MSC 94C99

SOLUTION OF THE PROBLEM OF HYDRAULIC CALCULATION IN NETWORKS USING THE METHOD OF STEPWISE DIMENSIONALITY REDUCTION

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The paper describes a developed by the authors method of stepwise dimensionality reduction of the Kirchhoff equations for the flow distribution problem (hydraulic calculation) in networks. Our method allows to replace the original system of equations by an equivalent system with lower dimension from the solution of which you can recover the solution of the original system. Thus, the developed method allows to improve the convergence and searching speed of solution by existing iterative methods for solving of this problem. It is given the justification of the method and algorithm of its implementation. The examples of the implementation for developed algorithm are provided for various schemes of hydraulic networks.

Keywords: hydraulic network, flow distribution, Kirchhoff equations, method of contour expenses, method of stepwise dimensionality reduction.

Introduction

The flow distribution problem in hydraulic networks consists of determining of the liquid expenses and the pressures drops at its various sections with known network configuration, given resistance area, pressure and fluid expenses at the nodes.

The solution to this problem is reduced to solving of a system of Kirchhoff equations [1]:

$$\begin{cases} Ax = q, \\ By = 0, \\ y + h = SXx, \end{cases} \tag{1}$$

where x – unknown vector of expenses on network branches; y – unknown vector of pressures drops on the branches; q – vector of expenses in the nodes; A – matrix of compounds for nodes and branches; B – matrix of independent contours; h – vector of pressures on the branches; matrix X has the form

$$X = \begin{vmatrix} |x_1| & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & |x_n| \end{vmatrix},$$

where n – number of network branches.

The third equation of system (1) is nonlinear, and in general has no analytical solution. Therefore, iterative methods are used for its solution. Currently, methods for solving of the flow distribution problem in pipe networks have a number of drawbacks: either by convergence and the choice of the initial approximation, or by the speed of finding of a solution.

The most widespread methods for solving of such problems are the hydraulic linkage method, developed by M.M. Andriyashev, V.G. Lobachyov and H. Cross [1], and generalized method of contour expenses (GMCE), developed by V.Ja. Hasilev [2].

The advantages of the hydraulic linkage method is the simplicity of implementation on a computer and high speed convergence. However, it has significant drawbacks, namely poor convergence or even the lack of convergence for certain schemes of large dimension with several sources of acting pressure. So now the GMCE is used most frequently for solving of the flow distribution problem, in which the more faster convergence and obtaining solutions for practically any network topology are provided by taking into account the mutual influence of contours in the calculation of corrections on expenses.

GMCE has also a number of disadvantages, which includes:

1) convergence speed of algorithms of this type depends on such features as: initial approximation, the dominance degree of coefficients, relating to the contour expenses, over the coefficients for other branches, and therefore, from the choice of the system of independent circuits;

2) more longer execution time of each iteration (compared with the method of hydraulic linkages);

3) solving time significantly increases for the large dimension of graph of water supply network;

4) need for redundancy of significantly more computer memory.

We propose a method that allows significantly to reduce the dimension of the original system of equations (1), that increases the solution speed for the flow distribution problem for a complex branched water supply network by several times compared with the most iterative algorithms. This method can be used in conjunction with any other method, and in particular cases, it allows to find the analytical solution of the system.

1. Method of stepwise dimensionality reduction (MSDR)

Suppose a graph of water supple network in which n branches, m nodes and $c = n - m + 1$ contours, is defined by its incidence matrix $\overline{A} = \{\overline{a}_{i,j}\}$ with dimension $m \times n$ (in contrast to the matrix A of system of equations (1), in which there are $m - 1$ linearly independent rows, matrix $\overline{A}$ contains one row more).

