

# SHORT NOTES

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## ON MODELLING THE CONVECTING POLAR IONOSPHERE

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In addition to the model of polar ionosphere in terms of ordinary differential equation with random coefficients we construct and investigate two more models: described in terms of a stochastic differential equation in Ito form and in terms of stochastic equation with current velocities (symmetric Nelson's mean derivatives). The existence of solution theorems for those equations are proved.

*Keywords: models of polar ionosphere; mean derivatives; current velocities; stochastic equations in Ito form.*

### Introduction and Setting up the Problem

Traditional models of distribution of electron concentration  $N_e$  in the  $F$ -region of polar ionosphere are based on two determining factors: ionization  $q$  and the large-scale electrical field of magnetospheric convection generating the transfer  $\bar{v}$  of ionospheric plasma [1]. Note that  $q$  takes two values: at day time it is a positive constant (without loss of generality we can set it equal to 1) and at night time it is equal to zero. Thus, in the polar day we consider it as 1 and in polar night as zero. The vector  $\bar{v}$  is equal to  $\frac{E \times B}{B^2}$  where  $E$  and  $B$  are electric and magnetic strengths, respectively. Everything is considered over some neighbourhood of the North Pole, so that  $x$  is a two-dimensional vector with coordinates  $x^1$  and  $x^2$ . The same notation for coordinates we keep for coordinates of vector  $\bar{v}$ ,  $w(t)$ ,  $dw(t)$ , etc.

In [2] the continuity equation  $\frac{\partial N_e}{\partial t} + \bar{v} \cdot \nabla N_e = q - \beta N_e$  is considered with the assumption that  $\bar{v}$  is the sum of the deterministic summand (for which we keep the notation  $\bar{v}$ ) and the stochastic summand that is supposed in [2] to be the Wiener process  $w(t)$  (the Brownian motion) with a certain real function coefficient  $\sigma(x)$ ,  $\beta$  is the recombination coefficient, and  $q$  is the ionization (see above). The function  $\sigma$  may be piece-wise constant. Thus the equation describing this model takes the form

$$\frac{\partial N_e}{\partial t} + (\bar{v}(t, x) + \sigma(x)w(t)) \cdot \nabla N_e = q - \beta N_e \quad (1)$$

or in coordinate form

$$\frac{\partial N_e}{\partial t} + (\bar{v}^1 + \sigma w^1(t)) \frac{\partial N_e}{\partial x^1} + (\bar{v}^2 + \sigma w^2(t)) \frac{\partial N_e}{\partial x^2} = q - \beta N_e. \quad (2)$$

The main aim of this paper is to pass from ordinary differential equation (1) and (2) with random coefficients to stochastic differential equations in the Ito form and

to stochastic differential equations with current velocities (Nelson's symmetric mean derivatives). In the framework of stochastic analysis such equations are considered as more adequately describing the behaviour of physical processes. Note that the current velocities are natural analogues of ordinary physical velocity of deterministic processes. We construct such equations and prove the existence of solution theorems for them.

At the present moment the analytical formulae for the solutions of those equations cannot be found, but the numerical methods (and in particular, the Monte Carlo method) can be applied. This will be done in the forthcoming publications together with comparison with the results of experiments. After that it will be possible to understand which model and in which situation gives the adequate description of real  $N_e$ .

## 1. Model Based on Equation in Ito Form

We refer the reader to [3] for detailed description of the theory of stochastic differential equation. We transform (2) into

$$dN_e = \left( q - \beta N_e - v^1 \frac{\partial N_e}{\partial x^1} - v^2 \frac{\partial N_e}{\partial x^2} \right) dt - \sigma \frac{\partial N_e}{\partial x^1} dw^1(t) - \sigma \frac{\partial N_e}{\partial x^2} dw^2(t). \quad (3)$$

Here the vector  $\nabla N_e$  is considered as a linear operator sending  $\mathbb{R}^2$  where the Wiener process is given, to  $\mathbb{R}^1$  where the solution  $N_e$  lives.

