SHORT NOTES

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FINITE DIFFERENCE METHOD FOR MODIFIED BOUSSINESQ EQUATION

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In this paper a numerical of the solution of the Cauchy problem for the nonlinear modified Boussinesq equation (or IMBq equation) is studied. This equation with the boundary conditions models the propagation of waves in shallow water, taking into account capillary effects and the preservation of mass in the layer, filtration of water in the soil, as well as shock waves. In the case when the equation is nondegenerate, a global solution and a solution in the form of solitons are obtained. In the degenerate case, the existence of a unique local solution was proved by the methods of phase space and the theory of relatively limited ones developed by G. Sviridyuk and his students, as well as the theory of differentiable Banach manifolds. A numerical study of this problem by the modified Galerkin method has already been carried out earlier. However, the operation time of algorithms based on modified Galerkin method it rapidly increases with an increase amount of Galerkin sum. In this article, a numerical study is carried out by the finite difference method. The Cauchy – Dirichlet problem for the IMBq equation is reduced to an implicit difference problem. A comparison is made of the speed of the modified Galerkin method and the finite difference method.

Keywords: Sobolev type equations; finite difference method; Galerkins method.

Introduction

The paper will carry out a numerical study of the mathematical model of wave propagation in shallow water using the Galerkin method and the grid method. A mathematical model of wave propagation in shallow water is based on the modified Boussinesq equation (IMBq equation), the Dirichlet boundary condition, and for unique solvability we add the initial Cauchy condition. In addition, the following conditions are imposed on the process: the wavelength is much greater than the depthat a standstill, capillary effects can not be neglected, the liquid is incompressible, the bottom surface is solid and flat [1]. Thus, we get the problem

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1), \tag{1}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T), \tag{2}$$

$$(\lambda - \Delta)u_{tt} = \alpha^2 \Delta u + \Delta(u^3), \tag{3}$$

where function $u = u(x,t)$ is a wave height at x at time t, $\Delta u = u_{xx}$ is a one-dimensional Laplace operator. The constants α, λ – characterize such fluid and medium parameters as depth, gravitational constant, and Bond number. This mathematical model relates to mathematical models of Sobolev type of high order [2]. In [1], equation (3) was investigated in the case of a non-degenerate case, when $\lambda \notin \sigma(\Delta)$, particular solutions were found in the form of solitons, and it is noted that in a more general case, this problem allows studying collisions plane waves. In [3], problem (1)–(3) was studied in the case when $\lambda \notin \sigma(\Delta)$, and the existence of uniqueness of the global solution was proved.

The problem (1)–(3) in suitable spaces can be reduced to the Cauchy problem

$$u(0) = u_0, \dot{u}(0) = u_1,$$

for semilinear sobolev type equation of the second order

$$L\ddot{u} = Mu + N(u),$$

where operators $L, M \in \mathcal{L}(\mathfrak{U};\mathfrak{F}), N \in C^{\infty}(\mathfrak{U};\mathfrak{F})$, $\mathfrak{U}$, $\mathfrak{F}$ are Banach spaces. In [4], the existence of a unique local solution of problem (1)–(3) was proved, when equation (3) is an Sobolev type equation [5], and solution u of class $C^{\infty}(0, \tau, W_2^m(\Omega))$, in our case $\Omega = (0, 1)$. The applicability of the grid method to solving linear Sobolev type equations in special cases was shown in [6].

1. Implicit Difference Scheme

On the rectangle $[0, T] \times [0, 1]$, we introduce a uniform three–layer with a step h in the variable x and with a step in s in the variable t a grid

$$w_{i,j} = \{(x_i, t_j) : x_i = ih, t_j = js, i = 0, 1, \dots, n, j = 0, 1, \dots, m\}.$$

We introduce the grid function $u_{ij} = u(x_i, t_j)$ on the grid $w_{i,j}$. On the grid we define linear normed space with norm $\|u_{hs}\| = \max\limits_{0\le i\le n}\left|\max\limits_{0\le j\le m}|u_{ij}|\right|$.