Suppose also it is given a matrix of resistances S, column vector of expenses in the nodes q and pressures on the branches h:

$$\overline{A} = \begin{pmatrix} \overline{a}_{1,1} & \overline{a}_{1,2} & \dots & \overline{a}_{1,n} \\ \overline{a}_{2,1} & \overline{a}_{2,2} & \dots & \overline{a}_{2,n} \\ \dots & \dots & \dots & \dots \\ \overline{a}_{m,1} & \overline{a}_{m,2} & \dots & \overline{a}_{m,n} \end{pmatrix},\ S = \begin{pmatrix} S_{1,1} & 0 & \dots & 0 \\ 0 & S_{2,2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_{n,n} \end{pmatrix}, \quad (2)$$
$$q = \begin{pmatrix} q_1 & q_2 & \dots & q_i & \dots & q_m \end{pmatrix}^T,\ h = \begin{pmatrix} h_1 & h_2 & \dots & h_j & \dots & h_n \end{pmatrix}^T.$$

Suppose x and y are a expenses vector and a vector of pressure drops on the branches respectively, that are solutions of Kirchhoff's equations (1), constructed for a given water supply network, characterized by parameters (2).

MSDR allows for a small number of operations (no more than n) to obtain from the original system an equivalent system with smaller dimension by replacing branches and nodes on the equivalent branches. Moreover, the resulting system will have the following property. Denote an unknown vector of the expenses for the new system x'. Then vector x of original system can be obtained by the following formula:

$$x = D \cdot x', \tag{3}$$

where matrix $D = \{d_{i,j}\}$ with dimension $n \times n'$ (n' is the number of branches in the graph obtained by the algorithm MSDR) is a special matrix that is formed during the implementation of MPSR; in future we call it an equivalent transformation matrix. At the same time new system will be characterized by matrices $\overline{A'}$, B', S' and by vectors h' and q'.

Unknown vector y can be easily got through the third equation of system (1).

Unknown vector x' of the reduced system is determined by any iterative method of solving of the following system of Kirchhoff equations:

$$\begin{cases} A'x' = q', \\ B'y' = 0, \\ y' + h' = S'X'x', \end{cases} \tag{4}$$

where n' is a number of branches, m' is a number of nodes and $c' = n' - m' + 1$ is a number of contours, with $n' \leq n$, $m' \leq m$ and $c' \leq c$. Matrices A' and S' are transformed MSDR matrices of the original system A and S (matrix A' is obtained from the matrix $\overline{A'}$ by deleting of the last row). Vectors h' and q' are transformed MSDR vectors h и q. Matrix B' is a matrix of contours for a new system, it can be obtained easily, using the matrix $\overline{A'}$, defining the configuration of the graph.

Algorithm of the method of stepwise dimensionality reduction (MSDR):

Step 0.

Setting the initial conditions. Before starting of the algorithm, all matrices and vectors for reduced system are set equal to the appropriate matrices and vectors of the original system, matrix D is defined as the unit matrix $n \times n$: $D =$ E. Thus, before computing equation (3) is executed, because $x = E \cdot x$. Note that expression (3) is executed after each step of the algorithm.

Step 1.

Reduction of sequential branches. In this step the nodes connected to two and only two branches of the graph, and the branches related to them are replaced by a single generic branch. Wherein the expense in the reduced node must be zero. Thus, for each operation in the graph one branch and one node are removed, and number of contours remains unchanged. In this case the matrices $\overline{A'}$, S', D and the vector h' are changed so that the solution of system (4) for the remaining sections of network (the branches of the graph) remains unchanged. Therefore we can say that the generalized section of the network is equivalent to the replaced one. It is also obvious that the fluid expense on replaced branches and on generalized is same by the law of conservation of mass and zero expense in reduced node. If the simple chain of nodes is present in the graph, then their replacement is performed alternately for each node. There is a variant of the algorithm, where such substitution occurs for a chain of the graph as a whole, however this option is difficult to implement and it does not lead to a significant acceleration of the algorithm.

1.1. In matrix $\overline{A'}$ it is carried out the search of row, containing two and only two elements with a nonzero value, which corresponds to a node having contact with only two branches of the graph. We denote this node-row as I. Nodes with non-zero expense $q_I \neq 0$ are not considered.

If the rows containing two and only two elements with non-zero values for which the property $q_I \neq 0$ is implemented, are missing, then go to step 2.

The smaller of two branches in the graph is denoted as F, and the number of the second branch – as U.

1.2. Values of resistance and pressure are calculated for the combined branch:

$$S'_{F,F} = S'_{F,F} + S'_{U,U}, \tag{5}$$

$$h'_F = h'_F + h'_U. \tag{6}$$

The coefficients of the matrices $\overline{A'}$ and D are recalculated.

