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# THE NUMERICAL SOLUTION TO THE OPTIMAL CONTROL PROBLEM FOR THE NONSTATIONARY DZEKTSER MODEL

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Of concern is a numerical solution to the optimal control problem for the operator-differential equation, unsolved with respect to the derivative by time, with Showalter–Sidorov condition. Such equations are called Sobolev type equations. Sobolev type equations now constitute a vast area of nonclassical equations of mathematical physics. So in this article we construct a numerical solution to the optimal control problem for the nonstationary Dzektser model with Showalter – Sidorov condition. Besides the introduction and bibliography article comprises three parts. The first part provides essential information regarding the theory of relatively $p$-sectorial operators. Also in this part the existence of solutions for optimal control problem with Showalter–Sidorov condition. The optimal control problem over solutions of Dzektser model is described in the second part. The third one contains the results of the numerical solution of optimal control problem for Dzektser model considered on a rectangle.

*Keywords: non-stationary Sobolev type equation, the optimal control problem, Showalter – Sidorov condition, Dzektser model.*

## Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary $\partial\Omega$ of $C^\infty$ class. In the cylinder $\Omega \times \mathbb{R}$ consider the Dirichlet problem for Sobolev type equation [1]

$$(\lambda - \Delta)x_t = a(t)(\alpha\Delta x - \beta\Delta^2 x) + u, \tag{1}$$

that models the evolution of the surface of the filtered fluid [2]. Here $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $\beta > 0$ are equation's parameters and a scalar function $a : [0, T] \to \mathbb{R}_+$ characterizes the environment. Vector-function $u : \mathbb{R} \to L_2(\Omega)$ is a control function, and it characterizes the external influence on the system. Equation (1) belongs to a class of Sobolev type equations that are the base of a large amount of non-classical models of mathematical physics [4]. A detailed historical review of Sobolev type equations and an extensive bibliography can be found in [3]. Let us note that in contrast to earlier (1) studies of equation (1), we consider it with a coefficient that depends on time.

Let $\mathfrak{Z}$ be a Hilbert space, operator $C \in \mathcal{L}(\mathfrak{X}; \mathfrak{Z})$. Introduce the cost functional

$$J(x,u) = \sum_{q=0}^{1} \int_0^T \|z^{(q)} - z_d^{(q)}\|_{\mathfrak{Z}}^2 dt + \sum_{q=0}^{k} \int_0^T \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt, \qquad z = Cx, \tag{2}$$

where $0 \le k \le p+1$, $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0, 1, \ldots, p+1$, are self-adjoint and positive defined operators, $z_d = z_d(t, s)$ is a required state from a Hilbert space $\mathfrak{Z}$.

We are interested in optimal control over solutions of Dirichlet problem for (1) with the *Showalter – Sidorov condition* [5]

$$P(x(0) - x_0) = 0, \tag{3}$$

which is in this situation more applicable than the traditional Cauchy condition (for more see [5, 6]). Namely we search an optimal control $v \in H^1_\partial(\mathfrak{U})$ which satisfies the condition

$$J(v) = \inf_{u \in H^1_\partial(\mathfrak{U})} J(x(t,u)). \tag{4}$$

The first results on the optimal control problem for linear Sobolev type equation can be found in [1]. The optimal control problem for non-linear Sobolev type equation is considered in [7]. Recently there were considered the optimal control problems for various stationary Sobolev-type models [8], [9]. The numerical solution of the optimal control problem for non-stationary Sobolev type equations was constructed in [10] in the case of relatively $p$-bounded operator [1].

Besides the introduction and the bibliography article comprises three parts. The first part provides essential information regarding the theory of relatively $p$-sectorial operators [1] and the existence of solutions for optimal control problem with Showalter–Sidorov condition [11]. The optimal control problem over solutions of Dzektser model is described in the second part. The third one contains the results of the numerical solution of optimal control problem for Dzektser model considered on a rectangle. References do not purport to compleaness and reflect only the authors' tastes and preferences.

## 1. Abstract results

Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces, operator $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ with nontrivial kernel ($\ker L \neq \{0\}$), operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$.

The sets $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y};\mathfrak{X})\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ are called *L-resolvent set* and *L-spectrum* of the operator $M$ respectively . If $\ker L \cap \ker M \neq \{0\}$, then $\rho^L(M) = \emptyset$.

Operator-functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1} L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called *L-resolvent*, *right L-resolvent*, and *left L-resolvent* of the operator $M$ with respect to the operator $L$ respectively.

