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NUMERICAL SOLUTION FOR NON-STATIONARY LINEARIZED HOFF EQUATION DEFINED ON GEOMETRICAL GRAPH

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The non-stationary linearized Hoff equation is considered in the article. For this equation, a solution is obtained both on the domain and on the geometric graph. For the five-edged graph, the Sturm – Liouville problem is solved to obtain a numerical solution of the non-stationary linearized Hoff equation on the graph. A numerical method for solving this equation on a graph is described. The graphics for obtained numerical solution are constructed at different instants of time for given values of the equation parameters and functions. The article besides the introduction and the bibliography contains four parts. The first part contains information on abstract non-stationary Sobolev type equations, and solutions for the non-stationary linearized Hoff equation on the domain are constructed. In the second one we consider the Sturm – Liouville problem on a graph and construct necessary spaces and operators on graphs. In the third one we study the solvability of the non-stationary linearized Hoff equation on the five-edged graph, and finally, in the last part we describe the numerical solution of the equation on the graph and the graphics of these solutions at different instants of time.

Keywords: Sobolev type equation; relatively bounded operator; Sturm – Liouville problem; Laplace operator on graph.

Introduction

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of C^∞ class. A semilinear Hoff equation

$$(\lambda - \Delta)u_t = \alpha u + u^3 \tag{1}$$

describes buckling of an I-beam under constant load. Here α is a real number, which is characterized material properties of the I-beam; the number $\lambda \in \mathbb{R}_+$ corresponds to the load on the beam [1]. If $\lambda \in \sigma(\Delta)$ then the left part of equation (1) becomes zero. Equations, that are unsolved with respect to the highest time-derivative, are referred to the Sobolev type equations [2]

$$L\dot{u} = Mu + N(u), \qquad \mathrm{ker} L \neq \{0\},$$

where operators L and M are linear bounded operators acting from the Banach space $\mathfrak{U}$ into the Banach space $\mathfrak{F}$, and the operator N is a nonlinear smooth operator acting in the same spaces. Even in the first papers devoted to these equations, their feature was noted in the fact that for them the Cauchy problem

$$u(0) = u_0$$

is fundamentally insolvable for arbitrary initial data u_0, even from dense set in $\mathfrak{U}$. In order for solutions of the Sobolev type equation to exist, it is necessary that the initial data belong to some set of admissible initial values (phase space) for these equations. In order not to check this condition, we shall use the Showalter – Sidorov condition

$$L(u(0) - u_0) = 0.$$

This condition coincides with the Cauchy condition, in the case of a nondegenerate operator L, and in the case of a degenerate operator L, it eliminates the need for matching the initial data. At present, the theory of Sobolev type equations is rapidly developing in various direction. For example, applying the methods of this theory [3, 4] to reconstruct a dynamically distorted sensor signal [5] served as the basis for creating a theory of optimal dynamical measurements [6, 7].

For the first time, the methods of the Sobolev type equations theory [2] were used to study the equation (1) in [8]. Later, within the framework of this theory, the Hoff equation (1) was investigated in various aspects (see, for example, [9–14]). In this paper we consider the non-stationary linearized Hoff equation of the form

$$(\lambda - \Delta)u_t = \alpha(t)u + g(t) \tag{2}$$

on a finite connected oriented graph $\mathbf{G}$, where the vector function $g : \mathbb{R} \to \mathfrak{F}$ characterizes the external action on the system, and the scalar function $\alpha : [0, T] \to \mathbb{R}_+$ characterizes the time variation of the parameters of this equation. Non-stationary Sobolev type equations were first considered in [15] and the proposed methods were applied to investigate various problems, for example, in [16–18].

The Sobolev type equations on graphs were first considered in the paper [19] and the proposed methods were applied to various models on graphs (see, for example, [10, 11, 13, 20–25]), including the Hoff model [10, 11, 13, 23, 24]. The Hoff equation, which is defined on a finite connected oriented graph $\mathbf{G}$, models the dynamics of the structure consisting of I-beams.

The main goal of this paper is to construct a numerical solution of the non-stationary linearized Hoff equation (2), considered on a five-edged connected oriented graph. For this, the methods proposed in [10] was applied, and in constructing the numerical solution we used the methods from [23–25].

