SHORT NOTES

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SPECTRAL PROBLEM FOR A MATHEMATICAL MODEL OF HYDRODYNAMICS

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Spectral problems of the form $(T + P)u = \lambda u$ have a huge range of applications: hydrodynamic stability problems, elastic vibrations of a membrane, a set of possible states of systems in quantum mechanics, and so forth. The self-adjoint operators perturbed by bounded operators are most thoroughly studied. In applications, the perturbed operator is usually represented by the Sturm − Liouville or Schrodinger operator. At present moment, the researchers are very interested in the equations not solved with respect to the highest derivative $L\dot{u} = Tu + f$, which are known as Sobolev type equations. The study of Sobolev type equations leads to spectral problems of the form $Tu = \lambda Lu$. In many cases, the operator T can be perturbed by an operator P, and then the spectral problem takes the form $(T + P)u = \lambda Lu$. The study of such problems allows to construct a solution of the equation, as well as to investigate various parameters of mathematical models. Previously, such spectral problems with the perturbed operator were not studied. In this paper, we propose the method for investigating and solving the direct spectral problem for a hydrodynamic model.

Keywords: potential; discrete self-adjoint operator; spectral problem; relative spectrum.

Introduction

The method of investigating the problem is based on the theory of regularized traces. The theory was obtained by applying the following fundamental result of linear algebra to a class of infinite-dimensional operators.

Theorem 1. (Lidsky's theorem) *If the operator A is nuclear, then for any pair $(\{\varphi_n\}_{n=1}^{\infty}, \{\psi_n\}_{n=1}^{\infty})$ of orthonormal bases the following an equality holds:*

$$\sum_{n=1}^{\infty}(A\varphi_n, \varphi_n) = \sum_{n=1}^{\infty}(A\psi_n, \psi_n) = \sum_{n=1}^{\infty}\lambda_n.$$

Consider the operators for which Theorem 1 is not true. For such operators we search for pairs of bases such that

$$\sum_{n=1}^{\infty}[(A\varphi_n, \varphi_n) - (A\psi_n, \psi_n)] = 0.$$

To this end, choose a basis formed by eigenvectors of the operator A, and represent the operator A as a sum of two operators, i.e. $A = A_0 + B$, where the operator B is in some sense subordinated to the operator A. The problem takes the following form:

$$\sum_{n=1}^{\infty}[(A\varphi_n, \varphi_n) - (A\psi_n, \psi_n)] =$$

$$= \sum_{n=1}^{\infty}[((A_0+B)\varphi_n,\varphi_n) - ((A_0+B)\psi_n,\psi_n)] =$$

$$= \sum_{n=1}^{\infty}[(A\varphi_n,\varphi_n) - ((A_0+B)\psi_n,\psi_n) + (B\varphi_n,\varphi_n)] =$$

$$\sum_{n=1}^{\infty}[\lambda_n - \mu_n + (B\varphi_n,\varphi_n)] = 0.$$

The formula of the trace

$$\lim_{m\to\infty}\sum_{i=0}^{n_i}(\mu_i - \lambda_i - (B\varphi_i,\varphi_i)) = 0$$

is proved for operators having a nuclear resolvent and perturbed by a bounded operator B [2]. The cases of a relatively compact perturbation and a non-nuclear operator are studied in [3]: if there exists a number $\delta \in [0,1)$ such that the operator BA^{-1} extends to a bounded operator, and a number $\omega \in [0,1), \omega+\delta < 1$ such that $A^{-(1-\delta-\omega)}$ is a nuclear operator, then there exists a subsequence of numbers $\{n_m\}_{m=1}^{\infty}$ such that for $\omega \geq \frac{\delta}{l}$

$$\lim_{m\to\infty}\left(\sum_{j=0}^{n_m}(\mu_j-\lambda_j) + \frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{k=1}^{l}\frac{(-1)^{k-1}}{k}Tr((BR_0(\lambda))^k)d\lambda\right) = 0.$$

Similar results were obtained in the case of an existence of «large» gaps in the spectrum of the operator A.