**Theorem 1.** *Let operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, then the L-resolvent, right and left L-resolvents of the operator $M$ are analytic in $\rho^L(M)$.*

**Definition 1.** Let $\lambda_q \in \rho^L(M), q = 0, 1, \ldots, p$. Operator-functions

$$R^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} R^L_{\lambda_k}(M) \quad \left( L^L_{(\lambda,p)}(M) = \prod_{k=0}^{p} L^L_{\lambda_k}(M) \right)$$

are called *right p-resolvent (left p-resolvent)* of the operator $M$ with respect to operator $L$ respectively (briefly, *right* $(L,p)$*-resolvent* and *left* $(L,p)$*-resolvent* of the operator $M$).

**Definition 2.** Operator $M$ is called *strongly* $(L,p)$*-sectorial*, if there exist constants $K > 0$, $b \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, such that the sector $S^L_{b,\theta}(M) = \{\mu \in \mathbb{C} : |\arg(\mu - b)| < \theta, \mu \neq b\} \subset \rho^L(M)$, for every $\mu_k \in S^L_{b,\theta}(M)$, $k = \overline{0,p}$,

$$\max\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{X})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{Y})}\} \le \frac{K}{\prod\limits_{k=0}^{p} |\mu_k - b|},$$

and there exists a dense lineal $\overset{\circ}{\mathfrak{Y}}$ in space $\mathfrak{Y}$, such that for every $y \in \overset{\circ}{\mathfrak{Y}}$ and $\lambda, \mu_1, \ldots, \mu_p \in S^L_{b,\theta}(M)$

$$\|M(\lambda L - M)^{-1} L^L_{(\mu,p)}(M) y\|_{\mathfrak{Y}} \leq \frac{\mathrm{const}(y)}{|\lambda - b| \prod\limits_{k=0}^{p} |\mu_k - b|},$$

$$\|R^L_{(\mu,p)}(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathfrak{Y},\mathfrak{X})} \leq \frac{K}{|\lambda - b| \prod\limits_{k=0}^{p} |\mu_k - b|}.$$

**Theorem 2.** *Let operator $M$ be strongly $(L,p)$-sectorial, then*

(i) *there exists a strongly right-continuous at zero semigroup $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \{0\} \cup \Sigma\}$, analytic in the sector $\Sigma = \{t \in \mathbb{C} : |\arg t| < \theta - \frac{\pi}{2}\}$ and solving equation $M\dot{x}(t) = Lx(t)$ $\Big(\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$ for $L(\nu L - M)^{-1}\dot{y}(t) = M(\nu L - M)^{-1} y(t)$ where $\nu \in \rho^L(M)\Big)$, of the form*

$$X^t = \frac{1}{2\pi i} \int\limits_{\Gamma} R^L_\mu(M) e^{\mu t} d\mu \qquad \left( Y^t = \frac{1}{2\pi i} \int\limits_{\Gamma} L^L_\mu(M) e^{\mu t} d\mu \right),$$

*were contour* $\Gamma = \{\mu \in \mathbb{C} : \mu = s - |z| + i\mathrm{tg}\theta z, \quad z \in \mathbb{R}\}, \quad s > b$;

(ii) $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$, *where* $\mathfrak{X}^0 = \ker X^\bullet$, $\mathfrak{X}^1 = \mathrm{im} X^\bullet$, $\mathfrak{Y}^0 = \ker Y^\bullet$, $\mathfrak{Y}^1 = \mathrm{im} Y^\bullet$, *projectors of this splitting are identities of these semigroups* $P = s\text{-}\lim\limits_{t\to 0+} X^t$, $Q = s\text{-}\lim\limits_{t\to 0+} Y^t$;

(iii) $L_k \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $M_k \in \mathcal{C}l(\mathfrak{X}^k; \mathfrak{Y}^k)$, (*here $L_k$ ($M_k$) is a restriction of $L$ ($M$) on* $\mathfrak{X}^k(\mathrm{dom} M \cap \mathfrak{X}^k)$), $k = 0, 1$;

(iv) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0; \mathfrak{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1; \mathfrak{X}^1)$, *and the operator* $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{X}^0)$ *is nilpotent and the degree of its nilpotency does not exceed* $p \in \mathbb{N}$.

Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{U}$ be Hilbert spaces. Denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Consider the space $H^{p+1}(\mathfrak{Y}) = \{\xi \in L_2(0,T;\mathfrak{Y}) : \xi^{(p+1)} \in L_2(0,T;\mathfrak{Y}),\ p \in \mathbb{N}_0\}$ that is a Hilbert space (because $\mathfrak{Y}$ is a Hilbert space) with inner product $[\xi, \eta] = \sum\limits_{q=0}^{p+1} \int\limits_0^T \left\langle \xi^{(q)}, \eta^{(q)} \right\rangle_{\mathfrak{Y}} dt.$

Consider the solution of Showalter–Sidorov problem (3) for the non-stationary Sobolev equation

$$L\dot{x}(t) = a(t) M x(t) + u(t), \tag{5}$$

where $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, scalar function $a \in C^{p+1}([0,T];\mathbb{R}_+)$ and function $u : \mathbb{R} \to \mathfrak{Y}$ is a control function.

**Definition 3.** Vector-function $x \in H^1(\mathfrak{X}) = \{x \in L_2(0,T;\mathfrak{X}) : \dot{x} \in L_2(0,T;\mathfrak{X})\}$ is called *a strong solution of equation* (5), if it transforms (5) to an identity almost everywhere on $(0,T)$. Strong solution $x = x(t)$ of (5) is called *a strong solution of Showalter – Sidorov problem* (3), (5), if it satisfies (3).

**Theorem 3.** *Let the operator $M$ be strongly $(L,p)$-sectorial, $p \in \mathbb{N}_0$, then for every $x_0 \in \mathfrak{X}$ and $u \in H^{p+1}(\mathfrak{Y})$, $a \in C^{p+1}([0,T];\mathbb{R}_+)$, separated from zero, there exists a unique solution*

$x \in H^1(\mathfrak{X})$ *of* (3), (5) *given by*

$$x = X^{\int_0^t a(\zeta)d\zeta} P x_0 + \int_0^t X^{\int_s^t a(\zeta)d\zeta} L_1^{-1} Q f(s) ds - \sum_{k=0}^{p} H^k M_0^{-1}(I-Q)(AD)^k Au(t), \qquad (6)$$

*where* $(Ah)(t) = a^{-1}(t)h(t)$ *and* $(Dh)(t) = \frac{dh}{dt}(t)$.

The proof of this Theorem is given in [11]. Consider the optimal control problem. Distinguish in space $H^{p+1}(\mathfrak{U})$ a closed and convex subset $H_\partial^{p+1}(\mathfrak{U})$ and call it the admissible controls.

**Definition 4.** The vector function $v \in H_\partial^{p+1}(\mathfrak{U})$ is *an optimal control* over the solutions of problem (3), (5) with functional (2) if

$$J(v) = \inf_{u \in H_\partial^{p+1}(\mathfrak{U})} J(x(t,u)), \qquad (7)$$

where $x \in H^1(\mathfrak{X})$ is the strong solution of (3), (5).

**Theorem 4.** *Let the operator $M$ be strongly $(L,p)$-sectorial, $p \in \mathbb{N}_0$, the function $a \in C^{p+1}([0,T];\mathbb{R}_+)$ be separated from zero. Then for every $x_0 \in \mathfrak{X}$ and every required state $z_d \in H_\partial^1(\mathfrak{Z})$ there exists a unique optimal control $v \in H_\partial^{p+1}(\mathfrak{U})$ for the problem* (3), (5), (7) *with functional* (2).

Following [12] let us describe the approximate solution of the optimal control problem. Replace the control space for finite-dimensional space $\mathfrak{U}^l = H_l^{p+1}(\mathbb{R}^n)$ of vector-polynomials of the form $u^l = u^l(t)$, where

$$u^l = \operatorname{col}\left(\sum_{j=0}^{l} c_{1j}t^j, \sum_{j=0}^{l} c_{2j}t^j, \ldots, \sum_{j=0}^{l} c_{nj}t^j, \ldots\right). \qquad (8)$$

Taking into account (6), it is necessary that $l > p$. Substituting $u^l$ instead of $u$ in (5), (2) and considering the optimal control problem

$$J(v^l) = \min_{H_\partial^{p+1}(\mathfrak{U}^l)} J(u^l),$$

we obtain the solution $(v^l, x^l)$, where $x^l = x(v^l, t)$.

## 2. Dzektser model

Groundwater is free (gravitational) water from the first surface of the Earth stable aquifer enclosed in unconsolidated sediments or fractured upper part of the bedrock overlying the ground from the surface area exposed for a waterproof layer. Their catchment area coincides with distribution area of water-permeable rock. The upper boundary of the phreatic zone is called the phreatic line. A saturated rock with water is called the water-bearing horizon with capacity determined by the vertical distance from the phreatic line to the confining layer. Groundwater supply appears due to infiltration of precipitation, sometimes due to infiltration of water rivers and other surface water bodies, and from deeper water-bearing horizon [2].