If $\overline{a'}_{I,F} = -\overline{a'}_{I,U}$, which corresponds to the case where in the node I one branch gets in, and another one gets out, then:

$$\overline{a'}_{i,F} = \overline{a'}_{i,F} + \overline{a'}_{i,U},\ \overline{a'}_{i,U} = 0,\ i = \overline{1,m}; \tag{7}$$

$$d'_{i,F} = d'_{i,F} + d'_{i,U},\ d'_{i,U} = 0,\ i = \overline{1,n}. \tag{8}$$

Otherwise, if $\overline{a'}_{I,F} = \overline{a'}_{I,U}$, that corresponds to the case where both branches go out from the node I or both are included in it, then:

$$\overline{a'}_{i,F} = \overline{a'}_{i,F} - \overline{a'}_{i,U},\ \overline{a'}_{i,U} = 0,\ i = \overline{1,m}, \tag{9}$$

$$d'_{i,F} = d'_{i,F} - d'_{i,U},\ d'_{i,U} = 0,\ i = \overline{1,n}. \tag{10}$$

If in matrix $\overline{A'}$ no more rows in which not all values of the elements are zero, then go to step 3.

Otherwise – go to step 1.1.

Step 2.

Reduction of parallel branches. On this step in the graph the branches located between the same pair of nodes are merged into the single generalized branch. The branches may be unidirectional or multidirectional. Only branches on which the pressure equals to zero can be combined. Thus, several branches are excluded from the graph for each such operation, therefore the number of contours is also reduced by the number of removed branches. At the same time matrices $\overline{A'}$, S' и D' are changed in such a way that the solution of system (4) for the remaining sections of network (the branches of the graph) remains unchanged. Therefore in this case the generalized section of network is also equivalent to a group of replaced sections. Thus the liquid expense on the generalized section in absolute value is equal to the liquid expense on the replaced sections.

2.1. In matrix $\overline{A'}$ columns with non-zero values in the same rows are searched; graphically, this corresponds to the parallel branches in the graph, i.e. branches, set by the same pair of nodes, regardless of their order. Branches with a nonzero pressure $h'_i \neq 0,\ \ i = 1, 2, \ldots, n'$ are not considered.

If these columns are missing and after the last doing of step 1 step 2.2 was doing at least one time, then go to step 1.

If these columns are missing and after the last doing of step 1 step 2.2 was never doing, then go to step 3.

Assume that the branches $I_1, I_2, \ldots, I_N$ are parallel and $h'_{I_1} = 0,\ h'_{I_2} = 0, \ldots,\ h'_{I_N} = 0$, then there exist such r and t, that for I_1 $\overline{a'}_{t,I_1} = 1,\ \overline{a'}_{r,I_1} = -1$ will be true by the properties of the incidence matrix of a directed graph, that is the expression below will be executed:

$$\begin{cases} \overline{a'}_{t,I_1} = 1,\ \overline{a'}_{t,I_2} = \pm 1, \ldots,\ \overline{a'}_{t,I_K} = \pm 1; \\ \overline{a'}_{r,I_1} = -1,\ \overline{a'}_{r,I_2} = \mp 1, \ldots,\ \overline{a'}_{r,I_K} = \mp 1; \\ h'_{I_1} = 0,\ h'_{I_2} = 0, \ldots,\ h'_{I_K} = 0. \end{cases} \tag{11}$$

2.2. In matrix $\overline{A'}$ coefficients corresponding to the removed branches $I_2, I_3, \ldots, I_N$ are set to by zero:

$$\begin{cases} \overline{a'}_{t,I_2} = 0,\ \overline{a'}_{t,I_3} = 0, \ldots,\ \overline{a'}_{t,I_K} = 0; \\ \overline{a'}_{r,I_2} = 0,\ \overline{a'}_{r,I_3} = 0, \ldots,\ \overline{a'}_{r,I_K} = 0. \end{cases} \tag{12}$$

This step corresponds to the deletion of branches $I_1, I_2, I_3, \ldots, I_N$ from the graph. Next, it is necessary to recalculate the characteristics for the merged section and the coefficients of the matrix for the equivalent transformation.

