA and $kerA$ – the image and the kernel of matrix A, respectively. We omit mentioning the dependence of the matrix on time t if it does not yield a misunderstanding.

Definition 2. *[2, 5] An* $n \times m$ *matrix* $A^+(t)$ *is called pseudo-inverse to* $m \times n$*-matrix* $A(t)$ *if for any* t

$$A(t)A^+(t)A(t) = A(t), \;\; A^+(t)A(t)A^+(t) = A^+(t),$$
$$[A(t)A^+(t)]^T = A(t)A^+(t), \;\; [A^+(t)A(t)]^T = A^+(t)A(t).$$

where the symbol ^{T} *denotes the transposition.*

Theorem 1. *[2] Let the* $n \times n$ *matrix* $A(t)$ *be* $C^i[0,T]$*-smooth,* $i = 0, 1, ...$ *and* $rankA(t) = const = \rho$ *on* T*. Then there exists a pseudo-inverse matrix* $A^+(t) \in C^i[0,T]$*.*

Lemma 1. *Let*

(i) $m \times n$*-matrices* $A(t)$ *and* $B(t)$ *belong to* $C^0[0,T]$ *and* C *be a constant real* $m \times n$*-matrix;*

(ii) the matrices $A^+(t), (B(t)P_0(t))^+, (CP_0)^+, (CP_2(t))^+$ *and* $(B(t)Q_2(t))^+$ *belong to* $C^0[0,T]$ *, where* $P_0 = E - A^+A, P_2 = P_0 - P_1, Q_2 = P_0 - Q_1, P_1 = P_0(BP_0)^+BP_0$ *and* $Q_1 = P_0(CP_0)^+CP_0$*.*

Then the matrices A^+A*,* P_0 *and* P_1*,* $P_2 = P_0 - P_1$*,* $P_3 = P_2(CP_2)^+CP_2$ *and* $P_4 = P_2 - P_3$ *are continuous projectors onto the sets* imA*,* $kerA$*,* $kerA \cap kerB$ *and* $kerA \cap kerB \cap kerC$*, respectively. The matrices* Q_1*,* $Q_2 = P_0 - Q_1$*,* $Q_3 = Q_2(BQ_2)^+BQ_2$ *and* $Q_4 = Q_2 - Q_3$ *are continuous projectors onto the sets* $kerA$*,* $kerA \cap kerC$ *and* $kerA \cap kerB \cap kerC$*, respectively.*

Proof.

All the matrices in the hypothesis of Lemma are continuous since the product and the sum of continuous matrices is continuous. From the definitions of projector and of pseude-inverse matrix we obtain

$$A^+(t)A(t)A^+(t)A(t) = A^+(t)[A(t)A^+(t)A(t)] = A^+(t)A(t)$$

and $A(t)A^+(t)A(t) = A(t)$. Hence, $A^+(t)A(t)$ is a projector onto imA. Then $P_0(t) = E - A^+(t)A(t)$ is a projector onto $kerA$. On the other hand,

$$P_1^2 = P_0[(BP_0)^+BP_0P_0(BP_0)^+]BP_0 = P_0(BP_0)^+BP_0 = P_1$$

and $AQ_1 = AP_0[(BP_0)^+BP_0] = 0$. So, P_1 is a projector onto $kerA$. Further on,

$$P_2^2 = (P_0 - P_1)^2 = P_0^2 + P_1^2 - 2P_0P_1 = P_0 + P_1 - 2P_1 = P_0 - P_1 = P_2$$

and

$$A(P_0 - P_1) = A(E - A^+A - P_0(BP_0)^+BP_0) = A - AA^+A-$$
$$-AP_0[(BP_0)^+BP_0] = A - A + 0 = 0,$$
$$B(P_0 - P_1) = B - BA^+A - BP_0(BP_0)^+BP_0 = B - BA^+A - BP_0 =$$
$$= B - BA^+A - B + BA^+A = 0.$$

Thus, $P_2 = P_0 - P_1$ is a projector onto $kerA \cap kerB$.

$$P_3^2 = P_2(CP_2)^+CP_2P_2(CP_2)^+CP_2 = P_2[(CP_2)^+CP_2(CP_2)^+]CP_2 =$$
$$P_2(CP_2)^+CP_2 = P_3$$

and $AP_3 = BP_3 = 0$ (since P_2 is a projector onto $kerA \cap kerB$). Then P_3 is a projector onto $kerA \cap kerB$.