The equation

$$(\lambda - \Delta)x_t = \alpha\Delta x - \beta\Delta^2 x \tag{9}$$

is of a great practical interest in the theory of groundwater flow. It is a generalization of equation of the groundwater flow with free surface and models the evolution of free filtered-fluid surface.

The parameter $\alpha$ in (9) is determined by the following formula

$$\alpha = \frac{(\varepsilon_\alpha + k)^2}{kh_0\mu},$$

where $\mu$ is a void fraction, $\varepsilon_\alpha$ is a flow's module power via free surface, $k$ is a coefficient of permeability, $h_0$ is a pressure on the free surface [2]. The parameters $\lambda$ and $\beta$ are determined by the following formula

$$\lambda = \frac{2(\varepsilon_\alpha + k)}{k^2h_0^2}, \quad \beta = \frac{h_0}{3\mu}.$$

Represent equation (9) as follows

$$\left(\frac{2(\varepsilon_\alpha + k)}{k^2h_0^2} - \Delta\right)x_t = \frac{1}{\mu}\left(\frac{(\varepsilon_\alpha + k)^2}{kh_0}\Delta x - \frac{h_0}{3}\Delta^2 x\right).$$

The void fraction $\mu$ characterizes the ratio of pores volume to the volume of its mineral part. Considering that in many instances this ratio changes in time, then this parameter is a scalar function depending on time. Rename the coefficients of equation (9) as follows $\beta = \frac{h_0}{3}$, $\alpha = \frac{(\varepsilon_\alpha+k)^2}{kh_0}$, $a(t) = \frac{1}{\mu(t)}$, then

$$(\lambda - \Delta)\, x_t = a(t)\left(\alpha\Delta x - \beta\Delta^2 x\right).$$

Taking into account the formula determining these parameters we obtain $\alpha, \lambda \in \mathbb{R}$, $\beta \in \mathbb{R}_+$.

Consider the problem of optimal control over solutions for the Dzektser non-stationary equation with Showalter – Sidorov condition.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of $C^\infty$ class. Consider the Dirichlet problem

$$x(s,t) = \Delta x(t,s) = 0, \qquad (s,t) \in \Omega \times \mathbb{R} \tag{10}$$

in cylinder $\Omega \times [0,T]$ for partial differential equation

$$(\lambda - \Delta)x_t = a(t)(\alpha\Delta x - \beta\Delta^2 x) + u \tag{11}$$

with Showalter – Sidorov condition (3). Coefficients $\lambda$, $\alpha$, $\beta$ and a scalar function $a : [0,T] \to \mathbb{R}_+$ were described above, and the control vector function $u(t)$ corresponds to the external influence on the system.

Reduce the problem (10), (11) to equation (5). For this purpose take Sobolev spaces

$$\mathfrak{X} = \{x \in W_2^{r+2}(\Omega) : x(s) = 0,\, s \in \partial\Omega\} \quad \text{and} \quad \mathfrak{U} = \mathfrak{Y} = W_2^r(\Omega),$$

where $r \in \{0\} \cup \mathbb{N}$, $W_2^r(\Omega)$ is a Sobolev space.

Define operators $L$, $M$ by the formulas

$$L = \lambda - \Delta, \quad M = \alpha\Delta - \beta\Delta^2,$$

wereas the domain of operator $M$: $\operatorname{dom} M = \{x \in W_2^4(\Omega) : x(s) = \Delta x(s) = 0, s \in \partial\Omega\}$.

**Lemma 1.** *For every $\lambda \in \mathbb{R} \setminus \{\frac{\alpha}{\beta}\}$ and $\beta > 0$ operators $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$.*

*Proof.*

This Lemma follows from the properties of the Laplace operator (namely linearity and continuity) and formulas determining $L$ and $M$.

□

By $\sigma(\Delta)$ denote the spectrum of homogeneous Dirichlet problem in the domain $\Omega$ for the Laplace operator $\Delta$. The spectrum $\sigma(\Delta)$ is negative, discrete, finite-multiple and it is concentrated only at $-\infty$, namely it is limited from right. The set of eigenvalues numbered in the non-increasing order with allowance for their multiplicity is denoted by $\{\lambda_k\}$. The family of the corresponding eigenfunctions is orthonormalized (in the sense of the space $L_2(\Omega)$, $\psi_k \in C^\infty$, $k \in \mathbb{N}$ with the scalar product $\langle \cdot, \cdot \rangle$) and is denoted by $\{\psi_k\}$.

