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OPTIMAL CONTROL OF SOLUTIONS TO THE SHOWALTER–SIDOROV PROBLEM IN A MODEL OF LINEAR WAVES IN PLASMA

***A. A. Zamyshlyaeva*[1],** zamyshliaevaaa@susu.ru,
***O. N. Tsyplenkova*[1],** tcyplenkovaon@susu.ru
[1]South Ural State University, Chelyabinsk, Russian Federation

In this article an optimal control problem for a high order Sobolev type equation is investigated under the assumption that the operator pencil is relatively polynomially bounded. The results are applied to the study of the optimal control of the solutions to the Showalter–Sidorov problem for a model of linear waves in a plasma in an external magnetic field. Showalter–Sidorov conditions are a generalization of Cauchy conditions. As it is well known, the Cauchy problem for Sobolev type equations is fundamentally insoluble for arbitrary initial values. We use the phase space method developed by G. A. Sviridyuk, a theory of relatively polynomially bounded operator pencils developed by A. A. Zamyshlyaeva. The mathematical model considered in the article describes ion-acoustic waves in plasma in an external magnetic field and was first obtained by Yu. D. Pletner.

Keywords: Sobolev type equations of higher order; model of linear waves in plasma; Showalter–Sidorov problem; relatively polynomially bounded operator pencil; strong solutions; optimal control.

Introduction

Let $\Omega = (0, a) \times (0, b) \times (0, c) \subset \mathbb{R}^3$. In this article we study the optimal control of solutions in a mathematical model of linear waves in a magnetized plasma [1], [2]. In the cylinder $\Omega \times \mathbb{R}$ consider equation

$$(\lambda - \Delta)x_{tttt}(s,t) = (\Delta - \lambda')x_{tt}(s,t) + \alpha\frac{\partial^2 x(s,t)}{\partial s_3^2} + u(s,t),\ s \in \Omega,\ t \in \mathbb{R} \tag{1}$$

with the boundary condition

$$x(s,t) = 0, \quad (s,t) \in \partial\Omega \times \mathbb{R}. \tag{2}$$

Model (1), (2) describes ion-acoustic waves in plasma in an external magnetic field. The parameters in the equation relate the ion frequency, the Langmuir frequency, and the Debye radius. The function $x(s,t)$ represents the generalized potential of the electric field, and the function $u(s,t)$ is an external influence.

The optimal control problem for (1), (2) will be investigated in the framework of the theory of relatively polynomially bounded pencils of operators [3]. Consider an abstract Sobolev type equation of high order

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x + y + Cu, \tag{3}$$

where the operators $A,\ B_{n-1}, \ldots, B_0 \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, the functions $u : [0,\tau) \subset \mathbb{R}_+ \to \mathfrak{U}$, $y : [0,\tau) \subset \mathbb{R}_+ \to \mathfrak{Y}\ (\tau < \infty)$, and $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{U}$ are Hilbert spaces.

Set the Showalter–Sidorov condition [4]

$$P\left(x^{(m)}(0) - x_m\right) = 0,\ m = \overline{0, n-1}, \tag{4}$$

where P is a relative spectral projector in space $\mathfrak{X}$. We are interested in the optimal control problem of finding a pair $(\hat{x}, \hat{u})$, where $\hat{x}$ is a solution of (3), (4), and the control $\hat{u} \in \mathfrak{U}_{ad}$ satisfies the relation

$$J(\hat{x}, \hat{u}) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u). \tag{5}$$

Here $J(x, u)$ is some specially constructed penalty functional, $\mathfrak{U}_{ad}$ is a closed convex set in the space $\mathfrak{U}$ of controls.

The basis of many non-classical models of mathematical physics are Sobolev type equations [5] – [10]. The study of the optimal control problem for linear first order Sobolev type equations was first started by G.A. Sviridyuk and A.A. Efremov [11]. Optimal control for semilinear Sobolev type equations was considered in [12]. Our approach is based on results [13] – [15]. In the article, instead of Cauchy conditions, the more general Showalter–Sidorov conditions are considered. Other equations with Showalter–Sidorov conditions were studied in [16] – [18].