2.3. Wicking coefficients are calculated for merged branches proportionally to their resistances. To this first we calculate the conductivity for merged branches $\sigma = \frac{1}{\sqrt{s}}$:

$$\sigma'_{I_1} = \frac{1}{\sqrt{S'_{I_1,I_1}}},\ \sigma'_{I_2} = \frac{1}{\sqrt{S'_{I_2,I_2}}}, \ldots,\ \sigma'_{I_K} = \frac{1}{\sqrt{S'_{I_K,I_K}}}. \tag{13}$$

Then we calculate the wicking coefficients for each of them according to the formula:

$$K_{I_i} = \frac{\frac{1}{\sqrt{S'_{I_i,I_i}}}}{\sum\limits_{j=1}^{N} \frac{1}{\sqrt{S'_{I_j,I_j}}}} = \frac{\sigma'_{I_i}}{\sum\limits_{j=1}^{N} \sigma'_{I_j}},\ i = \overline{1, K}. \tag{14}$$

In the equivalent transformation matrix the values in the columns $I_1, I_2, \ldots, I_N$ are multiplied by the corresponding wicking coefficients for them:

$$d_{k,I_i} = d_{k,I_i} \cdot K_{I_i},\ i = \overline{1, K};\ k = \overline{1, n}. \tag{15}$$

2.4. In the equivalent transformation matrix the values in the column of removed branch move in the column of the generalized branch, column of removed branch is zero.

That is, for all $i = \overline{2, K}$: if $a'_{t,I_i} = -1$, then

$$d_{k,I_1} = d_{k,I_1} - d_{k,I_i},\ \ d_{k,I_i} = 0,\ \ k = \overline{1, n}; \tag{16}$$

otherwise, if $a'_{t,I_i} = 1$, then

$$d_{k,I_1} = d_{k,I_1} + d_{k,I_i}, \quad d_{k,I_i} = 0, \quad k = \overline{1,n}. \tag{17}$$

2.5. The resistance values for the summing branch are calculated by the formula for calculating of the resistance for the parallel sections:

$$S'_{I_1,I_1} = \left(\sum_{k=1}^{K} \sigma_{I_k} \right)^{-2}. \tag{18}$$

Go to step 2.1.

Step 3.

In the absence of the possibility of further reducing of system for sequential and parallel branches the removing of elements of vectors and matrices which are responsible for the combined sections of the network is carried out.

3.1. Removing of the nodes. Containing only zero values the rows of the matrix $\overline{A'}$, corresponding to removed nodes from the graph, as well as corresponding elements of the vector q'_∂ are removed, surviving nodes are renumbered.

3.2. Removing of the branches. In matrix $\overline{A'}$ columns containing only zero elements corresponding to removed branches are reduced. After that, the renumbering of the branches of the graph is carried out. Removing of columns corresponding to the removed branches in the equivalent transformation matrix D, as well as of columns and rows in the matrix S' and elements of the vector h' is produced.

Step 4.

For the obtained reduced system the matrix B' is constructed, then the system of equations (4) is solved by any existing method for solving of the flow distribution problem in water supply networks, for example by GMCE. If the original system was reduced to a single branch, closed on itself, then the solution of system (4) can be obtained from the formula:

$$x'_1 = \frac{h'_1}{S'^2_{1,1}}. \tag{19}$$

Step 5.

Solutions of the original system is found by the formula (3).

Step 6.

If it is necessary, the vector of pressure drops on the branches y is determined by the third equation of system (1).

It is important to note that the implementation of step 1 of the algorithm to reduce the sequential sections often leads to the emergence of new parallel sections that can be cut in step 2. Conversely, step 2 of the algorithm often leads to a new sequential sections that can be cut in step 1. Thus, by cyclic doing of step 1 and step 2, you can greatly reduce the dimension of the graph of the hydraulic network or even to reduce it to a single branch. In the case of a reducing to one branch, the resulting solution can be considered as analytical. However, an analytical solution can be obtained only if there are no nodes with nonzero fluid expense in hydraulic network, besides the graph of system must satisfy certain properties, the discussion of which is beyond the scope of this article.