**Lemma 2.** *For every $\lambda \in \mathbb{R} \setminus \{\alpha/\beta\}$ and $\beta > 0$ operator $M$ is strongly $(L, 0)$-sectorial.*

*Proof.*

If there exist $\lambda_k$ such that $\lambda - \lambda_k = 0$ and $\alpha\lambda_k - \beta\lambda_k^2 = 0$ then $\ker L \cap \ker M \neq \{0\}$. Therefore the condition $\lambda \neq \frac{\alpha}{\beta}$ is necessary. Construct the $L$-spectrum of operator $M$.

$$\mu L - M = \sum_{k=1}^{\infty} \left(\mu(\lambda - \lambda_k) - (\alpha\lambda_k - \beta\lambda_k^2)\right) \langle \cdot, \psi_k \rangle \psi_k.$$

Whence it follows that the $L$-spectrum of operator $M$ takes the form

$$\sigma^L(M) = \left\{ \mu_k = \frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k},\ k \in \mathbb{N} \setminus \{l : \lambda_l = \lambda\} \right\}.$$

Since the spectrum of the Laplace operator $\{\lambda_k\}_{k=1}^{\infty}$ is negative, discrete, finite-multiple and it is concentrated only at $-\infty$ then $\sigma^L(M)$ has the same properties, namely it is limited from right.

□

Under the conditions of Lemma 2 construct the $L$-resolvent and the right $L$-resolvent of operator $M$:

$$(\mu L - M)^{-1} = \sum_{k \in \mathbb{N} : \lambda \neq \lambda_k} \frac{\langle \cdot, \psi_k \rangle \psi_k}{\mu(\lambda - \lambda_k) - (\alpha\lambda_k - \beta\lambda_k^2)},$$

$$R_\mu^L(M) = (\mu L - M)^{-1} L = \sum_{k \in \mathbb{N} : \lambda \neq \lambda_k} \frac{\langle \cdot, \psi_k \rangle \psi_k}{\mu + \frac{\beta\lambda_k^2 - \alpha\lambda_k}{\lambda - \lambda_k}}.$$

Projector $P$ is defined as follows

$$P = \sum_{k \in \mathbb{N} \setminus \{l : \lambda_l = \lambda\}} \langle \cdot, \psi_k \rangle \psi_k$$

and consequently the Showalter – Sidorov condition (3) takes the form

$$\sum_{k\in\mathbb{N}\setminus\{l:\lambda_l=\lambda\}} \langle (x(0)-x_0),\psi_k\rangle\,\psi_k = 0. \tag{12}$$

Under the conditions of Lemma 2 for every $x_0 \in \mathfrak{X}$, $u \in H^1(\mathfrak{U})$ and $a \in C^1([0,T];\mathbb{R}_+)$, separated from zero, there exists a unique solution $x \in H^1(\mathfrak{X})$ of (10), (11), (12) of the form

$$x(t) = -\sum_{l\in\mathbb{N}:\lambda_l=\lambda} \frac{\langle u,\psi_l\rangle\psi_l}{a(t)(\alpha\lambda-\beta\lambda^2)} + \sum_{k\in\mathbb{N}\setminus\{l:\lambda_l=\lambda\}} e^{\left(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int\limits_0^t a(\zeta)d\zeta\right)}\langle x_0,\psi_k\rangle\psi_k +$$
$$+\sum_{k\in\mathbb{N}\setminus\{l:\lambda_l=\lambda\}} \int\limits_0^t e^{\left(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int\limits_s^t a(\zeta)d\zeta\right)} \frac{\langle u(s),\psi_k\rangle\psi_k}{\lambda-\lambda_k}\,ds.$$

This follows from Theorems 2 and 3.

**Theorem 5.** *Let $\lambda \in \mathbb{R}\setminus\{\alpha/\beta\}$ and $\beta > 0$. Then for every $x_0 \in \mathfrak{X}$, $z_d \in H^1_\partial(\mathfrak{Z})$ and $a \in C^1([0,T];\mathbb{R}_+)$, is separated from zero, there exists a unique solution $v \in H^1_\partial(\mathfrak{U})$ of optimal control problem* (4), (10), (11), (12) *with functional* (2).