The article consists of three parts. The first one contains the main principles of the theory of relatively polynomially bounded operators pencils and projectors. In the second part, we present a theorem on the existence and uniqueness of a strong solution of the Showalter–Sidorov problem for a nonhomogeneous high order Sobolev type equation, which was previously proved by the authors earlier [19]. In the third section, we reduce the problem under study to an abstract optimal control problem for a nonhomogeneous Sobolev type equation of high order with the Showalter–Sidorov condition. The possibility of using of abstract results is shown and theorem of existence of a unique optimal control is proved.

1. Polynomially A-Bounded Operator Pencils and Projectors

By $\vec{B}$ denote the pencil formed by operators $B_{n-1}, \ldots, B_0$.

Definition 1. *The sets*

$$\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X})\}$$

and $\sigma^A(\vec{B}) = \mathbb{C} \backslash \rho^A(\vec{B})$ *are called an A-resolvent set and an A-spectrum of pencil* $\vec{B}$, *respectively.*

Definition 2. *The operator function*

$$R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}$$

of a complex variable with domain $\rho^A(\vec{B})$ *is called an A-resolvent of pencil* $\vec{B}$.

Definition 3. *The pencil* $\vec{B}$ *is called polynomially bounded with respect to operator* A *(or polynomially A-bounded), if there exists a constant* $a \in \mathbb{R}_+$ *such that for each* $\mu \in \mathbb{C}$ *the inequality* $(|\mu| > a)$ *implies the inclusion* $(R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X}))$.

Introduce an additional condition

$$\int_\gamma \mu^m R^A_\mu(\vec{B})d\mu \equiv \mathbb{O}, m = \overline{0, n-2}, \tag{A}$$

where the contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1. [3] *If the pencil $\vec{B}$ is polynomially A-bounded, and condition (A) is satisfied, then the following operators*

$$P = \frac{1}{2\pi i}\int_\gamma R^A_\mu(\vec{B})\mu^{n-1}Ad\mu, \quad Q = \frac{1}{2\pi i}\int_\gamma \mu^{n-1}AR^A_\mu(\vec{B})d\mu$$

are projectors in $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Put $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \text{im } P$, $\mathfrak{Y}^1 = \text{im } Q$. It follows from Lemma 1 that $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$. By A^k (B^k_l) denote the restriction of the operator A (B_l) to $\mathfrak{X}^k$, $k = 0, 1$; $l = \overline{0, n-1}$.

The following assertion was proved in [3].

Theorem 1. *Let the assumptions of Lemma 1 be satisfied. Then*
(i) $A^k \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{X}^k; \mathfrak{Y}^k)$, $k = 0, 1$, $l = 0, 1, \dots, n-1$;
(iii) there exists an operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{Y}^1; \mathfrak{X}^1)$;
(iv) there exists an operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{Y}^0; \mathfrak{X}^0)$.

Construct operators $H_0 = (B^0_0)^{-1}A^0 \in \mathcal{L}(\mathfrak{U}^0)$, $H_1 = (B^0_0)^{-1}B^0_1 \in \mathcal{L}(\mathfrak{U}^0)$, …, $H_{n-1} = (B^0_0)^{-1}B^0_{n-1} \in \mathcal{L}(\mathfrak{U}^0)$ and $S_0 = (A^1)^{-1}B^1_0 \in \mathcal{L}(\mathfrak{U}^1)$, $S_1 = (A^1)^{-1}B^1_1 \in \mathcal{L}(\mathfrak{U}^1)$, …, $S_{n-1} = (A^1)^{-1}B^1_{n-1} \in \mathcal{L}(\mathfrak{U}^1)$.