We write problem (1)–(3) in a differential form. We approximate the derivatives using a five–point cross–type pattern

$$u_t(x_i, t_j) = \frac{u_{i,j+1} - u_{i,j}}{s} + O(s),$$

$$u_{tt}(x_i, t_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{s^2} + O(s^2),$$

$$u_{xx}(x_i, t_j) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2).$$

Thus, equation (3) takes the form

$$\frac{1}{h^2s^2}u_{i+1,j+1} + \left(\alpha^2\mathfrak{s}^{-}{}_{2}h^2s^2\right)u_{i+1,j} + \frac{1}{h^2s^2}u_{i+1,j-1} - \left(\frac{2}{h^2s^2} + \lambda\right)u_{i,j+1} -$$

$$-\left(\frac{2\alpha^2s^2 - 2\lambda h^2 - 4}{h^2s^2}\right)u_{i,j} - \left(\frac{2}{h^2s^2} + \lambda\right)u_{i,j-1} + \frac{1}{h^2s^2}u_{i-1,j-1} +$$

$$+\frac{1}{h^2}((u_{i+1,j})^3 - 2(u_{i,j})^3 + (u_{i-1,j})^3) = 0$$

or in operator form

$$(Lu)_{ij} + N(u)_{ij} = 0. \tag{4}$$

The boundary conditions (2) take the form

$$u_{0,j} = u_{n,j} = 0 \tag{5}$$

and initial conditions (3)

$$u_{i,0} = u_0(x_i), \quad u_{i,1} = u_0(x_i) + s u_1(x_i). \tag{6}$$

The boundary and initial conditions are approximated exactly, therefore, in general, the scheme has the second order of approximation.

2. Numerical Research

A series of computational experiments were carried out in which the same problem was solved by the finite difference method and the modified Galerkin method. We present the results of two experiments. The first shows that the solutions obtained by various methods are close. The second shows that if the split points of a region are chosen incorrectly using the finite difference method, a solution different from the Galerkin solution can be obtained.

Example 1. In a cylinder $[0,1] \times [0,2]$ consider problem (1)–(3) when $\alpha = 1, \lambda = 1$.

$$(1-\Delta)\ddot{u} = \Delta u + \Delta(u^3), \tag{7}$$

$$u(0,t) = u(1,t) = 0, \tag{8}$$

$$\begin{aligned} u(x,0) &= 2\sin(x) + 11\sin(2x) - 3\sin(3x) + 4\sin(4x), \\ \dot{u}(x,0) &= \sin(x) - \sin(2x) + 10\sin(3x) - 5\sin(4x). \end{aligned} \tag{9}$$

For given parameters, equation (7) is not a Sobolev type equation. We will solve it by the grid method for $n = 10$, $m = 1500$ and construct a graph of the approximate solution of fig. 1. As well as using the modified Galerkin method $m = 4$ (the number of Galerkin terms) and construct a graph of the solution in fig. 2 when building a solution using the grid method and the Galerkin method, the same steps $s = 0,2$ were used with respect to the variable t and $h = \frac{\pi}{1500}$, $h = 0.002$ by variable x, respectively. The processor time spent in both cases is comparable and averaged 45 sec and 30 sec. The calculations were carried out in the Maple system on an FX-6300 processor.

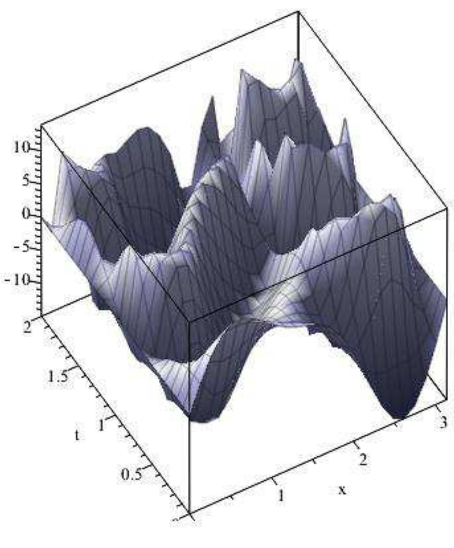

Fig. 1. Approximate solution graph

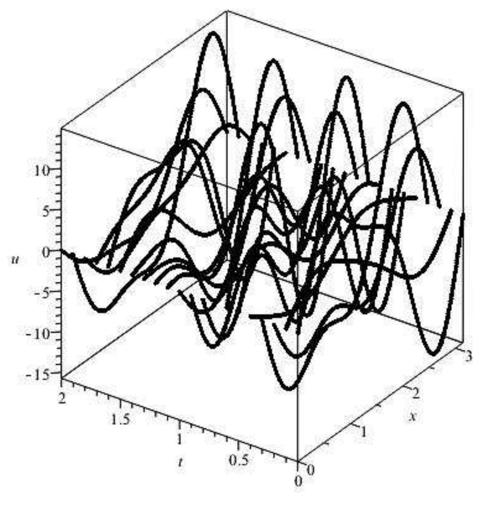

Fig. 2. Approximate solution graph

Example 2. In a cylinder $[0,1] \times [0,2]$ consider problem (1)–(3) when $\alpha = 1, \lambda = -1$.