1. The Solvability of the Non-Stationary Linearized Hoff Equation on the Domain

1.1. Solutions of the Abstract Non-Stationary Sobolev Type Equations

We recall the basic concepts of the theory of Sobolev type equations, which can be found, for example, in [2].

Let $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ (linear bounded operators acting from $\mathfrak{U}$ into $\mathfrak{F}$) and $\ker L \neq \{0\}$. Sets $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ are called *L-resolvent set* and *L-spectrum* of operator M correspondingly.

For complex value $\mu \in \mathbb{C}$ we define operator-valued functions of forms $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ with domain $\rho^L(M)$. These functions we call *L-resolvent, right* and *left L-resolvent* of operator M correspondingly.

The operator M is called *spectrally bounded with respect to the operator* L (or, shortly, *(L, σ)-bounded*), if $\exists r > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r) \Rightarrow (\mu \in \rho^L(M))$.

For the (L,σ)-bounded operator M we choose in complex plane $\mathbb{C}$ the closed circuit of form $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > r\}$. Then next integrals

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)d\mu$$

make sense, as the integrals of analytic functions on a closed circuit. Moreover, operators $P : \mathfrak{U} \to \mathfrak{U}$ and $Q : \mathfrak{F} \to \mathfrak{F}$ are projectors [2]. Denote

$$\mathfrak{U}^0 = \ker P, \quad \mathfrak{F}^0 = \ker Q, \quad \mathfrak{U}^1 = \mathrm{im}\ P, \quad \mathfrak{F}^1 = \mathrm{im}\ Q.$$

And so we get that $\quad \mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1, \quad \mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1.$

By L_k, M_k we denote the restrictions of operators L, M on the subspace $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [2] *Let the operator M be (L,σ)-bounded. Then*
(i) $L_k, M_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0, 1$;
(ii) *there exist operators* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$, $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$.

If the operator M is a (L,σ)-bounded, then by virtue of Theorem 1 there are exist operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Definition 1. (L,σ)-bounded operator M is called
(i) $(L, 0)$-*bounded*, if $H = \mathbb{O}$;
(ii) (L, p)-*bounded*, if $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$ for some $p \in \mathbb{N}$;
(iii) (L, ∞)-*bounded*, if $H^p \neq \mathbb{O}$ for all $p \in \mathbb{N}$.

Here and below we set that $p \in \mathbb{N}_0$, where $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$. Consider the solution of Showalter – Sidorov problem

$$P(u(0) - u_0) = 0, \qquad u_0 \in \mathfrak{U} \tag{3}$$

for non-stationary equation

$$L\dot{u}(t) = a(t)Mu(t) + g(t), \tag{4}$$

where $a : [0, T] \to \mathbb{R}_+$ is a scalar function, that characterizes the time variation of the interaction parameters of the system under study, and the vector function $g : [0, T] \to \mathfrak{F}$ describes the external action.

Definition 2. A vector function $u \in C^1(\mathbb{R}; \mathfrak{U})$ is called a *solution* of equation (4), if it satisfies this equation on $\mathbb{R}$. The solution of equation (4) is called a *solution of Showalter – Sidorov problem* (3), (4), if in addition it satisfies (3).

Construct the solution of the problem (3) for equation (4).

Theorem 2. [17] *Let the operator M be (L,p)-bounded ($p \in \mathbb{N}_0$) and $a \in C^{p+1}([0,T]; \mathbb{R}_+)$. Then for arbitrary $u_0 \in \mathfrak{U}$ and $g \in C^{p+1}([0,T]; \mathfrak{F})$ there exists the unique solution $u \in C^1([0,T]; \mathfrak{U})$ of Showalter – Sidorov problem (3) for equation (4), and it has the form*

$$u(t) = e^{S\int\limits_0^t a(\mathfrak{z})d\mathfrak{z}} Pu_0 + \int\limits_0^t e^{S\int\limits_s^t a(\mathfrak{z})d\mathfrak{z}} L_1^{-1}Qg(s)ds - \sum_{k=0}^{p} H^k M_0^{-1}(\mathbb{I} - Q)(AD)^k Ag(t),$$

where $(Ah)(t) = a^{-1}(t)h(t)$ и $(Dh)(t) = \frac{dh}{dt}(t)$.