Definition 4. *Introduce the family of operators $\{K^1_q, K^2_q, \dots, K^n_q\}$ as follows:*

$$\begin{gathered} K^s_0 = \mathbb{O},\ s \neq n,\ K^n_0 = \mathbb{I}, \\ K^1_1 = H_0,\ K^2_1 = -H_1, \dots, K^s_1 = -H_{s-1}, \dots, K^n_1 = H_{n-1}, \\ K^1_q = K^n_{q-1}H_0,\ K^2_q = K^1_{q-1} - K^n_{q-1}H_1, \dots, K^s_q = K^{s-1}_{q-1} - K^n_{q-1}H_{s-1}, \dots, \\ K^s_q = K^{n-1}_{q-1} - K^n_{q-1}H_{n-1},\ q = 1, 2, \dots. \end{gathered}$$

Then the A-resolvent of pencil $\vec{B}$ can be represented as a Laurent series [20]

$$(\mu^n A - \mu^{n-1}B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} = -\sum_{q=0}^{\infty} \mu^q K^n_q (B^0_0)^{-1}(\mathbb{I} - Q) +$$

$$+\sum_{q=1}^{\infty} \mu^{-q}(\mu^{n-1}S_{n-1} + \dots + \mu S_1 + S_0)^q L_1^{-1}Q.$$

Using this representation, we classify the character of the infinity point for the A-resolvent of the operator pencil $\vec{B}$.

Definition 5. *The point ∞ is called:*

- *a removable singular point of the A-resolvent of pencil $\vec{B}$, if $K_1^s \equiv \mathbb{O}, s = 1, 2, \ldots, n$;*
- *a pole of order $p \in \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, if there exists $p : K_p^s \not\equiv \mathbb{O}$, $s = 1, 2, \ldots, n$, but $K_{p+1}^s \equiv \mathbb{O}$ for any s;*
- *an essentially singular point of the A-resolvent of pencil $\vec{B}$, if $K_q^n \not\equiv \mathbb{O}$ for any $q \in \mathbb{N}$.*

Definition 6. *The pencil $\vec{B}$ is called (A, p)-bounded, if it is polynomially A-bounded and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent of pencil $\vec{B}$.*

2. Strong Solutions

Consider linear homogeneous Sobolev type equation

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x. \tag{6}$$

Definition 7. *The mapping $V^\bullet \in C^\infty(\mathbb{R}; \mathcal{L}(\mathfrak{X}))$ is called a propagator of (6), if for all $x \in \mathfrak{X}$ the function $x(t) = V^t x$ is a solution of (6).*

Assume that the pencil $\vec{B}$ is polynomially A-bounded, and condition (A) is satisfied. Fix a contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$ and for all $t \in \mathbb{R}$ consider the family of operators

$$X_k^t = \frac{1}{2\pi i} \int_\gamma R_\mu^A(\vec{B})(\mu^{n-k-1}A - \mu^{n-k-2}B_{n-1} - \ldots - B_{k+1})e^{\mu t} d\mu,$$

where $k = \overline{0, n-1}$.

Lemma 2. [3] *Let the assumptions of Lemma 1 be satisfied. Then*
(i) for every $k = \overline{0, n-1}$ the operator function X_k^t is a propagator of (6);
(ii) for every $k = \overline{0, n-1}$ the operator function X_k^t is an entire function;
(iii) $\left.\frac{d^l}{dt^l} X_k^t\right|_{t=0} = \begin{cases} P, \; l = k; \\ \mathbb{O}, \; l \neq k; \end{cases} \quad k = \overline{0, n-1}, \; l = 0, 1, \ldots.$

Definition 8. *The set $\mathfrak{P}$ is called a phase space of equation (6), if*
(i) every solution $x = x(t)$ of (6) lies in $\mathfrak{P}$, i.e. $x(t) \in \mathfrak{P} \; \forall t \in \mathbb{R}$,
(ii) for arbitrary $x_m \in \mathfrak{P}, m = \overline{0, n-1}$ there exists a unique solution to (6), satisfying

$$x^{(m)}(0) = x_m, \; m = \overline{0, n-1}. \tag{7}$$

Theorem 2. [3] If the pencil $\vec{B}$ is polynomially (A, p)-bounded, and condition (A) is satisfied, then the phase space of equation (6) coincides with the image of projector P.