The physical meaning of the optimal control problem is to regulate effectively the groundwater flows in the system of layers.

## 3. Calculative experiment

Introduce a numerical method for solving of the optimal control problem for non-stationary Dzektser model in the rectangle based on the obtained results.

Consider the basic steps of an algorithm for finding of the optimal control problem solutions.

**Step 1.** Input parameters $\lambda$, $a(t)$, $N$, polynomial degree $l$ and the required state $z_d$.

**Step 2.** Generation of components of optimal control in the form (8).

**Step 3.** Computation of the solution of Showalter – Sidorov problem (11) for equation (3) with condition (10) in the form

$$x(t) = -\sum_{l\in\mathbb{N}:\lambda_l=\lambda} \frac{\langle u,\psi_l\rangle\psi_l}{a(t)(\alpha\lambda-\beta\lambda^2)} + \sum_{k\in\mathbb{N}\setminus\{l:\lambda_l=\lambda\}} e^{\left(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int\limits_0^t a(\zeta)d\zeta\right)}\langle x_0,\psi_k\rangle\psi_k +$$
$$+\sum_{k\in\mathbb{N}\setminus\{l:\lambda_l=\lambda\}} \int\limits_0^t e^{\left(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int\limits_s^t a(\zeta)d\zeta\right)} \frac{\langle u(s),\psi_k\rangle\psi_k}{\lambda-\lambda_k}\,ds.$$

**Step 4.** Construction of the functional (2) and a closed convex subset of admissible controls $H^1_\partial(\mathfrak{U}^l)$ by the condition $\|u^l(t)\|^2_{\mathfrak{U}} < 1$.

**Step 5.** Calculation of minimum point of the functional $J(u^l)$ on the subset of admissible controls with built-in procedure for finding of extrema of functions of several variables in Maple 14.

Consider an example illustrating results obtained above. It is required to find the solution of (10)–(12).

Let $l = 2$, $N = 2$. Domain $\Omega = \{(s_1, s_2) \in \mathbb{R}^2 : 0 \leq s_1 \leq l_1, 0 \leq s_2 \leq l_2, \} \subset \mathbb{R}^2$. Parameters $x_0$ and a required state $z_d$ are given in forms

$$x_0(s_1, s_2) = \sin(\pi s_1)\sin(\pi s_2) + \sin(2\pi s_1)\sin(\pi s_2),$$

$$z_d(s_1, s_2, t) = (t+1)(\sin(\pi s_1)\sin(\pi s_2) + \sin(2\pi s_1)\sin(\pi s_2)).$$

The function $a(t) = \frac{1}{t+1}$ and parameters $\alpha = 1$, $\beta = \frac{1}{\pi^2}$, $\lambda = 3\pi^2$, $C = \mathbb{I}$, $N_q = \mathbb{I}$. Substituting these parameters in (9) and solving the optimal control problem (7) with the functional (2) we found the function $v^l \in H^1_\partial(\mathfrak{U}^l)$. The resulting solution of the optimal control problem in the final time is shown in figure 1.

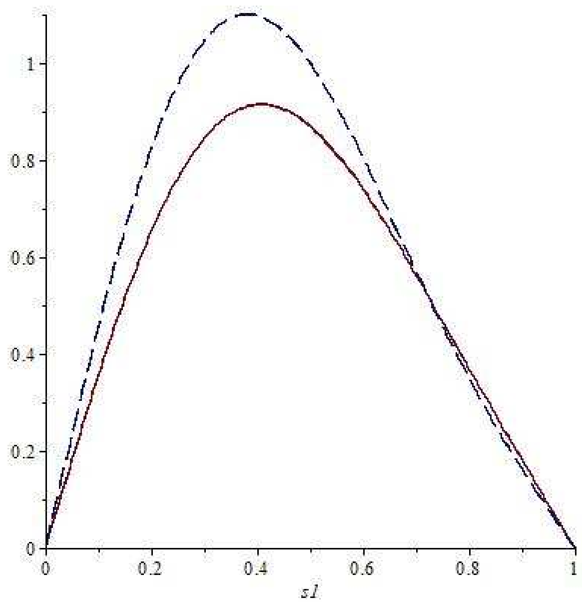


**Fig. 1**. The required state (dash line) and the solution of the optimal control problem (solid line) at the final time ($s_2 = \frac{1}{2}$)

Figure 1 shows the graphs of solution of the optimal control problem (solid line) and the required state (dashed line). It is seen that the results obtained by numerical experiment and the required state are closed in the integral sense.

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