1.2. Solutions of the Non-Stationary Linearized Hoff Equation

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of C^∞ class. In cylinder $\Omega \times \mathbb{R}$ we consider Dirichlet problem

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R} \tag{5}$$

for Hoff equation of the form

$$(\lambda - \Delta)u_t = \alpha(t)u + g(t) \tag{6}$$

with Showalter – Sidorov condition

$$(\lambda - \Delta)(u(0) - u_0) = 0. \tag{7}$$

Here the vector function $g(t)$ characterizes external action on the described system.

Reduce problem (5)–(7) to problem (3), (4). In order we get the spaces

$$\mathfrak{U} = \{u \in W_q^{m+2}(\Omega) : u(x) = 0,\ x \in \partial\Omega\}, \qquad \mathfrak{F} = W_q^m(\Omega), \tag{8}$$

where $W_q^m(\Omega)$ are Sobolev spaces with $2 \le q < \infty$ and $m = 0, 1, \ldots$. Operators L and M we define by the next formulas

$$L = \lambda - \Delta, \qquad M = \mathbb{I}. \tag{9}$$

By $\sigma(\Delta)$ we denote a spectrum of homogeneous Dirichlet problem on the domain Ω for the Laplace operator Δ. The spectrum of $\sigma(\Delta)$ is negative, discrete, finite and condensed only to $-\infty$. By $\{\lambda_k\}$ we denote the set of eigenvalues, which numbered by nonincreasing order with their multiplicity. And by $\{\psi_k\}$ we define the family of corresponding eigenfunctions orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle$ in space $L_2(\Omega)$, $\psi_k \in C^\infty$, $k \in \mathbb{N}$.

Lemma 1. [8] *Let spaces $\mathfrak{U}$ and $\mathfrak{F}$ be from* (8)*, and operators L and M be from* (9)*. Then for arbitrary $\lambda \in \mathbb{R}\backslash\{0\}$ the operator M is a $(L,0)$-bounded one.*

If $\lambda \in \sigma(\Delta)$ has multiplicity $r \in \mathbb{N}$, we put $\ker L = \text{span}\{\psi_1, \psi_2, \ldots, \psi_r\}$, where ψ_l are the eigenfunctions of the Laplace operator Δ corresponding to the eigenvalue λ, and they can be chosen orthogonal in the sense of the inner product $\langle\cdot,\cdot\rangle$ в $L_2(\Omega)$. Then $\text{im}L = \{g \in \mathfrak{F} : \langle g, \psi_l\rangle = 0,\ l = 1, 2, \ldots, r\}$.

It is clear that if $\lambda \neq 0$ then the L-spectrum of operator M can be represent as

$$\sigma^L(M) = \{\mu \in \mathbb{C} : \mu_k = \frac{1}{\lambda - \lambda_k}, \lambda_k \neq \lambda\}.$$

If $\lambda = 0$ then the L-spectrum of operator M is such that $\sigma^L(M) = \mathbb{C}$, so everywhere else we take $\lambda \neq 0$. Since the points of the spectrum of the Laplace operator $\{\lambda_k\}$ are real, discrete, had finite multiplicity and condensed only to $-\infty$, then the relative spectrum of $\sigma^L(M)$ is obviously bounded.

Construct projectors P and Q. If $\lambda \notin \sigma(\Delta)$ then the projector $P = \mathbb{I}$, and if $\lambda \in \sigma(\Delta)$ then $P = \mathbb{I} - \sum\limits_{l=1}^{r} \langle\cdot, \psi_l\rangle\psi_l$. The projector Q has the same form, but is defined on the space $\mathfrak{F}$. By Theorem 2 and Lemma 1, the following theorem is true.

Theorem 3. *Let the conditions of Lemma* 1 *be fulfilled,* $\lambda \in \mathbb{R}\backslash\{0\}$, $\alpha \in C^1([0,T];\mathbb{R}_+)$, $g \in C^1([0,T];\mathfrak{F})$ *and*

(i) $\lambda \notin \sigma(\Delta)$. *Then for arbitrary* $u_0 \in \mathfrak{U}$ *there exists the unique solution* (5)–(7), *which has the form*

$$u(t) = \sum_{k=1}^{\infty} e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_0^t \alpha(\tau)d\tau\right)} \langle u_0, \psi_k\rangle\psi_k + \sum_{k=1}^{\infty}\int\limits_0^t e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_s^t \alpha(\tau)d\tau\right)} \frac{\langle g(s), \psi_k\rangle\psi_k}{\lambda-\lambda_k}ds;$$

