COMPUTATIONAL MATHEMATICS

MSC 35F99

UZAWA ALGORITHM IMPLEMENTATION FOR STEADY INCOMPRESSIBLE NEWTONIAN LIQUIDS

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In this article Uzawa algorithm for steady incompressible Newtonian liquids was implemented. The flow model of these liquids is described by Navier − Stokes equation. Uzawa method involves the Delaunay triangulation of a set and computation of values in the middle of every triangle's edge. The method is iterative and the proper implicit scheme that describes the flow of an incompressibe Newtonian liquid is introduced. For the computational experiment the centrifuge model was taken. The abstract example is about stiring the incompressible Newtonian liquid inside the centrifuge. The result of the computational experiment corresponds to practise: the pressure increase towards the wall, the lowest pressure is in the middle. The results of this research will be helpful for the further research of steady incompressible Non-Newtonian liquids in the same condition.

Keywords: Mathematical Physics Equations, Partial Differential Equations, Newtonian fluid, Uzawa algorithm

Introduction

The problem of Stokes equations approximation was considered in [1] – [4]. It is claimed in [5] that discretization of the Stokes equations does not solve completely the problem of numerical approximation of these equations. In the following sections we will try to implement the classical algorithm of optimization introduced by Uzawa [6]. The proof of convergence result from optimization theory.

Firstly, we will introduce the Theorem from [5].

Let Ω be an open set of $\mathbb{R}^n$ with boundary Γ. $H_0^1(\Omega)$ is a Hilbert space for the scalar product

$$\sum_{i=1}^{n}(\nabla u_i, \nabla v_i) = ((u, v)). \tag{1}$$

Let $f \in L^2(\Omega)$ be a given vector function in Ω. We seek a vector function $u = (u_1, ..., u_n)$ representing the velocity of the fluid, and a scalar function p representing the pressure, which are defined in Ω and satisfy the following equations and boundary conditions

$$-\nu\Delta u + \nabla p = f \; in \; \Omega \;\; (\nu > 0), \tag{2}$$

$$\text{div u} = 0 \;\; \text{in} \; \Omega, \tag{3}$$

$$u = 0 \; on \; \Gamma. \tag{4}$$

There in [5] it appears the following statement

$$u \text{ belongs to } V \text{ and satisfies } \nu((u, v)) = (f, v), \forall \nu \in V. \tag{5}$$

Lemma 1. *Let Ω be an open bounded set of class C^2. The following conditions are equivalent*

(i) $u \in V$ satisfies (5).

(ii) u belongs to $H_0^1(\Omega)$ and satisfies (2) – (4) *in the following weak sense*

$$\text{there exists } p \in L^2(\Omega) \text{ such that } -\nu\Delta u + \nabla p = f$$

$$\text{in the distribution sense in } \Omega; \tag{6}$$

$$\text{div u} = 0 \text{ in the distribution sense in } \Omega; \tag{7}$$

$$\xi_0 u = 0. \tag{8}$$

Theorem 1. *For any open set Ω which is bouded in some direction, and for every $f \in L^2(\Omega)$, the problem* (5) *has a unique solution u. (The result is also valid if f is given in $H^{-1}(\Omega)$.) Moreover, there exists a function $p \in L_l^2oc(\Omega)$ such that* (2) – (3) *are satisfied. If Ω is an open bounded set of class C^2, then $p \in L^2(\Omega)$ and* (2) – (4) *are satisfied by u and p.*

In next several sections we shall discuss the variety of equations, ordered by complexity, the Uzawa method is used for. The complete computational experiment will be provided in 2-dimensional case and the results will be shown.

1. Function value on a Triangle

Let u and p be the functions defined by Theorem 1; we will obtain u, p as limits of sequences u^m, p^m which are much easier to compute than u and p.

The algorithm starts with an arbitrary element p^0,

$$p^0 \in L^2(\Omega) \tag{9}$$

When p^m is known, we define u^{m+1} and p^{m+1} ($m \geq 0$), by the conditions $u^{m+1} \in H_0^1(\Omega)$ and

$$\nu((u^{m+1}, v)) - (p^m, divv) = (f, v),\ \forall v \in H_0^1(\Omega), \tag{10}$$

$$p^{m+1} \in L^2(\Omega) \text{ and}$$

$$(p^{m+1} - p^m, q) + \rho(div\, u^{m+1}, q) = 0,\ \forall q \in L^2(\Omega). \tag{11}$$

We suppose that $\rho > 0$ is a fixed number.