One can easily see that $P_4^2 = (P_2 - P_3)^2 = P_2^2 + P_3^2 - 2P_3^2 = P_2 - P_3$, $C(P_2 - P_3) = CP_2 - CP_2(CP_2)^+CP_2 = 0$. Then P_4 is a projector onto $kerC$. Since P_2 and P_3 are projectors onto $kerA \cap kerB$, P_4 is a projector onto $kerA \cap kerB \cap kerC$.

The other assertions of the Lemma are proved by complete analogy of the above arguments.

□

Consider the system

$$AX = B, \tag{3}$$

where X is a vector we are looking for, A and B are a matrix and a vector of appropriate size (i.e., such that system (3) is well-posed). Then the classical Kronecker-Capelli theorem on solvability of system (3) takes place:

Theorem 2. *[1] System* (3) *is solvable if and only if the equality*

$$(E - AA^+)B = 0$$

holds.

We should present the theroem on the presentation of solutions of system (3).

Theorem 3. *[1] If system* (3) *is solvable, its general solution is described by the formula*

$$X = A^+B + (E - A^+A)U,$$

where U is an arbitrary vector.

2. The study of the system with the use of pseudo-inverse matrices

For investigatilon of system (1), (2) we introduce the matrices

$$P_0 = E - L^+L,$$

$$P_1 = P_0(MP_0)^+MP_0, \;\; Q_1 = P_0(SP_0)^+SP_0,$$
$$P_2 = P_0 - P_1, \;\; Q_2 = Q_0 - Q_1,$$

$$P_3 = P_2(SP_2)^+SP_2, \;\; Q_3 = Q_2(MQ_2)^+MQ_2,$$
$$P_4 = P_2 - P_3, \;\; Q_4 = Q_2 - Q_3.$$

We assume that the matrices L^+ and $(MP_0)^+$ are continuous. Then by Lemma 1 matrix P_0 is a projector of n-dimensional Euclidean space of vectors to zero space of the matrix L, while P_2 and P_3 are projectors to the intersection of the kernels of matrices L and M. The matrix P_4 is a projector onto the intersection $kerL \cap kerM \cap kerS$. The matrices Q_1, $Q_2 = Q_0 - Q_1$ and $Q_3 = Q_2(MQ_2)^+MQ_2$, $Q_4 = Q_2 - Q_3$ are projectors onto the stes $kerL$, $kerL \cap kerS$ and $kerL \cap kerM \cap kerS$, respectively.

Using projectors and properties of pseudo-inverse matrix, it is established that the process $\xi(t)$ is represented in the following two ways:

$$\xi = L^+\eta + u_1 + v_1 + h_1, \tag{4}$$
$$\xi = L^+\eta + u_2 + v_2 + h_2, \tag{5}$$

where

$$\eta = L\xi, \tag{6}$$
$$u_1 = P_1\xi, \;\; v_1 = P_3\xi, \;\; h_1 = P_4\xi, \tag{7}$$
$$u_2 = Q_1\xi, \;\; v_2 = Q_3\xi, \;\; h_2 = Q_4\xi \tag{8}$$

and

$$u_1 \in \ker L, \;\; v_1 \in \ker L \cap \ker M, \;\; u_2 \in \ker L, \;\; v_2 \in \ker L \cap \ker S, \tag{9}$$

Since the substitution of vectors (4) and (5) to problem (1) and (2) should give the same result, we require that equalities

$$Mv_2 = M(u_1 - u_2), \tag{10}$$
$$Sv_1 = S(u_2 - u_1) \tag{11}$$

hold simultaneously.

Substitute vector (4) to the expression for u_1 and v_1 of (7), then taking into account (9), we first obtain

$$[E - P_0(MP_0)^+M]u_1 = 0. \tag{12}$$

Then taking into account (11) and (12), we have

$$v_1 = P_2(SP_2)^+S(u_2 - u_1). \tag{13}$$

Substitution of vector (5) to (8) yields the equalities

$$[E - P_0(SP_0)^+S]u_2 = 0, \tag{14}$$
$$v_2 = Q_2(MQ_2)^+M(u_1 - u_2). \tag{15}$$

Since equalities (10) and (11) should hold, from (13) and (15) it follows that

$$[E - SP_2(SP_2)^+]S(u_1 - u_2) = 0, \tag{16}$$
$$[E - MQ_2(MQ_2)^+]M(u_1 - u_2) = 0. \tag{17}$$