(ii) $\lambda \in \sigma(\Delta)$. *Then for arbitrary* $u_0 \in \mathfrak{U}$ *there exists the unique solution* (5)–(7), *which has the form*

$$u(t) = -\sum_{l\in\mathbb{N}:\lambda_l=\lambda} \frac{\langle g(s), \psi_l\rangle\psi_l}{\alpha(t)} + \sum_{k\in\mathbb{N}\backslash\{l:\lambda_l=\lambda\}}^{\infty} e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_0^t \alpha(\tau)d\tau\right)} \langle u_0, \psi_k\rangle\psi_k +$$

$$+ \sum_{k\in\mathbb{N}\backslash\{l:\lambda_l=\lambda\}} \int\limits_0^t e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_s^t \alpha(\tau)d\tau\right)} \frac{\langle g(s), \psi_k\rangle\psi_k}{\lambda-\lambda_k}ds.$$

2. The Sturm – Liouville Problem for Laplace Operator Defined on a Graph

Let $\mathbf{G} = \mathbf{G}(\mathfrak{V};\mathfrak{E})$ is a finite connected oriented graph, where $\mathfrak{V} = \{V_k\}$ is a set of vertices, and $\mathfrak{E} = \{E_l\}$ is a set of edges. Each edge is characterized by two numbers $l_l,\ d_l \in \mathbb{R}_+$, denoting the length and the cross-sectional area of the edge E_l respectively. On the graph $\mathbf{G}$ consider the Sturm – Liouville problem in a next formulation. We set equations

$$u_{jxx} = bu_j \ \text{ for all } x \in\ (0, l_j),\ t \in \mathbb{R}, \tag{10}$$

for which in each vertex of graph we set conditions

$$\sum_{E_j\in\ E^{\alpha}(V_k)} d_j c_j(x) u_{jx}(0,t) - \sum_{E_m\in\ E^{\omega}(V_k)} d_m c_m(x) u_{mx}(l_m, t) = 0, \tag{11}$$

$$u_l(0,t) = u_k(0,t) = u_m(l_m,t) = u_n(l_n,t), \tag{12}$$

where $E_j, E_l \in E^{\alpha}(V_k)$, $E_m, E_n \in E^{\omega}(V_k)$, $t \in \mathbb{R}$. Here by $E^{\alpha(\omega)}(V_k)$ is denoted the set of edges with begin (end) in vertex V_k. Condition (11) indicates that the flow through each vertex must be equal to zero, and condition (12) indicates that the solution $u = (u_1, u_2, ..., u_j, ...)$ at each vertex must be continuous. In particular, when the graph $\mathbf{G}$ consists of one edge and two vertices, the condition (12) disappears, and the condition (11) becomes a homogeneous Neumann condition.

Consider the Hilbert space

$$L_2(\mathbf{G}) = \{g = (g_1, g_2, ..., g_j, ...) : g_j \in L_2(0, l_j)\}$$

with inner product $\langle g, h\rangle = \sum\limits_{E_j\in\mathfrak{E}} d_j \int\limits_0^{l_j} g_j(x)h_j(x)dx,$

and the Hilbert space

$$\mathfrak{U} = \{u = (u_1, u_2, ..., u_j, ...) : u_j \in W_2^1(0, l_j) \text{ and Condition (12) is satisfied}\}$$

$$\text{with inner product} \quad [u, v] = \sum_{E_j \in \mathfrak{E}} d_j \int_0^{l_j} (u_{jx}v_{jx} + u_j v_j)dx.$$

By the Sobolev embedding theorems, the space $W_2^1(0, l_j)$ (up to measure zero) consists of absolutely continuous functions, and hence the space $\mathfrak{U}$ is correctly defined. Moreover, this space is densely and compactly embedded in $L_2(\mathbf{G})$. We identify $L_2(\mathbf{G})$ with its conjugate and denote by $\mathfrak{F}$ the space conjugate to $\mathfrak{U}$ with respect to the duality of $\langle \cdot, \cdot \rangle$ in $L_2(\mathbf{G})$. Obviously, $\mathfrak{F}$ is a Banach space with norm $||f||_{\mathfrak{F}} = \sup\limits_{u \in \mathfrak{U}\{0\}} |\langle f, u \rangle|/||u||_{\mathfrak{U}}$, and the embedding $||u||_{\mathfrak{U}} \to ||f||_{\mathfrak{F}}$ is compact.