Proceed to linear inhomogeneous Sobolev type equation

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x + y. \tag{8}$$

Consider sets

$$\mathcal{M}_f^m = \{x \in \mathfrak{X} : \; (\mathbb{I} - P)x = -\sum_{l=0}^{p} K_l^n (B_0^0)^{-1} \frac{d^{l+m}}{dt^{l+m}} (\mathbb{I} - Q)y(0)\}, \text{ where } m = \overline{0, n-1}.$$

Theorem 3. [3] If the pencil $\vec{B}$ is polynomially (A, p)-bounded, condition (A) is satisfied, the function $y : [0, \tau) \to \mathfrak{Y}$ is such that $y^0 = (\mathbb{I} - Q)y \in C^{p+n}([0, \tau); \mathfrak{Y}^0)$ and $y^1 = Qy \in C([0, \tau); \mathfrak{Y}^1)$, then for arbitrary $x_m \in \mathcal{M}_f^m$, $m = \overline{0, n-1}$ there exists a unique solution to problem (7), (8) for $t \in [0, \tau)$ given by

$$x(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \frac{d^q}{dt^q} y^0(t) + \sum_{m=0}^{n-1} X_m^t P x_m + \int_0^t X_{n-1}^{t-s} (A^1)^{-1} y^1(s) ds. \quad (9)$$

Corollary 1. *If the pencil $\vec{B}$ is polynomially (A, p)-bounded, and condition (A) is satisfied, then for arbitrary $x_m \in \mathfrak{X}$, $m = \overline{0, n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique solution to problem (4), (8) for $t \in (-\tau, \tau)$ given by (9).*

Definition 9. *The vector-function $x \in H^n(\mathfrak{X}) = \{x \in L_2(0, \tau; \mathfrak{X}) : \ x^{(n)} \in L_2(0, \tau; \mathfrak{X})\}$ is called a strong solution to (8), if it turns the equation to an identity almost everywhere on interval $(0, \tau)$.*

This is correctly defined by virtue of the continuity of the embedding $H^n(\mathfrak{X}) \hookrightarrow C^{n-1}([0, \tau]; \mathfrak{X})$. The term "strong solution" has been introduced to distinguish a solution of (8) in this sense from classical solution. Note that classical solution (9) is also a strong solution to problem (8).

Construct the spaces

$$H^{p+n}(\mathfrak{Y}) = \{v \in L_2(0, \tau; \mathfrak{Y}) : v^{(p+n)} \in L_2(0, \tau; \mathfrak{Y}), p \in \{0\} \cup \mathbb{N}\}.$$

The space $H^{p+n}(\mathfrak{Y})$ is a Hilbert space with inner product

$$[v, w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{Y}} dt.$$

Let $y \in H^{p+n}(\mathfrak{Y})$. Introduce the operators

$$A_1 y(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \frac{d^q}{dt^q} y^0(t),$$

$$A_2 y(t) = \int_0^t X_{n-1}^{t-s} (t-s) (A^1)^{-1} y^1(s) ds, t \in (-\tau, \tau),$$

and the function

$$k(t) = \sum_{m=0}^{n-1} X_m^t P x_m.$$

Lemma 3. [3] *If the pencil $\vec{B}$ is polynomially (A, p)-bounded, and condition (A) is satisfied, then*

(i) $A_1 \in \mathcal{L}(H^{p+n}(\mathfrak{Y}); H^n(\mathfrak{X}))$;

(ii) for arbitrary $x_m \in \mathcal{M}_y^m$, the vector function $k \in C^n([0, \tau); \mathfrak{X})$;

(iii) $A_2 \in \mathcal{L}(H^{p+n}(\mathfrak{Y}); H^n(\mathfrak{X}))$.