Substitution of any of equations (4) and (5) to formula (6) yields

$$(E - LL^{+})\eta = 0. \tag{18}$$

On substituting vector (4) to equation (1) and vector (5) to condition (2), we obtain

$$d\eta(t) = M(t)L^{+}(t)\eta(t)dt + M(t)u_1(t)dt + f(t)dt + N(t)dw(t), \tag{19}$$

$$L^{+}(0)\eta(0) = a - Su_2(0). \tag{20}$$

Thus, problem (1), (2) is reduced to problem (19) and (20) with conditions (12), (14), (16), (17) and (18). Note that if $u_1 \in kerM$, from equation (12) it follows that equality $u_1 = 0$ holds. Analogously, from equality (14) it follows that if $u_2 \in kerS$ we obtain $u_2 = 0$.

The system of equations (12), (14), (16) and (17) with respect to vectors u_1 and u_2 can be solved explicitly. To do this, by introducing the notation $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we rewrite it in the form of two equivalent systems

$$\begin{pmatrix} E - P_0(MP_0)^{+}M & 0 \\ 0 & E - P_0(SP_0)^{+}S \end{pmatrix} u = 0, \tag{21}$$

$$\begin{pmatrix} E - SP_2(SP_2)^{+} & 0 \\ 0 & E - MQ_2(MQ_2)^{+} \end{pmatrix} \cdot \begin{pmatrix} S & -S \\ M & -M \end{pmatrix} u = 0. \tag{22}$$

The general solution of system (21) takes the form

$$u = \begin{pmatrix} P_0(MP_0)^{+}M & 0 \\ 0 & P_0(SP_0)^{+}S \end{pmatrix} p, \tag{23}$$

where $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is an arbitrary vector. This formula is derived with the application of the fact that the matrices $P_0(MP_0)^{+}M$ and $P_0(SP_0)^{+}S$ are projectors. Since systems (21) and (22) are equivalent, by substituting solution (23) of the first one to (22), we obtain a condition, to which p must satisfy:

$$\Xi p = 0, \tag{24}$$

where

$$\Xi = \begin{pmatrix} E - SP_2(SP_2)^{+} & 0 \\ 0 & E - MQ_2(MQ_2)^{+} \end{pmatrix} \cdot \begin{pmatrix} S & -S \\ M & -M \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} P_0(MP_0)^{+}M & 0 \\ 0 & P_0(SP_0)^{+}S \end{pmatrix}.$$

Equality (24) is satisfied by p such that:

$$p = (E - \Xi^{+}\Xi)r, \tag{25}$$

where $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ is an arbitrary vector.

Finally, substituting (25) to (23), we get the general formula for the solution of system (21), (22), namely: $u = \Phi r$, where

$$\Phi = \begin{pmatrix} P_0(MP_0)^{+}M & 0 \\ 0 & P_0(SP_0)^{+}S \end{pmatrix} (E - \Xi^{+}\Xi)$$

and r is an arbitrary vector. Moreover, the components of vector u are calculated by formulae:

$$u_1 = \Phi_1 r, \;\; u_2 = \Phi_2 r, \tag{26}$$

where

$$\Phi_1 = \begin{pmatrix} P_0(MP_0)^+M & 0 \end{pmatrix}(E - \Xi^+\Xi),$$
$$\Phi_2 = \begin{pmatrix} 0 & P_0(SP_0)^+S \end{pmatrix}(E - \Xi^+\Xi),$$

Formulae (13) and (15) for calculation of vectors v_1 and v_2 now take the form:

$$v_1 = P_2(SP_2)^+S(\Phi_2 - \Phi_1)r, \tag{27}$$
$$v_2 = Q_2(MQ_2)^+M(\Phi_1 - \Phi_2)r. \tag{28}$$

Substitution of vectors (26) to equation (19), (20) yields

$$d\eta(t) = M(t)L^+(t)\eta(t)dt + M(t)\Phi_1(t)r(t)dt + f(t)dt + N(t)dw(t), \tag{29}$$
$$L^+(0)\eta(0) = a - S\Phi_2(0)r(0) \tag{30}$$