The algorithm of Uzawa based on a triangulation of the set Ω. We will consider the values of the function u on midpoints of each triangle T.

Let T be a triangle with vertices M_1, M_2, M_3, mid-edges P_1, P_2, P_3, and let ν_1, ν_2, ν_3 denote the corresponding basis functions ($\nu_i\ (P_i) = \sigma_{ij}$). Let us introduce $\hat{T}$, a reference triangle, and the linear mapping transforming $\hat{T}$ into T (fig. 1, 2).

Suppose T_n is the set of all triangles to triangulate Ω. $\pi_n^0 : T_n \to \mathbb{R}$ is the initial approximation to pressure. $\Omega_h = \bigcup_{T \in T_h} T$ is the initial approximation to the set Ω after triangulation, $\partial\Omega_h$ is the boundary,

transforms into reference triangle $\hat{T}$,

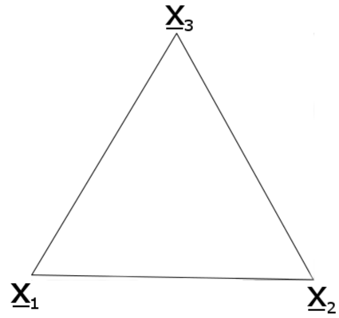

Fig. 1. Original triangle

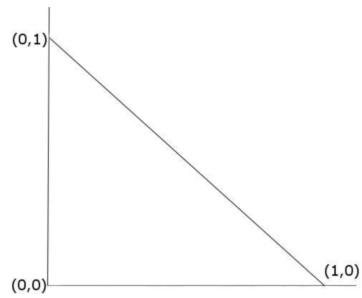

Fig. 2. Reference triangle

$$\underline{x} = \underline{x_1} + \left(\underline{x_2} - \underline{x_1} \,|\, \underline{x_3} - \underline{x_1}\right) \underline{X}, \tag{12}$$

where $\underline{x}$ is a coordinate vector corresponding to the triangle T and $\underline{X}$ is a coordinate vector corresponding to the reference triangle $\hat{T}$. Now we need to construct the basis functions. Let λ_i denote the barycentric coordinates in T with respect to the $\underline{X}_j$. We have

$$\nu_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad \nu_2 = \lambda_2 + \lambda_3 - \lambda_1, \qquad \nu_3 = \lambda_3 + \lambda_1 - \lambda_2,$$

where ν_1, ν_2 and ν_3 are corresponding basis functions. On the other hand,

$$\lambda_1(\underline{X}) = 1 - X_1 - X_2, \;\; \lambda_2(\underline{X}) = X_1, \;\; \lambda_3(\underline{X}) = X_2,$$

where X_1 and X_2 are coordinates of the midpoint on the reference triangle $\hat{T}$.

Now, there are three possible midpoints for each side of the triangle.

Case 1. The point belongs to the bottom edge.

$$\nu_1 = 1 - 2X_2.$$

Case 2. The point is on the left edge.

$$\nu_2 = 1 - 2X_1.$$

Case 3. The point belongs to hypothenuse.

$$\nu_3 = 2(X_1 + X_2) - 1$$

For the system of triangles the following conditions are fulfilled:

- Pressure p is constant on each triangle
- $\underline{u}$ is the sum of basis functions

$$\underline{u} = \sum_{i=1}^{N_m} \begin{pmatrix} U_i \\ V_i \end{pmatrix} w_i,$$

where N_m is the number of internal points.
- w_i is defined on the neighbouring triangles to mid-point 'i', it is zero everywhere else.

Now we can find the function w (fig. 3).

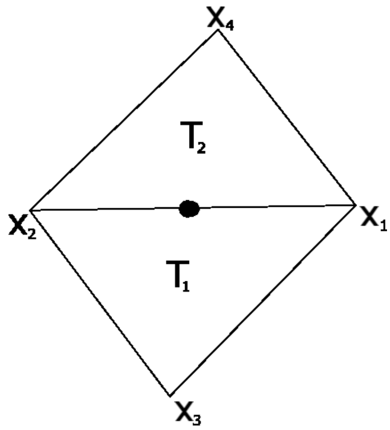

Fig. 3. Neighbouring triangles

We have two linear mappings transforming T_1 into $\hat{T}$ and T_2 into $\hat{T}$

$$\gamma_1 = \underline{x_3} + (\underline{x_1} - \underline{x_3} \mid \underline{x_2} - \underline{x_3})\underline{X}, \tag{13}$$

and

$$\gamma_2 = \underline{x_4} + (\underline{x2} - \underline{x_4} \mid \underline{x1} - \underline{x_4})\underline{X}. \tag{14}$$

$$w(T_1) = \nu(\gamma_1^{-1}(\underline{x})), \;\; w(T_2) = \nu(\gamma_2^{-1}(\underline{x})).$$

Depending on a position of the considered midpoint, w can be different, as ∇w.