$$(-1 - \Delta)\ddot{u} = \Delta u + \Delta(u^3), \tag{10}$$

$$u(0,t) = u(1,t) = 0, \tag{11}$$

$$\begin{aligned} u(x,0) &= 11\sin(2x) - 3\sin(3x) + 4\sin(4x), \\ \dot{u}(x,0) &= \sin(2x) + 10\sin(3x) - 5\sin(4x). \end{aligned} \tag{12}$$

The equation (10) is a Sobolev type equation for given parameters. Therefore, the initial data (12) must be chosen so that they belong to the phase space [7] of equation (10). We will solve it by the grid method for $n = 10$, $m = 1500$ and construct a graph of the approximate solution of fig. 3. As well as using the modified Galerkin method $m = 4$ (the number of Galerkin terms) and plot the graph of the solution in fig. 4. When we construct a graph of approximate solution by the finite difference method and by the Galerkin method, we used step $s = 0,2$ with respect to the variable t in both case and steps $h = \frac{\pi}{1500}$, $h = 0,002$ by variable x, respectively. When we constructing an approximate solution using the grid method on the time layer $t = 1,4$, the system of algebraic equations has no solution. At the same time, the solution by the Galerkin method exists. However, if we take the grid step with the variable x less, for example, $h = \frac{\pi}{1000}$, then the solution will already exist on all time layers in the segment $[0,2]$.

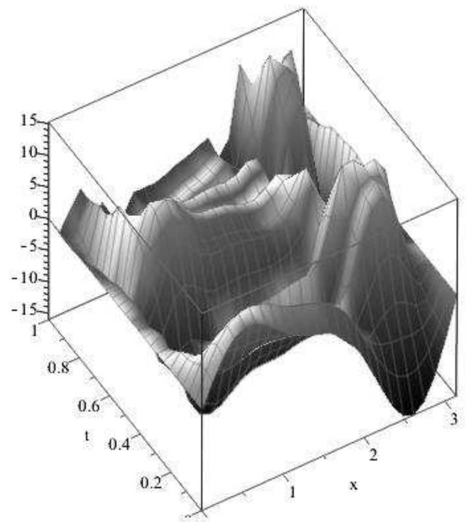

Fig. 3. Approximate solution graph

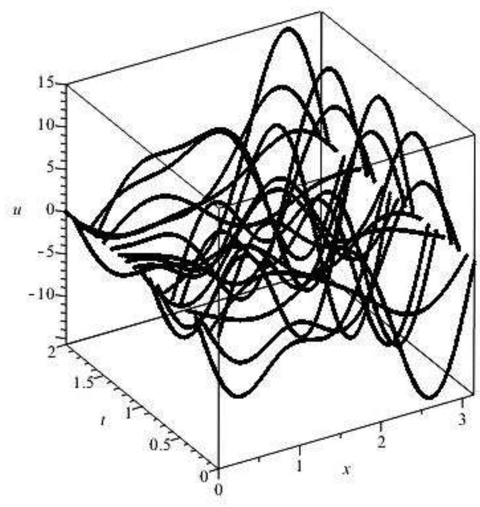

Fig. 4. Approximate solution graph

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МЕТОД КОНЕЧНЫХ РАЗНОСТЕЙ ДЛЯ МОДИФИЦИРОВАННОГО УРАВНЕНИЯ БУССИНЕСКА

Е. В. Бычков

В статье проведено численное исследование решения задачи Коши для нелинейного модифицированного уравнения Буссинеска (или IMBq уравнения). Данное уравнение вместе с краевыми условиями моделирует распространение волн на мелкой воде, с учетом капиллярных эффектов и сохранении массы в слое, фильтрации воды в грунте, а также ударных волн. В том случае, когда уравнение невырожденно получено глобальное решение и решение в виде солитонов. В вырожденном случае было доказано существование единственного локального решения методами фазового пространства и теории относительно ограниченных, разработанных Свиридюком Г.А. и его учениками, а также теории дифференцируемых банаховых многообразий. Ранее уже проводилось численное исследование данной задачи модифицированным методом Галеркина. Однако, время работы алгоритмов основанных на модифицированном методе Галеркина быстро возрастает при увеличении количества Галеркинских слагаемых. В данной статье численное исследование проводится методом конечных разностей. Задача Коши – Дирихле для IMBq уравнения редуцируется к неявной разностной задаче. Проводится сравнение скорости работы модифицированного метода Галеркина и метода конечных разностей.

Ключевые слова: уравнения соболевского типа; метод конечных разностей; метод Галеркина.

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