Definition 3. Vector finction $u = (u_1, u_2, ..., u_j, ...)$ such that $u_j \in C^2(0, l_j) \bigcap C^1[0, l_j]$ is called a *solution of Sturm – Liouville problem* (10)–(12), if it satisfies the equations (10) and boundary conditions (11), (12).

Definition 4. The function $u = (u_1, u_2, ..., u_j, ...)$, which is not identically zero, is called the *eigenfunction of the problem* (10)–(12) for the operator $A = (A_1, A_2, ..., A_j, ...)$, $A_j = d^2/dx^2$ if there exists a number b such that u is a solution of this problem. The number b is called the *eigenvalue* corresponding to the eigenfunction u.

Let an operator $B : \mathfrak{U} \to \mathfrak{F}$ has the form $\langle Bu, v \rangle = \sum\limits_{E_j \in \mathfrak{E}} d_j \int\limits_0^{l_j} (u_{jx}v_{jx} + bu_j v_j)dx.$ Since the inequality $c_1||u||_{\mathfrak{U}}^2 \leqslant \langle Bu, u \rangle \leqslant c_2||u||_{\mathfrak{U}}^2$ holds for all $b \in \mathbb{R}_+$, $u \in \mathfrak{U}$ and some $c_1, c_2 \in \mathbb{R}_+$, it follows that the linear operator $B : \mathfrak{U} \to \mathfrak{F}$ is bijective and continuous. Hence by the Banach theorem there is the operator $B^{-1} : \mathfrak{F} \to \mathfrak{U}$. Since the embedding $\mathfrak{U} \hookrightarrow \mathfrak{F}$ is compact, the operator $B^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ is compact, and hence the spectrum of B is discrete, real, finite, and condensed only to $+\infty$. We fix $b \in \mathbb{R}$ and obtain the operator $A = B - b\mathbb{I}$, for which the following theorem is true.

Theorem 4. [19] *The operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, and the spectrum $\sigma(A)$ of the operator A is discrete, finite, non-negative and condensed only to $+\infty$.*

Note that the first eigenvalue of the operator A is zero, and this value is single-valued. Indeed, $\langle Au, u \rangle = \sum\limits_{E_j \in \mathfrak{E}} d_j \int\limits_0^{l_j} u_{jx}^2 dx \geq 0$ for all $u \in \mathfrak{U}$ and equals to zero for such $u = (u_1, u_2, ..., u_j, ...)$, that $u_1 = u_2 = ... = u_j = ... = const$. We introduce the eigenfunction $\phi_1 = (\sum\limits_{E_j \in \mathfrak{E}} d_j l_j)^{-1/2}(1, 1, ..., 1, ...)$ of the operator A, which is normalized in the sense of $L_2(\mathbf{G})$ and corresponding to the first eigenvalue.

We denote by $\{\lambda_n\}_{n=2}^{\infty}$ the set of remaining eigenvalues of the operator A, numbered by nondecreasing order with their multiplicity; and by $\{\phi_n\}_{n=2}^{\infty}$ we denote the corresponding eigenfunctions orthonormal in the sense of $L_2(\mathbf{G})$. Note that the linear span of $\{\phi_k : k \in \mathbb{N}\}$ is dense in $\mathfrak{U}$, $L_2(\mathbf{G})$, $\mathfrak{F}$.

3. Solutions of the Non-Stationary Linearized Hoff Equation on the Five-Edged Graph

Let $\mathbf{G} = \mathbf{G}(\mathfrak{V}; \mathfrak{E})$ is a finite connected oriented graph (see Fig. 1), where $\mathfrak{V} = \{V_k\}$ $(k = 1, 2, 3, 4, 5, 6)$ is a set of vertices, and $\mathfrak{E} = \{E_l\}$ $(l = 1, 2, 3, 4, 5)$ is a set of edges. To each edge we associate two numbers $l_l, d_l \in \mathbb{R}_+$, which define the length $l_1 = l_2 = l_3 = l_4 = l_5 = l$ and the cross-sectional area $d_1 = d_2 = d_3 = d_4 = d_5 = 1$ of this edge.