Now take into account the fact that any solution of equation (29) satisfies the relation

$$\eta(t) = X(t)\eta(0) + \theta(t), \tag{31}$$

where matrix $X(t)$ and vector $\theta(t)$ are solutions of Cauchy problems:

$$\frac{dX(t)}{dt} = M(t)L^+(t)X(t), \;\; X(0) = E$$

and

$$d\theta(t) = M(t)L^+(t)\theta(t)dt + M(t)\Phi_1(t)r(t)dt + f(t)dt + N(t)dw(t),$$
$$\theta(0) = 0.$$

respectively. One can easily see that $\theta(t) = X(t)\chi(t)$, where

$$\chi(t) = \int_0^t X^{-1}(s)[M(s)\Phi_1(s)r(s) + f(s)]ds + \int_0^t X^{-1}(s)N(s)dw(s),$$

therefore, equality (31) can be rewritten in the form

$$\eta(t) = X(t)\eta(0) + X(t)\chi(t). \tag{32}$$

Substituting (32) to conditions (18) and (30), we obtain

$$[E - L(t)L^+(t)]X(t)\eta(0) = -[E - L(t)L^+(t)]X(t)\chi(t),$$
$$0 \le t \le T, \tag{33}$$
$$\Lambda\eta(0) = a - S\Phi_2(0)r(0), \tag{34}$$

where

$$\Lambda = SL^+(0)X(0).$$

We have to apply the following lammas from the theory of algebraic systems.

Lemma 2. *[6] Let the matrix $A(t)$ be continuous on the interval $[0,T]$ and*

$$G=\int_0^T A^*(s)A(s)ds. \tag{35}$$

Then any solution c of the system

$$Gc=0 \tag{36}$$

is a constant solution of the system

$$A(t)c=0, 0\le t\le T. \tag{37}$$

And vice versa, any constant solution of system (37) is a solution of system (36)

Lemma 3. *[7] Let in the system of equations*

$$A(t)y=B(t), 0\le t\le T, \tag{38}$$

matrices $A(t)$ and $B(t)$ be continuous. Then system (38) has a constant (independent of $t\in[0,T]$) solution y if and only if for all $t\in[0,T]$ the equality

$$A(t)G^+\int_0^T A^*(s)B(s)ds=B(t) \tag{39}$$

holds, where G is matrix (35).

Lemma 4. *[8] If the hypothesis of Lemma 3 and equality (39) hold, the general solution of system (38) has the form*

$$y=G^+\int_0^T A^*(s)B(s)ds+(E-G^+G)c, \tag{40}$$

where c is an arbitrary vector.

Corollary 1. *[1] If there exists a constant solution y of system (38), the equality*

$$(E-GG^+)\int_0^T A^*(s)B(s)ds=0$$

holds.

Now apply the above-mentioned Lemmas to equation (33). We have

$$A(t)=[E-L(t)L^+(t)]X(t),$$
$$B(t)=-A(t)\chi(t)$$

and since the matrix $E-L(t)L^+(t)$ is a self-adjoint projector, the equality

$$A^*(t)A(t)=X^*(t)[E-L(t)L^+(t)]X(t)$$

holds. Hence,

$$G = \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)ds, \tag{41}$$

$$\int_0^T A^*(s)B(s) = \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds,$$

then by formula (40) we obtain

$$\eta(0) = -G^+ \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds + (E - G^+G)c, \tag{42}$$

where c is an arbitrary vector, and G is matrix (41). By Lemma 3, here the condition of solvability (33) with respect to $\eta(0)$ should hold:

$$[E - L(t)L^+(t)]X(t)G^+ \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds =$$
$$[E - LL^+]X(t)\chi(t), 0 \le t \le T. \tag{43}$$

By substituting vector (42) in equation (34), we obtain the equation for finding vector c:

$$(\Lambda - \Lambda G^+G)c = a - S\Phi_2(0)r(0)+$$
$$+\Lambda G^+ \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds. \tag{44}$$

By Theorem 3 the solution of equation (44) is written in the form :

$$c = (\Lambda - \Lambda G^+G)\{a - S\Phi_2(0)r(0)+$$
$$+\Lambda G^+ \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds\}+$$
$$[E - (\Lambda - \Lambda G^+G)^+(\Lambda - \Lambda G^+G)]\beta, \tag{45}$$

where β is an arbitrary vector. By Theorem 3, formula (45) takes place if the condition of compatibility for system (44) is satisfied

$$[E - (\Lambda - \Lambda G^+G)(\Lambda - \Lambda G^+G)^+]\{a - S\Phi_2(0)r(0)+$$
$$+\Lambda G^+ \int_0^T X^*(s)[E - L(s)L^+(s)]X(s)\chi(s)ds\} = 0. \tag{46}$$