Note that all the models and equations under consideration are applicable for describing the ionosphere only over a small enough neighbourhood of the North Pole. That is why we can suppose that the domain of the coefficients in (3) is compact. To avoid investigating the behaviour of the processes on the boundary, we may consider the domain as flat torus. We suppose all functions in (3) to be smooth and since the domain is compact, they all together with their derivatives are bounded.

Denote by  $\mathcal{F}_t$  the filtration that defines the Wiener process. Consider equation (3) either with  $q = 1$  or with  $q = 0$ .

**Theorem 1.** *For every initial condition  $N_e(0) = N_e^0$  where  $N_e^0$  is measurable with respect to  $\mathcal{F}_0$ , equation (3) has a unique strong solution well-defined for all  $t \geq 0$ .*

The proof follows from the fact that under the above assumptions equation (3) satisfies all conditions of existence and uniqueness of strong solution theorem for stochastic differential equations in Ito form [3].

## 2. Model Based on Current Velocities

We refer the reader, e.g., to [4] for rather detailed introduction to the theory of equations with mean derivatives. But since this theory is much less known than the Ito equations, we have introduce to some definition and facts from this theory here.

Consider a stochastic process  $\xi(t)$  in  $\mathbb{R}^n$ ,  $t \in [0, T]$ , given on a certain probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and such that  $\xi(t)$  is an  $L_1$  random element for all  $t$ . Denote by  $\mathcal{N}_t^\xi$  the minimal  $\sigma$ -subalgebra of  $\mathcal{F}$  that contains the preimages of Borel sets from  $\mathbb{R}^n$  under the mapping  $\xi(t) : \Omega \rightarrow \mathbb{R}^n$ . This  $\sigma$ -subalgebra is supposed to be complete. We denote by  $E_t^\xi$  the conditional expectation  $E(\cdot | \mathcal{N}_t^\xi)$  with respect to  $\mathcal{N}_t^\xi$  for  $\xi(t)$ .

**Definition 1.** (i) The forward mean derivative  $D\xi(t)$  of  $\xi(t)$  at the time instant  $t$  is an  $L_1$  random element of the form

$$D\xi(t) = \lim_{\Delta t \rightarrow +0} E_t^\xi \left( \frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} \right), \quad (4)$$

where the limit is supposed to exist in  $L_1(\Omega, \mathcal{F}, \mathbf{P})$  and  $\Delta t \rightarrow +0$  means that  $\Delta t$  tends to 0 and  $\Delta t > 0$ .

(ii) The backward mean derivative  $D_*\xi(t)$  of  $\xi(t)$  at  $t$  is the  $L_1$ -random element

$$D_*\xi(t) = \lim_{\Delta t \rightarrow +0} E_t^\xi \left( \frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} \right) \quad (5)$$

where (as well as in (i)) the limit is assumed to exist in  $L^1(\Omega, \mathcal{F}, \mathbf{P})$  and  $\Delta t \rightarrow +0$  means that  $\Delta t \rightarrow 0$  and  $\Delta t > 0$ .

Introduce the differential operator  $D_2$  that differentiates an  $L_1$  random process  $\xi(t)$ ,  $t \in [0, T]$  according to the rule

$$D_2\xi(t) = \lim_{\Delta t \rightarrow +0} E_t^\xi \left( \frac{(\xi(t + \Delta t) - \xi(t))(\xi(t + \Delta t) - \xi(t))^*}{\Delta t} \right), \quad (6)$$

where  $(\xi(t + \Delta t) - \xi(t))$  is considered as a column vector (vector in  $\mathbb{R}^n$ ),  $(\xi(t + \Delta t) - \xi(t))^*$  is a row vector (transposed, or conjugate vector) and the limit is supposed to exist in  $L_1(\Omega, \mathcal{F}, \mathbf{P})$ . We emphasize that the matrix product of a column on the left and a row on the right is a matrix so that  $D_2\xi(t)$  is a symmetric semi-positive definite matrix function on  $[0, T] \times \mathbb{R}^n$ . We call  $D_2$  the quadratic mean derivative. Note that for a diffusion Markov process its quadratic mean derivative equals the diffusion coefficient.