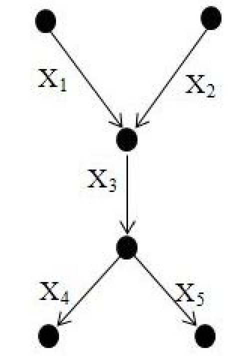

Fig. 1. Graph **G**

As a solution of the Sturm – Liouville problem for the Laplace operator on the graph $\mathbf{G}$ we looking for nontrivial functions $X(x) = (X_1(x), X_2(x), X_3(x), X_4(x), X_5(x))$ (eigenfunctions). In view of the conditions (11) and (12) we obtain the eigenvalues $\lambda_k = (\frac{\pi k}{l})^2$ and the eigenfunctions: $X^k(x) = (X_1^k(x), X_2^k(x), X_3^k(x), X_4^k(x), X_5^k(x))$ $(k \in \mathbb{N})$, where

$$\begin{cases} X_1^k(x) = C_1 \cos \frac{\pi(k-1)}{l} x, \\ X_2^k(x) = C_1 \cos \frac{\pi(k-1)}{l} x, \\ X_3^k(x) = C_1 \cos(\pi(k-1)x) \cos \frac{\pi(k-1)}{l} x, \\ X_4^k(x) = C_1 \cos^2(\pi(k-1)x) \cos \frac{\pi(k-1)}{l} x, \\ X_5^k(x) = C_1 \cos^2(\pi(k-1)x) \cos \frac{\pi(k-1)}{l} x. \end{cases}$$

The equation (6) on the graph $\mathbf{G}$ takes the form

$$\lambda u_{jt} - u_{jxxt} = \alpha(t) u_j + g_j \quad \text{for all} \quad x \in (0, l_j),\ t \in \mathbb{R}. \tag{13}$$

At each vertex V_i for equations (13) we define the boundary conditions (11), (12). The problem (11), (12), (13) can be considered as the Neumann problem for the equation (6) defined on the domain. If we supplement (11), (12) with the initial condition

$$u_j(x, 0) = u_{0j}(x) \quad \text{for all} \quad x \in (0, l_j), \tag{14}$$

then we get the initial–boundary value problem for equations (13).

We carry out the reduction of the problem (11), (12), (14) for the equations (13) to the Showalter – Sidorov problem (3) for the linear equation Sobolev type (4), where the operators $L, M \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, and $\mathfrak{U}$, $\mathfrak{F}$ are the Banach spaces from the previous section.

Fix $\lambda \in \mathbb{R}$ and construct the operators $L = \lambda \mathbb{I} - A$, $M = \mathbb{I}$, where the oparator A from Theorem 4. Then the next lemma is true.

Lemma 2. [10] *Let $\lambda \in \mathbb{R} \backslash \{0\}$ then the operator M is $(L, 0)$-bounded.*

Let $\{\lambda_k\}$ be the eigenvalues of the operator A, numbered by nondecreasing order with their multiplicity; and X^k are the corresponding eigenfunctions, which orthonormal in the sense of $L_2(\mathbf{G})$. We construct projectors

$$P = \begin{cases} \mathbb{I}, & \text{if} \quad \lambda \notin \sigma(A); \\ \mathbb{I} - \sum\limits_{\lambda_k = \lambda} \langle \cdot, X^k \rangle X^k, & \text{if} \quad \lambda \in \sigma(A); \end{cases} \qquad Q = \begin{cases} \mathbb{I}, & \text{if} \quad \lambda \notin \sigma(A); \\ \mathbb{I} - \sum\limits_{\lambda_k = \lambda} \langle \cdot, X^k \rangle X^k, & \text{if} \quad \lambda \in \sigma(A); \end{cases}$$

where $\sigma^L(M) = \left\{ \mu_k = \dfrac{1}{\lambda - \lambda_k}, k \in \mathbb{N} \right\}$. By Theorem 3 and Lemma 2 the next theorem is true.