By substituting (45) to (42), we derive to the last expression for vector $\eta(0)$, namely:

$$\eta(0) = -[E - (E - G^+G)(\Lambda - \Lambda G^+G)^+\Lambda]G^+ \int_0^T X^*(s)[E-$$
$$-L(s)L^+(s)]X(s)\chi(s)ds + (E - G^+G)(\Lambda - \Lambda G^+G)^+\cdot$$
$$\cdot\{a - S\Phi_2(0)r(0)\}+$$
$$+(E - G^+G)[E - (\Lambda - \Lambda G^+G)^+(\Lambda - \Lambda G^+G)]\beta, \tag{47}$$

where β is an arbitrary vector.

One can see that the solvability conditions (43), (46) have the form of a system of integral equations with respect to vector $r(t)$ since vector $\chi(t)$, included in this system, is associated with $r(t)$ by the integral formula

$$\chi(t) = \int_0^t X^{-1}(s)[M(s)\Phi_1(s)r(s) + f(s)]ds + \int_0^t X^{-1}(s)N(s)dw(s). \tag{48}$$

So, it is established that the General solution of problem (1), (2) can be represented in two forms: either (4) or (5), where the vectors u_1, u_2, v_1 and v_2 are calculated by formulae (26), (27) end (28). Vector $r(t)$ is an arbitrary solution of integral system (43), (46) and (48), vector $\eta(t)$ is a solution of equation (29) with initial condition (47). Since vectors $h_1(t)$ and $h_2(t)$ belong to the intersection $kerL \cap kerM \cap kerS$, for their calculation the formulae

$$h_1(t) = P_4\gamma(t), \;\; h_2(t) = Q_4\gamma(t),$$

take place where $\gamma(t)$ is an arbitrary continuous vector given on the interval $[0, T]$. In [7, 8] it is established that the matrix that is applied to vector β in formula (47), is a projector on the set $kerG \cap ker\Lambda$. Then the third term in the right-hand side of equality (47) (which we denote by α) is an arbitrary solution of the system

$$\Lambda\alpha = 0, \;\; G\alpha = 0, \tag{49}$$

The main difficulty in solving problem (1), (2) is in finding solution of integral system (43), (46), (48), to which the vector $r(t)$ satisfies. Now we can formulate the conditions under which the resulting complex system relative to the vector $r(t)$ admits a simple solution.

3. Theorems on the solvability of systems and formulae for solutions

Problem (1), (2) should be considered for matrices $L(t)$, $M(t)$, S that satisfy the identities

$$[E - L(t)L^+(t)]X(t)X^{-1}(s)M(s)\Phi_1(s) = 0, \tag{50}$$
$$[E - (\Lambda - \Lambda G^+G)(\Lambda - \Lambda G^+G)^+]S\Phi_2(t) = 0, \tag{51}$$
$$t, s \in [0, T].$$

Then for problem (1), (2) compatibility conditions

$$[E-L(t)L^+(t)]X(t)G^+\{\int_0^T X^*(s)[E-L(s)L^+(s)]z(s)ds+$$
$$+\int_0^T X^*(s)[E-L(t)L^+(t)]X(s)[\int_0^s X^{-1}(u)N(u)dw(u)]ds\}=$$
$$=[E-L(t)L^+(t)]\{z(t)+X(t)\int_0^t X^{-1}(s)N(s)dw(s)\}, \quad (52)$$
$$[E-(\Lambda-\Lambda G^+G)(\Lambda-\Lambda G^+G)^+]\{a+\Lambda G^+[\int_0^T X^*(s)[E-L(s)L^+(s)]z(s)ds+$$
$$\int_0^T X^*(s)[E-L(s)L^+(s)]X(s)[\int_0^s X^{-1}(u)N(u)dw(u)]ds]\}=0, \quad (53)$$

should hold where $z(s)$ is a solution of the following Cauchy problem:

$$dz(t)=M(t)L^+(t)z(t)dt+f(t)dt,$$
$$z(0)=0.$$

Thus, from the results of previous section we derive the following theorems.