**Remark 1.** From the properties of conditional expectation (see, e.g., [5]) it follows that there exist Borel mappings  $a(t, x)$ ,  $a_*(t, x)$  and  $\alpha(t, x)$  from  $R \times \mathbb{R}^n$  to  $\mathbb{R}^n$  and to the space of symmetric positive semi-definite matrices, respectively, such that  $D\xi(t) = a(t, \xi(t))$ ,  $D_*\xi(t) = a_*(t, \xi(t))$  and  $D_2\xi(t) = \alpha(t, \xi(t))$ . Following [5] we call  $a(t, x)$ ,  $a_*(t, x)$  and  $\alpha(t, x)$  the regressions.

Consider the vector fields

$$\begin{aligned} v^\xi(t, x) &= \frac{1}{2}(a(t, x) + a_*(t, x)), \\ u^\xi(t, x) &= \frac{1}{2}(a(t, x) - a_*(t, x)), \end{aligned}$$

where  $a(t, x)$  and  $a_*(t, x)$  are regressions for forward and backward derivatives, respectively.

**Definition 2.**  $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$  is called current velocity of  $\xi(t)$ ;  $u^\xi(t) = u^\xi(t, \xi(t)) = D_A\xi(t)$  is called osmotic velocity of  $\xi(t)$ .

Here  $D_S = \frac{1}{2}(D + D_*)$  and  $D_A = \frac{1}{2}(D - D_*)$ .

Let a Borel vector field  $a(t, x)$  and a Borel field of symmetric positive semi-definite matrices  $\alpha(t, x)$  be given on  $\mathbf{R}^n$ . The sustem of the form

$$\begin{cases} D_S\xi(t) = v(t, \xi(t)), \\ D_2\xi(t) = \alpha(t, \xi(t)), \end{cases} \quad (7)$$

where the equalities are fulfilled a.s., is called the first order equation with current velocities.

**Definition 3.** Equation (7) has a solution on the interval  $[0, T]$  if there exists a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and a process  $\xi(t)$  given on it for  $t \in [0, T]$ , so that (7) is satisfied.

For our model we introduce the equation with current velocities of the form

$$\begin{cases} D_S N_e = q - \bar{v} \cdot \nabla N_e - \beta N_e \\ D_2 N_e = \sigma^2 \left( \frac{\partial N_e}{\partial x^1} \right)^2 + \sigma^2 \left( \frac{\partial N_e}{\partial x^2} \right)^2 \end{cases} \quad (8)$$

As well as above we suppose that all functions in equation (8) are smooth. Since the domain is compact, they all together with their derivatives are bounded.

**Theorem 2.** Let the density  $\rho$  of the initial value of  $N_e$  be smooth and nowhere equal to zero. Then under the above assumptions there exists a solution of (8) with this initial condition.

*Proof.* In [6] an existence of solution theorem for equations with current velocities is proved under the assumption that  $\rho$  is smooth and nowhere equal to zero and under the assumptions of some inequalities for the right-hand sides of the equations together with their first partial derivatives. Thus, since all the functions in (8) are bounded together with their derivatives, the assertion of theorem follows from [6]. □

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## О МОДЕЛИРОВАНИИ КОНВЕКТИРУЮЩЕЙ ПОЛЯРНОЙ ИОНОСФЕРЫ

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В дополнение к модели полярной ионосферы в терминах обыкновенных дифференциальных уравнений со случайными коэффициентами мы строим и изучаем еще две модели, описанные в терминах стохастических дифференциальных уравнений в форме Ито и в терминах стохастических уравнений с текущими скоростями (симметрическими производными в среднем по Нельсону). Доказаны теоремы существования решений для указанных уравнений.

*Ключевые слова: модели полярной ионосферы; производные в среднем; текущие скорости; стохастические уравнения в форме Ито.*

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