Theorem 4. [19] *If the pencil $\vec{B}$ is polynomially (A,p)-bounded, condition (A) is satisfied, then for arbitrary $x_m \in \mathcal{M}_y^m, m = \overline{0,n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique strong solution to (7), (8).*

Corollary 2. *If the pencil $\vec{B}$ is polynomially (A,p)-bounded, condition (A) is satisfied, then for arbitrary $x_m \in \mathfrak{X}$ $m = \overline{0,n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique strong solution to (4), (8).*

3. Optimal Control

Introduce the control space

$$\overset{\circ}{H}{}^{p+n}(\mathfrak{U}) = \{u \in L_2(0,\tau;\mathfrak{U}) : u^{(p+n)} \in L_2(0,\tau;\mathfrak{U}), u^{(q)}(0) = 0, q = \overline{0,p}\},$$

$p \in \{0\} \cup \mathbb{N}$. It is a Hilbert space with inner product

$$[v,w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{U}} dt.$$

In the space $\overset{\circ}{H}{}^{p+n}(\mathfrak{U})$ we single out a closed convex subset $\overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, which will be called the set of admissible controls.

A vector function $\hat{u} \in \overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$ is called an optimal control of solutions to (3), (4), if relation (5) holds.

Our aim is to prove the existence of a unique control $\hat{u} \in \overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, minimizing the penalty functional

$$J(x,u) = \mu \sum_{q=0}^{n} \int_0^\tau ||x^{(q)} - \tilde{x}^{(q)}||^2 dt + \nu \sum_{q=0}^{p+n} \int_0^\tau \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt. \tag{10}$$

Here $\mu, \nu > 0$, $\mu + \nu = 1$, $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0, 1, \ldots, p+n$, are self-adjoint positively defined operators, and $\tilde{x}(t)$ is the target state of the system.

Theorem 5. [19] *If the pencil $\vec{B}$ is polynomially (A,p)-bounded, and condition (A) is satisfied, then for arbitrary $x_m \in \mathcal{M}_y^m, k = \overline{0,n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique optimal control to solutions of (3), (7).*

Corollary 3. *If the pencil $\vec{B}$ is polynomially (A,p)-bounded, and condition (A) is satisfied, then for arbitrary $x_m \in \mathfrak{X}$ $m = \overline{0,n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique optimal control to solutions of (3), (4).*

In order to reduce (1), (2) to (3) put

$$\mathfrak{X} = \{x \in W_2^{l+2}(\Omega) : x(s) = 0, s \in \partial\Omega\}, \quad \mathfrak{Y} = W_2^l(\Omega),$$

where $W_2^l(\Omega)$ is Sobolev space. Define operators $A = \Delta - \lambda$, $B_2 = (\lambda' - \Delta)$, $B_0 = -\alpha\frac{\partial^2}{\partial x_3^2}$, $B_3 = B_1 = \mathbb{O}$. Operators $A, B_3, B_2, B_1, B_0 \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ for all $l \in \{0\} \cup \mathbb{N}$.

Denote the eigenfunctions of the Dirichlet problem (2) for the Laplace operator by $\varphi_{ijk} = \left\{\sin\frac{\pi i x_1}{a}\sin\frac{\pi j x_2}{b}\sin\frac{\pi k x_3}{c}\right\}$, i, j, $k \in \mathbb{N}$, and denote the eigenvalues by $\lambda_{ijk} = -\left(\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2 + \left(\frac{\pi k}{c}\right)^2\right)$. Since $\{\varphi_{ijk}\} \subset C^\infty(\Omega)$, we obtain

$$\mu^4 A - \mu^3 B_3 - \mu^2 B_2 - \mu B_1 - B_0 =$$

$$= \sum_{i,j,k=1}^{\infty} [(\lambda_{ijk} - \lambda)\mu^4 + (\lambda_{ijk} - \lambda')\mu^2 - \alpha\left(\frac{\pi k}{c}\right)^2] < \varphi_{ijk}, \cdot > \varphi_{ijk},$$

where $< \cdot, \cdot >$ is the inner product in $L^2(\Omega)$.