The method of regularized traces is effective in the case of direct and inverse spectral problems of the form

$$(M+P)u = \lambda u,$$

where the operator M is linear and selfadjoint, and the perturbing operator P is linear and bounded. Later, the condition that the perturbing operator P is bounded was replaced by the condition that the perturbing operator P is subordinated to the operator M.

Direct and inverse spectral problems are used to solve the problems of hydraulic stability, to consider vibrations of a membrane and elastic bodies in the theory of shells, in quantum mechanics, to study electrical oscillations in an extended line, as well as in geophysical models of the terrestrial sphere and cosmology. The resolvent method allows to solve such problems effectively, if the perturbed operator has a nuclear resolvent and is perturbed by a bounded operator. In the case of an operator with a non-nuclear resolvent, this method is extremely difficult applied, because it is difficult to find a special function that transforms eigenvalues of the operator. The problems with an unbounded perturbing operator can be solved, but the operator must in some sense be subordinated to the perturbing operator. For the first time, direct and inverse spectral problems of such type for an operator with a single spectrum were considered in the papers [4], [5]. The spectral problems for the Dzieczer equation were considered and solved using the Galerkin method in [6].

In this paper, we consider a direct spectral problem for the Dzieczer equation, which describes evolution of the free surface of a filtered fluid. We describe theoretical basis of the research method and obtain a formula for finding relative eigenvalues of the considered problem. We assume that the perturbed operator has a multiple spectrum, and resolvent of the operator is not a nuclear operator. In order to solve the considered problem, we apply the resolvent method developed by V.A. Sadovnichy and V.V. Dubrovsky [8] to the relative or L-resolvent of the operator [10].

1. Statement of the Problem

Consider the Dzieczer equation in N-dimensional convex, simply connected domain $\Omega \subset \mathbb{R}_N$

$$(\lambda - \Delta)u_t = \alpha \Delta u - \beta \Delta^2 u + f.$$

The equation simulates an evolution of the free surface of a filtered fluid. The parameter α is defined by the formula

$$\alpha = \frac{\varepsilon_\alpha + k}{k h_0 a},$$

where a is a coefficient of porosity, ε_α is a module for feeding the flow through the free surface, k is a filtration coefficient, h_0 is a head on the free surface. The parameters λ and β are defined by the formulas

$$\lambda = \frac{2(\varepsilon_\alpha + k)}{k^2 H_0^2}, \quad \beta = \frac{h_0}{3a}.$$

We define the operators $T, L : \mathfrak{U} \to \mathfrak{F}$ by the formulas

$$T = \alpha \Delta - \beta \Delta^2, \quad \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}, \quad L = a^2 - \Delta, \tag{1}$$

and

$$\mathfrak{U} = \{u \in W_2^{k+2}(\Omega) : u|_{\partial\Omega} = 0\},$$

$$\mathfrak{F} = \{u \in W_2^{k}(\Omega)\}, k \in \{0\} \bigcup \mathbb{N}.$$

Let P be a bounded linear operator. Consider the operator $T + P$ and $\{\nu_n\}_{n=1}^{\infty} = \sigma^L(T + P)$, where ν_n are numbered in order of nonincreasing of real parts, taking into account algebraic multiplicity. The problem is to find $\sigma^L(T + P)$.