Theorem 5. *Let* $\lambda \in \mathbb{R}\backslash\{0\}$, $\alpha \in C^1([0,T];\mathbb{R}_+)$, $g \in C^1([0,T];\mathfrak{F})$ *and*

(i) $\lambda \notin \sigma(\Delta)$. *Then for arbitrary* $u_0 \in \mathfrak{U}$ *there exists the unique solution of problem* (13), (14), *which has the form*

$$u(t) = \sum_{k=1}^{\infty} e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_0^t \alpha(\tau)d\tau\right)} \langle u_0, X^k\rangle X^k + \sum_{k=1}^{\infty} \int\limits_0^t e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_s^t \alpha(\tau)d\tau\right)} \frac{\langle g(s), X^k\rangle X^k}{\lambda-\lambda_k} ds;$$

(ii) $\lambda \in \sigma(\Delta)$. *Then for arbitrary* $u_0 \in \mathfrak{U}$ *there exists the unique solution of problem* (13), (14), *which has the form*

$$u(t) = -\sum_{l\in\mathbb{N}:\lambda_l=\lambda} \frac{\langle g(s), X^l\rangle X^l}{\alpha(t)} + \sum_{k\in\mathbb{N}\backslash\{l:\lambda_l=\lambda\}}^{\infty} e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_0^t \alpha(\tau)d\tau\right)} \langle u_0, X^k\rangle X^k +$$

$$+ \sum_{k\in\mathbb{N}\backslash\{l:\lambda_l=\lambda\}} \int\limits_0^t e^{\left(\frac{1}{\lambda-\lambda_k}\int\limits_s^t \alpha(\tau)d\tau\right)} \frac{\langle g(s), X^k\rangle X^k}{\lambda-\lambda_k} ds.$$

4. The Numerical Solution of the Non-Stationary Linearized Hoff Equation on the Graph

On the basis of the obtained theoretical results, a numerical method for solving the linearized Hoff equation on a graph is developed, which is depended on the specified parameters of the equation, coefficients, and initial data. The constructed numerical method is based on the modified Galerkin method (see also [23–25]). The algorithm for constructing the approximate solution of the problem (13), (14) and plotting the graphics consists the following steps.

Step 1. The graph **G** is set.
Step 2. The Sturm – Liouville problem for the Laplace operator on **G** is solved.
Step 3. The required parameters of the Hoff equation are given.
Step 4. The general solution of the equation is constructed.
Step 5. We apply the initial conditions and the coefficients of expansion are determined.
Step 6. We obtain a general solution in expanded form.
Step 7. The graphic of the solution of the equation is constructed.

This method is implemented in the *Maple* 16 environment. To implement the algorithm, standard functions and operators *Maple* were used.

Let us find a numerical solution of problem (13), (14) defined on the five-edged graph (see Fig. 1) with parameters $\lambda = 2$, $\alpha(t) = 5t/2$. We take the initial data u_0 in form $u_{j0} = \cos 3\pi x$, and the vector funtion $g(s)$ in form $g_j(s) = (1+s)\cos 5\pi x$ $(j = 1,2,3,4,5)$.

For given functions, we find the values of the constants, determine the coefficients of the expansion, and put it in the solution. Since $\lambda = 2$ and, consequently, $\lambda \notin \sigma(A)$, by Theorem 5 the numerical solution has the form

$$u(t) = \sum_{k=1}^{K} \left(e^{\frac{1}{\lambda-\lambda_k}\int\limits_0^t a(t)dt} \langle u_0, X^k\rangle + \int\limits_0^t \frac{\langle g(s), X^k\rangle}{\lambda-\lambda_k} e^{\frac{1}{\lambda-\lambda_k}\int\limits_s^t a(t)dt} ds \right) X^k,$$

where λ_k and X^k are eigenvalues and eigenfunctions of the Sturm – Liouville problem for the Laplace operator on a geometric graph. We seek the sum of a finite number of terms, for this we set $K = 6$.

The solution $u(t)$ is represented in the form of graphics for $t = 0$ in the figure 2, and for $t = 1$ in the figure 3.