Application of the method of stepwise dimensionality reduction allows:

1) to reduce the dimension of the system, that improves the convergence of the methods used to solve it;

2) to reduce the search time for solving of the Kirchhoff equations, especially for systems of large dimension;

3) in some cases the method allows to obtain analytical solution of Kirchhoff equations.

The efficiency of the algorithm depends on the network topology, number of nodes with nonzero expenses and number of nodes with applicable pressures.

2. Example of implementation of algorithm MPSR for case of fully foldable system

Suppose the graph of the hydraulic network is given (see fig. 1).

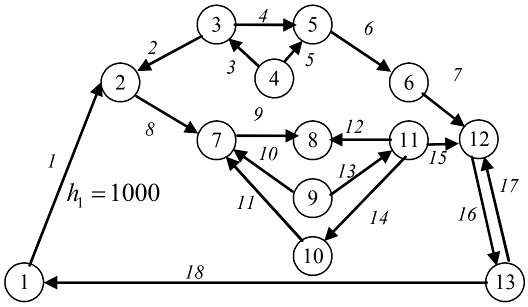

Fig. 1. Graph of the hydraulic network

The incidence matrix for this graph takes the form:

$$\overline{A}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1
\end{pmatrix}.$$

Suppose the network has one active source of pressure, that is $h_1 = 1000$. Pressure value on other branches is zero:

$$h = \begin{pmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Suppose also fluid expenses in all nodes of a graph of system equal to zero:

$$q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Assume that the values of the hydraulic resistance in all sections of the system are known. For example we take the convenient to calculate values of the coefficients: $S_{i,i} = i,\ i = \overline{1;18}$.

Step 0.

Initial parameters of the reduced system are taken equal to parameters of the original system:

$$\overline{A'} = \overline{A}, \quad S' = S, \quad h' = h, \quad q' = q.$$

We define the equivalent transformation matrix as the unit matrix with dimension $n \times n$ ($D = E$).

Step 1.

Reduction of sequential branches.

Step 1.1. (First round). We consider the first round of the algorithm in detail. We search the reduce sections. Since there are no nodes with fluid expenses in the system, all rows of matrix $\overline{A'}$ may be considered. Reduced nodes are 1, 4, 6, 8, 9, 10.

Step 1.2. Parameters of combined sections are recalculated:

$$\begin{array}{ll}
S'_{1,1} = S'_{1,1} + S'_{18,18} = 1 + 18 = 19; & h_1 = h_1 + h_{18} = 1000 + 0 = 1000;\\
S'_{3,3} = S'_{3,3} + S'_{5,5} = 3 + 5 = 8; & h_3 = h_3 + h_5 = 0 + 0 = 0;\\
S'_{6,6} = S'_{6,6} + S'_{7,7} = 6 + 7 = 13; & h_6 = h_6 + h_7 = 0 + 0 = 0;\\
S'_{9,9} = S'_{9,9} + S'_{12,12} = 9 + 12 = 21; & h_9 = h_9 + h_{12} = 0 + 0 = 0;\\
S'_{10,10} = S'_{10,10} + S'_{13,13} = 10 + 13 = 23; & h_{10} = h_{10} + h_{13} = 0 + 0 = 0;\\
S'_{11,11} = S'_{11,11} + S'_{14,14} = 11 + 14 = 25; & h_{11} = h_{11} + h_{14} = 0 + 0 = 0.
\end{array}$$

The coefficients of the matrices $\overline{A'}$ and D are recalculated by formulas (7)–(10):

$$\overline{A'} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0\\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{pmatrix},$$

$$
D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Graph of the system will reduced, as shown in fig. 2.

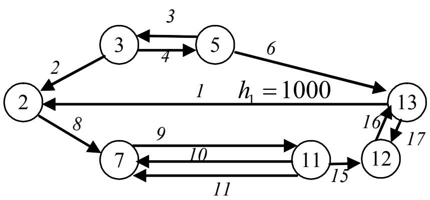

Fig. 2. Graph of the hydraulic network after the first doing of step 1 of MSDR

As seen, the matrix $\overline{A'}$ after conversion is reduced incidence matrix of the graph of the system.

Step 2.

Reduction of parallel branches.