2. Main Result

Let us formulate the theoretical basis of the method for solving the considered problem. Let operators T and L act in the separable Hilbert space $\mathfrak{H}$. Let operator T be discrete, selfadjoint, positive, semi-bounded below, and operator L be a linear, closed, continuously invertible operator. Let P be a bounded operator acting in the same space $\mathfrak{H}$. Denote L-resolvent of the operator M by $R_0(\mu) = (\mu L - M)^{-1}$, and L-resolvent of the perturbed operator $M + P$ by $R(\mu) = (\mu L - M - P)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be eigenvalues of the operator M numbered in order of nonincreasing of real parts, taking into account algebraic multiplicity, $\{\mu_n\}_{n=1}^{\infty} = \partial^L(M)$ be L-spectrum of the operator M, $\{\nu_n\}_{n=1}^{\infty} = \partial^L(M + P)$ be L -spectrum of the operator $M + P$. The eigenvalues of the operator $M+P$ are also numbered in order of nonincreasing of real parts, taking into account algebraic multiplicity.

Lemma 1. [7] *If $\|P\| < r/2$, where $0 < r \le r_0$, then the operator $M+P$ is discrete, and*

(i) if $R_0(\lambda) \in \mathfrak{S}_q$, then $R(\lambda) \in \mathfrak{S}_q$, $1 \le q < \infty$,

(ii) if $\lambda_n \in \mathbb{C} \setminus \Omega_{r_n}$, then $\mu_n^s \in \mathbb{C} \setminus \Omega_{r_n}$, $s = \overline{1, \eta_n}$, η_n is a multiplicity of the eigenvalue λ_n.

Lemma 2. [7] *If $\|PR_0(\mu)\| = q < 1$ for $\mu \in \gamma_n$, then the following equality holds:*

$$LR(\mu) = LR_0(\mu) + \sum_{k=1}^{\infty} [R_0(\mu)P]^k LR_0(\mu). \tag{2}$$

Theorem 2. [9] *Let M be a discrete, selfadjoint, positive, semibounded from below operator, L be linear, closed, continuously invertible operator such that L^{-1} is a bounded operator, $R_0(\mu)$ be a nuclear operator and $P \le \frac{r_0}{2}$, then there is a spectral identity:*

$$\sum_{q=1}^{\eta_n} \nu_n^q = \eta_n \mu_n + \sum_{q=1}^{\eta_n} (L^{-1}P\varphi_n, \varphi_n) + \alpha_n, \tag{3}$$

where

$$\alpha_n = \sum_{k=2}^{\infty} \frac{(-1)^k}{2\pi i} \int_{\gamma_n} \mu Sp[R_0(\mu)P]^k LR_0(\mu) d\mu.$$

The operators T, M, P defined in the statement of the problem satisfy all conditions of Theorem 2. Therefore, the relative eigenvalues of the spectral problem for the Dzieczer equation can be obtained by the formula (3).

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СУЩЕСТВОВАНИЕ РЕШЕНИЯ ОБРАТНОЙ СПЕКТРАЛЬНОЙ ЗАДАЧИ ДЛЯ ДИСКРЕТНОГО САМОСОПРЯЖЕННОГО ПОЛУОГРАНИЧЕННОГО СНИЗУ ОПЕРАТОРА

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Спектральные задачи вида $(T+P)u=\lambda u$ имеют огромную область приложений: задачи гидродинамической устойчивости, упругие колебания мембраны, определение множества возможных состояний систем в квантовой механике. Наиболее подробно изучены для самосопряженные операторы возмущенные ограниченным. В приложениях возмущенный оператор обычно представлен оператором Штурма – Лиувилля или Шредингера. В настоящее время огромный интерес исследователей вызывают уравнения не разрешенные относительно старшей производной $L\dot{u}=Tu+f$, которые известны как уравнения соболевского типа. Их изучение приводит к спектральным задачам вида $Tu=\lambda Lu$. Во многих случаях оператор T может быть возмущен некоторым оператором P и тогда спектральная задача примет вид $(T+P)u=\lambda Lu$. Изучение таких задач позволяет строить решение уравнения, исследовать различные параметры математических моделей. Ранее такие спектральные задачи с возмущенным оператором не изучались. В данной работе предложен метод исследования и решения прямой спектральной задачи для одной гидродинамической модели.

Ключевые слова: возмущенный оператор; дискретный самосопряженный оператор; потенциал.

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