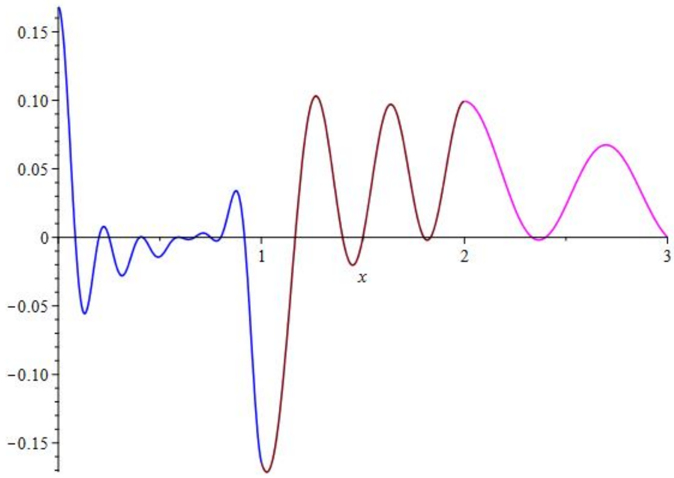

Fig. 2. The graphic of the solution of the problem (13), (14) at the time $t = 0$

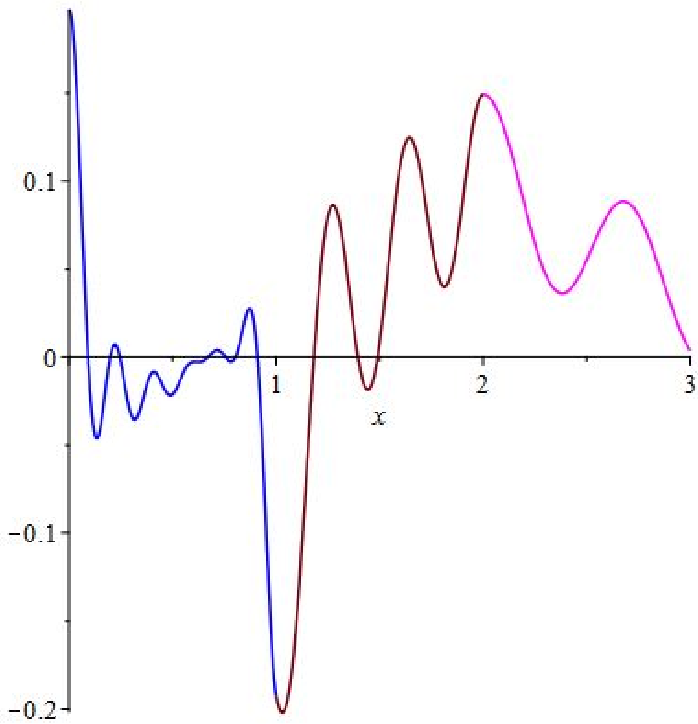

Fig. 3. The graphic of the solution of the problem (13), (14) at the time $t = 1$

In the figures 2, 3 the middle section (from 1 to 2) corresponds to the edge E_3; on the section from 0 to 1 are the coinciding parts of the solution on the edges E_1, E_2; and on the section from 2 to 3 are the coinciding parts of the solution on the edges E_4, E_5 (see Fig. 1).

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ЧИСЛЕННОЕ РЕШЕНИЕ НЕСТАЦИОНАРНОГО ЛИНЕАРИЗОВАННОГО УРАВНЕНИЯ ХОФФА НА ГЕОМЕТРИЧЕСКОМ ГРАФЕ

М. А. Сагадеева, А. В. Генералов

В статье рассматривается нестационарное линеаризованное уравнение Хоффа. Для этого уравнения получено решение как в области, так и на геометрическом графе. Для пятиреберного графа решена задача Штурма – Лиувилля для получения численного решения нестационарного линеаризованного уравнения Хоффа на графе. Описан численный метод решения для указанного уравнения на графе. Построены графики решения в различные моменты времени при заданных значениях параметров уравнения и функциях. Статья, кроме введения и списка литературы, содержит четыре части. В первой части приведены сведения об абстрактных нестационарных уравнения соболевского типа, а также построено решение для нестационарного линеаризованного уравнения Хоффа в области. Во второй рассматривается задача Штурма – Лиувилля на графе и строятся необходимые пространства и операторы на графах. В третьей исследуется разрешимость нестационарного линеаризованного уравнения Хоффа на графе, и, наконец, в последней части приведено описание численного решения исследуемого уравнения на графе и графики этих решений в различные моменты времени.

Ключевые слова: уравнения соболевского типа; относительно ограниченный оператор; задача Штурма – Лиувилля; оператор Лапласа на графе.

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