Step 2.1. (First round). In the graph of the system in this step there are 3 groups of parallel branches, that can be combined: {3,4}, {9,10,11} и {16,17}.

Step 2.2. Corresponding columns 3, 10, 11 and 17 in the matrix $\overline{A'}$ are zero, obtaining a graph of system with the corresponding number of the removed branches (see. fig. 3):

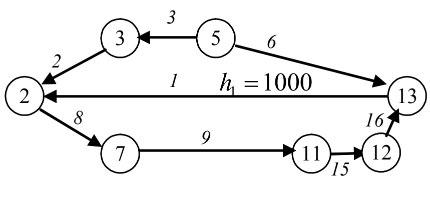

Fig. 3. Graph of the hydraulic network after the first doing of step 2 of MSDR

Step 2.3. We calculate the value of the conductivity and coefficient of wicking for reduced branches.

Group of branches {3,4}:

$$\sigma_3 = \frac{1}{\sqrt{S'_{3,3}}} = \frac{1}{\sqrt{8}} \approx 0.354; \; \sigma_4 = \frac{1}{\sqrt{S'_{4,4}}} = \frac{1}{\sqrt{4}} = 0.5; \; \sum_{j=1}^{K} \sigma_{I_j} = 0.354 + 0.5 = 0.854;$$

$$K_3 = \frac{\sigma_3}{\sum_{j=1}^{N} \sigma_{I_j}} = \frac{0.354}{0.854} = 0.415; \;\; K_4 = \frac{\sigma_4}{\sum_{j=1}^{N} \sigma_{I_j}} = \frac{0.5}{0.854} = 0.585.$$

Group of branches {9,10,11}:

$$\sigma_9 = \frac{1}{\sqrt{21}} = 0.218; \; \sigma_{10} = \frac{1}{\sqrt{23}} = 0.209; \; \sigma_{11} = \frac{1}{\sqrt{25}} = 0.2; \; \sum_{j=1}^{K} \sigma_{I_j} = 0.627;$$

$$K_9 = \frac{0.218}{0.627} = 0.348; \; K_{10} = \frac{0.209}{0.627} = 0.333; \; K_{11} = \frac{0.200}{0.627} = 0.319.$$

Group of branches {16,17}:

$$\sigma_{16} = \frac{1}{\sqrt{16}} = 0.25; \; \sigma_{17} = \frac{1}{\sqrt{17}} = 0.243; \; \sum_{j=1}^{K} \sigma_{I_j} = 0.493;$$

$$K_{16} = \frac{0.25}{0.493} = 0.507; \; K_{17} = \frac{0.243}{0.493} = 0.493.$$

Step 2.4. Coefficients of the matrix of equivalent transformation are transmuted:

$$D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.415 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.585 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.415 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.348 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.319 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.348 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.319 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.507 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.493 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Step 2.5. The resistance values for generalized branches are defined:

$$S'_{3,3} = \frac{1}{\left(\sum_{k=1}^{K} \sigma_{I_k}\right)^2} = \frac{1}{0.854^2} \approx 1.371;\ S'_{9,9} = \frac{1}{0.627^2} \approx 2.544;\ S'_{16,16} = \frac{1}{0.493^2} \approx 4.114.$$

Further again steps 1 and 2 of the algorithm are doing until the graph of the system is no reduced to a single branch. The sequence of transformations of the graph is shown below.

Step 1 (second round).

Nodes 3, 5, 7, 11, 12 and related to them branches with the largest number are reduced (see fig. 4).

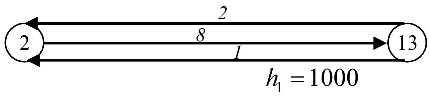

Fig. 4. Graph of the hydraulic network after the second doing of step 1 of MSDR

Step 2 (second round).

Branch 8 is reduced (see fig. 5).

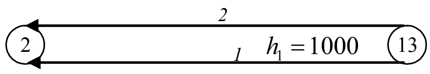

Fig. 5. Graph of the hydraulic network after the second doing of step 2 of MSDR

Step 1 (third round).

Node 13 and branch 2 are reduced (see fig. 6).