Theorem 4. *Let in problem* (1), (2) *for matrices* $L(t)$, $M(t)$, S *identities* (50), (51) *hold. Then for sovability of the problem it is necessary and sufficient that conditions* (52), (53) *are satisfied.*

Theorem 5. *If the matrices* $L(t)$, $M(t)$ *and* S *in problem* (1), (2) *satisfy equations* (50) *and* (51) *and the problem has a solution, then its general solution can be written in two ways:*

$$\xi(t)=L^+(t)\eta(t)+H_1(t)r(t)+P_4(t)r_0(t), \quad (54)$$
$$\xi(t)=L^+(t)\eta(t)+H_2(t)r(t)+Q_4(t)r_0(t), \quad (55)$$

where the matrices $H_1(t)$ *and* $H_2(t)$ *are calculated as follows:*

$$H_1=[E-P_2(SP_2)^+S]\Phi_1+P_2(SP_2)^+S\Phi_2,$$
$$H_2=[E-Q_2(MQ_2)^+M]\Phi_2+Q_2(MQ_2)^+M\Phi_1,$$

$r(t)$, $r_0(t)$ *are arbitrary continuous vectors and* $\eta(t)$ *is a solution of Ito equation*

$$d\eta(t)=M(t)L^+(t)\eta(t)dt+M(t)\Phi_1(t)r(t)dt+f(t)dt+N(t)dw(t) \quad (56)$$

with initial condition

$$\eta(0)=-[E-(E-G^+G)(\Lambda-\Lambda G^+G)^+\Lambda]G^+\{\int_0^T X^*(s)[E-L(s)L^+(s)]\theta(s)ds+$$
$$+\int_0^T X^*(s)[E-L(s)L^+(s)]X(s)[\int_0^s X^{-1}(u)N(u)dw(u)]ds\}+$$
$$+(E-G^+G)(\Lambda-\Lambda G^+G)^+\cdot\{a-S\Phi_2(0)r(0)\}+\alpha, \quad (57)$$

where α is a solution of the system

$$\Lambda\alpha = 0, \ \ G\alpha = 0, \tag{58}$$

and vector $\theta(t)$ satisfies the equation

$$d\theta(t) = M(t)L^{+}(t)\theta(t)dt + M(t)\Phi_1(t)r(t)dt + f(t)dt,$$
$$\theta(0) = 0, \tag{59}$$

(note that vector $r(t)$ in (56), (57), (59) is the same as in (54), (55)).

Theorem 6. *The solution of problem (1), (2) (if it exists) is unique if and only if system (58) has only zero solution $\alpha = 0$ and the equality*

$$P_4(t) = Q_4(t) = 0, \ \ 0 \le t \le T, \tag{60}$$

$$\Phi_1(t) = \Phi_2(t) = 0, \ \ 0 \le t \le T \tag{61}$$

holds.

Theorem 7. *Let in problem (1), (2) the compatibility conditions (52) and (53) hold and equality (61) take place. Then solutions of problem (1) and (2) exist, and the general solution has the form*

$$\xi(t) = L^{+}(t)\eta(t) + h(t), 0 \le t \le T,$$
$$L(t)h(t) = 0, \ \ M(t)h(t) = 0, \ \ Sh(t) = 0, \tag{62}$$

where the vector $\eta(t)$ is obtained from the following system

$$dz(t) = M(t)L^{+}(t)z(t)dt + f(t)dt, \ \ z(0) = 0,$$
$$d\eta(t) = M(t)L^{+}(t)\eta(t)dt + f(t)dt + N(t)dw(t)$$

with initial condition

$$\eta(0) = -[E - (E - G^{+}G)(\Lambda - \Lambda G^{+}G)^{+}\Lambda]G^{+}\{\int_0^T X^{*}(s)\left[E - L(s)L^{+}(s)\right] z(s)ds +$$
$$+\int_0^T X^{*}(s)[E - L(s)L^{+}(s)]X(s)[\int_0^s X^{-1}(u)N(u)dw(u)]ds\}+$$
$$+(E - G^{+}G)(\Lambda - \Lambda G^{+}G)^{+}a + \alpha,$$

where α is an arbitrary solution of the system

$$\Lambda\alpha = 0, \ \ G\alpha = 0,. \tag{63}$$

Remark 1. If systems (62) and (63) have only zero solutions, then under the conditions of Theorem 7 the solution of problem (1), (2) is unique and it is represented by the formula

$$\xi(t) = L^{+}(t)\eta(t).$$

In this case in equality (7) it is necessary to set $\alpha = 0$.

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