Lemma 4. [21] *Let one of the following conditions be fulfilled:*
(i) $\lambda \notin \sigma(\Delta)$*;*
(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$.
Then the pencil $\vec{B}$ *is polynomially* $(A, 0)$*-bounded, and condition* (A) *is satisfied.*

The A-spectrum of pencil $\vec{B}$ is made up of solutions μ_{ijk} to the equation

$$(\lambda_{ijk} - \lambda)\mu^4 + (\lambda_{ijk} - \lambda')\mu^2 - \alpha\left(\frac{\pi k}{c}\right)^2 = 0. \tag{11}$$

Construct a projector P:

$$P = \begin{cases} \mathbb{I}, \text{ if (i) is fulfilled;} \\ \mathbb{I} - \sum\limits_{\lambda_{ijk}=\lambda} \langle \cdot, \varphi_{ijk}\rangle \varphi_{ijk}, \text{if (iii) is fulfilled.} \end{cases}$$

The Showalter–Sidorov conditions take the form

$$\begin{array}{c} \sum_{\lambda_{ijk}\neq\lambda} < x(\cdot, 0) - x_0, \varphi_{ijk} > \varphi_{ijk} = 0, \\ \sum_{\lambda_{ijk}\neq\lambda} < x_t(\cdot, 0) - x_1, \varphi_{ijk} > \varphi_{ijk} = 0, \\ \sum_{\lambda_{ijk}\neq\lambda} < x_{tt}(\cdot, 0) - x_2, \varphi_{ijk} > \varphi_{ijk} = 0, \\ \sum_{\lambda_{ijk}\neq\lambda} < x_{ttt}(\cdot, 0) - x_3, \varphi_{ijk} > \varphi_{ijk} = 0. \end{array} \tag{12}$$

Theorem 6. *Let the assumptions of Lemma 4 be satisfied. Then for arbitrary* $x_m \in \mathfrak{X}$, $m = \overline{0, 3}$, *there exists a unique optimal control of the solutions* $(\hat{x}, \hat{y})$ *for the equation (1) with conditions (2), (12), minimizing functional (10).*

Proof.
By Lemma 4 the pencil $\vec{B}$ is polynomially $(A, 0)$-bounded condition (A) is satisfied, then theorem 5 for the given problem is valid.

□

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Alyona A. Zamyshlyaeva, DSc (Math), Head of the Department of Applied Mathematics and Computer Science, South Ural State University (Chelyabinsk, Russian Federation), zamyshliaevaaa@susu.ru.

Olga N. Tsyplenkova, PhD (Math), Department of Equations of Mathematical Physics, South Ural State University (Chelyabinsk, Russian Federation), tcyplenkovaon@susu.ru.

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ РЕШЕНИЯМИ ЗАДАЧИ ШОУОЛТЕРА–СИДОРОВА В МОДЕЛИ ЛИНЕЙНЫХ ВОЛН В ПЛАЗМЕ

А. А. Замышляева, О. Н. Цыпленкова

В работе исследована задача оптимального управления для уравнения соболевского типа высокого порядка в предположении относительно полиномиальной ограниченности пучка операторов. Результаты применены к исследованию оптимального управления решениями задачи Шоуолтера–Сидорова для модели линейных волн в плазме во внешнем магнитном поле. Условия Шоуолтера–Сидорова являются обобщением условий Коши. Как известно, задача Коши для уравнений соболевского типа является принципиально неразрешимой при произвольных начальных значениях. В работе применяется метод фазового пространства, разработанный Г. А. Свиридюком, теория относительно полиномиально ограниченных пучков операторов, разработанная А. А. Замышляевой. Математическая модель, рассмотренная в статье, описывает ионно-звуковые волны в плазме во внешнем магнитном поле, впервые была получена Ю. Д. Плетнером.

Ключевые слова: уравнения соболевского типа высокого порядка; модель линейных волн в плазме; задача Шоуолтера–Сидорова; относительно полиномиально ограниченный пучок операторов; сильные решения; оптимальное управление.

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Замышляева Алена Александровна, доктор физико-математических наук, доцент, заведующий кафедрой прикладной математики и информатики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), zamyshliaevaaa@susu.ru.

Цыпленкова Ольга Николаевна, кандидат физико-математических наук, доцент кафедры уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), tcyplenkovaon@susu.ru.

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