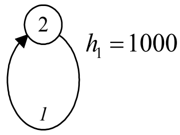

Fig. 6. Graph of the hydraulic network after the third doing of step 1 of MSDR

Step 3.

After reduction of the graph matrix S' and vector h' will contain one single element: $S'_{1,1} \approx 24.383$, $h_1 = 1000$. Node 2 is renumbered as node 1.

After reduction the matrix D takes the following form:

$$D = \big(1 \quad -0.57 \quad -0.24 \quad 0.34 \quad 0.24 \quad 0.57 \quad 0.57 \quad 0.43 \quad 0.15 \quad -0.14$$

$$-0.14 \quad -0.15 \quad 0.14 \quad -0.14 \quad 0.43 \quad 0.22 \quad -0.21 \quad 1 \big)^T.$$

Step 4.
Applying formula (19), we find the solution of reduced systems:

$$x'_1 = \frac{h'_1}{S'^2_{1,1}} = \frac{1000}{24.383^2} = 1.682.$$

Step 5.
Applying formula (3), we recover the solution of the original system:

$$x = D \cdot x' = \begin{pmatrix} 1.68 & -0.96 & -0.4 & 0.56 & 0.4 & 0.96 & 0.96 & 0.72 & 0.25 & -0.24 \\ -0.23 & -0.25 & 0.24 & -0.23 & 0.72 & 0.36 & 0.35 & 1.68 \end{pmatrix}^T.$$

This solution satisfies the system of equations (1), prepared in accordance with the original system, and hence it is its solution.

3. Example of implementation of algorithm MPSR for case of not fully foldable system

Suppose the graph of the hydraulic network is given (see fig. 7).

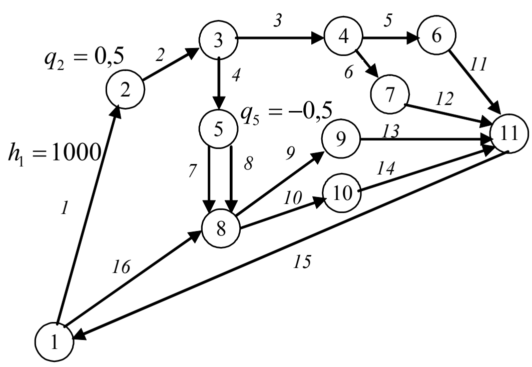

Fig. 7. Graph of the hydraulic network

Suppose the network has one active source of pressure, that is $h_1 = 1000$. Pressure on other branches is zero:

$$h = \begin{pmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Suppose also the fluid expenses in all nodes of graph of the system are zero, except for two following assemblies: $q_2 = 0.5,\ q_5 = -0.5$:

$$q = \begin{pmatrix} 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Assume that the values of the hydraulic resistance for all sections of the system are known. For example we take the convenient to calculate values of the coefficients: $S_{i,i} = i,\ i = \overline{1;16}$.

Step 0.

The initial parameters of reduced system and matrix of equivalent transformation are given:

$$\overline{A'} = \overline{A},\ S' = S,\ h' = h,\ \ q' = q,\ D = E.$$

Step 1.

Reduction of sequential branches (first round). On this step the nodes of the graphs 6, 7, 9, 10 and related to them branches with the highest number are reduced. Node 2 can not be reduced because $q_2 \neq 0$. After reduction we obtain the graph of the following form (see. fig. 8).

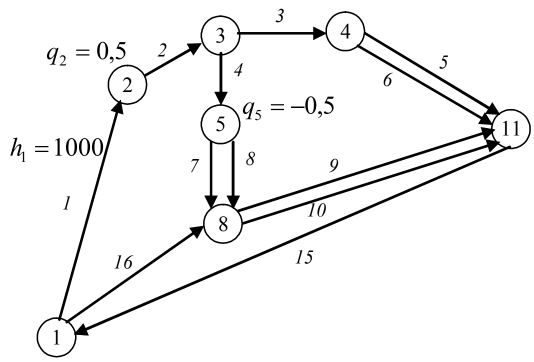

Fig. 8. Graph of the hydraulic network after the first doing of step 1 of MSDR

Step 2.

Reduction of parallel branches. In the graph of the system on this step there are three groups of parallel branches which can be subjected to combination: {5,6}, {7,8}, {9,10}. In each group the branches with the largest numbers 6, 8 and 10 are removed (see fig. 9).