2. Implementation for steady incompressible Newtonian liquids

Now we can derive the new equation using Navier – Stokes equation.
For the example of Stokes equation, described in [5], we had following equation

$$\int\limits_{\Omega_h} -w_k \nabla p + w_k \mu \nabla^2 \underline{u} + w_k f d^2 \underline{x} = \underline{0}.$$

In the case of steady Newtonian liquid the equation will be different

$$\int\limits_{\Omega_h} -w_k \nabla p + w_k \mu \nabla^2 \underline{u} + w_k f d^2 \underline{x} = \rho \left(\underline{u} \nabla \underline{u} \right) w_k.$$

For this equation we can obtain the following implicit scheme

$$\rho \int_{\Omega_h} w_k \underline{u}_h^n \nabla \underline{u}_h^{n+1} d^2\underline{x} + \mu \int_{\Omega_h} \nabla w_k \nabla \underline{u}_h^{n+1} d^2\underline{x} = \sum_T \mathrm{area(T)} \pi_\mathrm{h}^\mathrm{n}(\mathrm{T}) \nabla \mathrm{w_k} + \int_{\Omega_\mathrm{h}} \underline{\mathrm{f}} \mathrm{w_k} \mathrm{d}^2 \underline{\mathrm{x}}. \quad (15)$$

Also obtain the equivalent of this equation in radial coordinate system:

$$-\rho \frac{u^2(r)}{r} \underline{e_r} = -\frac{\partial p}{\partial r} \underline{e_r} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} u(r) - \frac{u}{r^2} \right) \underline{e_\theta} + f(r) \underline{e_\theta},$$

and finally, split the equation in two

$$\mu \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + f = 0$$

and

$$\frac{\partial p}{\partial r} = \rho \frac{u^2}{r}.$$

The force F should be

$$F = (-r \sin\theta, r \cos\theta)$$

This information is enough to make a computational experiment.

3. Computational experiment

To produce a computational experiment we compiled a code using Matlab with $\rho = 1$, $\nu = 1$, $\mu = 1$.

The result of computational experiment is shown below (fig. 4, 5)

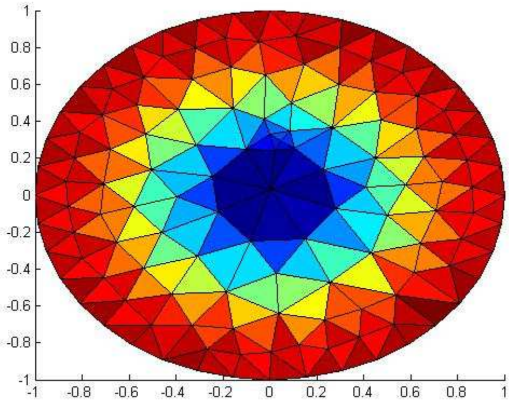

Fig. 4. Pressure distribution

The result of the computational experiment corresponds to practise: the pressure increase towards the wall, the lowest pressure is in the middle.

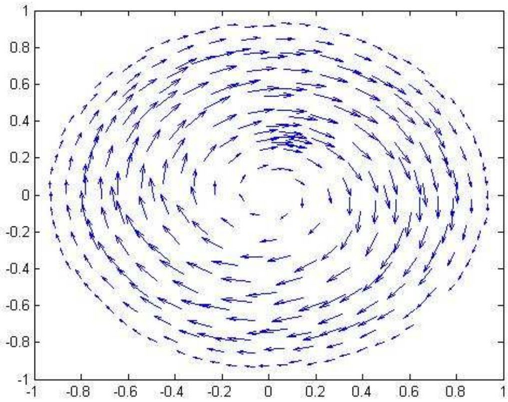

Fig. 5. Flow direction

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