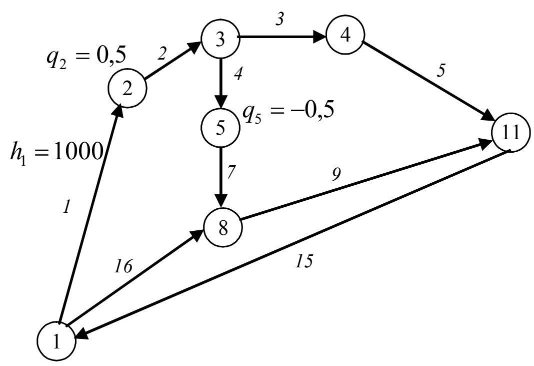

Fig. 9. Graph of the hydraulic network after the first doing of step 2 of MSDR

Step 1 (second round). Node 4 is reduced (see fig. 10).

Further reduction of the graph is not possible, no sequential and parallel reducible sections. Nodes and branches of the graph are renumbered (see fig. 11).

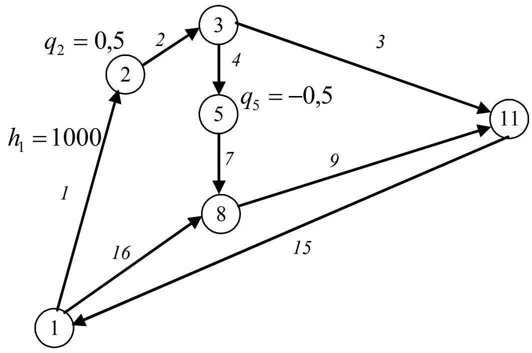

Fig. 10. Graph of the hydraulic network after the second doing of step 1 of MSDR

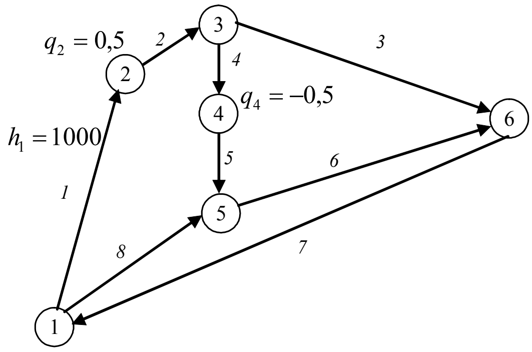

Fig. 11. Transformed MPSR hydraulic network graph with enumerated nodes and branches

Step 3.

In reduced graph there are 8 branches and 6 knots ($n' = 8,\ m' = 6$). Nonzero elements of matrix S' equal $S'_{1,1} = 1$; $S'_{2,2} = 2$; $S'_{3,3} = 7.24$; $S'_{4,4} = 4$; $S'_{5,5} = 1.87$; $S'_{6,6} = 5.74$; $S'_{7,7} = 15$; $S'_{8,8} = 16$.

Vectors h', q':

$$h' = \begin{pmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

$$q' = \begin{pmatrix} 0 & 0,5 & 0 & -0,5 & 0 & 0 \end{pmatrix}^T.$$

Matrix D takes the following form:

$$D = \left(\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0.515 & 0.485 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.517 & 0.483 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.511 & 0.489 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.515 & 0.485 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.511 & 0.489 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Step 4.

Forming matrices A', B' and solving system (4) GMCE, we find the following values of the expenses on the branches:

$$x' = \begin{pmatrix} 10.64 & 11.14 & 5.21 & 5.93 & 5.43 & 0.2 & 5.41 & -5.23 \end{pmatrix}^T.$$

Step 5.

Applying formula (3), we recover the solution for original system:

$$x = D \cdot x' = \begin{pmatrix} 10.64 & 11.14 & 5.21 & 5.93 & 2.68 & 2.53 & 2.81 & 2.62 & 0.1 & 0.1 \\ & & 2.68 & 2.53 & 0.1 & 0.1 & 5.41 & -5.23 & & \end{pmatrix}^T.$$

This solution satisfies system of equations (1), prepared in accordance with the original system, and hence it is its solution.

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