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SOLUTION TO THE INITIAL-FINAL VALUE PROBLEM FOR A NON-STATIONARY LEONTIEF TYPE SYSTEM

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The article is devoted to the construction of a solution to the initial-final value problem for a non-stationary Leontief type system. Such systems take place in dynamic balance models of the economy. A distinctive feature of Leontief type systems is the degeneracy of the matrix at the time derivative, due to the fact that some types of resources of economic systems cannot be stored. In addition, dynamic balance systems of the economy are often described using time-dependent coefficients. We use resolving streams of matrices to construct solutions for such systems. In addition, the initial-final value condition is used instead of the standard initial condition. For economic systems, the initial-final value condition can be interpreted as taking into account not only indicators at the initial moment of time, but also indicators that are achieved at the final moment of time.

Keywords: Sobolev type equations; spectral projector; relatively regular matrices; flows of solving matrices.

Introduction

In order to construct mathematical models of economic systems and processes, balance models [1, 2] are widely used. Dynamic models allow to describe the process of change of economic indicators, to establish a direct relationship between the previous and subsequent stages of development, and, therefore, to bring the analysis based on the economic-mathematical model to the real conditions of development of the economic system. In dynamic models, production capital investments are distinguished from the composition of the final product, their structure and impact on the growth of production are investigated.

In $\mathbb{R}^n$, consider the dynamic balance model in the form of the non-stationary Leontief type system

$$L\dot{u}(t) = a(t)Mu(t) + f(t), \tag{1}$$

where L and M are square matrices of order n, and $\det L = 0$. Here $a : [0, T] \to \mathbb{R}_+$ is a scalar function that describes in time the variation of parameters of interaction between the velocities of states changing and the states of the system under study, and the matrix M is (L, p)-regular (i.e., there exists $\mu \in \mathbb{C}$ such that $\det(\mu L - M) \neq 0$, and ∞ is a pole of $(\mu L - M)^{-1}$ of the order $p \in \mathbb{N}_0$, hereinafter $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$). The vector function $f : [0, T] \to \mathbb{R}^n$ describes external influences on the system. Note that the condition of degeneracy of the system, $\det L = 0$, is one of the distinguishing features of balance models of the economy, since resources of a certain type cannot be stored [2]. Moreover, note that balance models often have a non-stationary form, i.e. the matrices included in system (1) depend on time (see, for example, [3]). However, in this case, in order to obtain a constructive solution, special conditions are required for these matrices [4, 5].

Leontief type systems can be considered as a finite-dimensional analog of the Sobolev type equations [6, 7, 8]. This study was carried out in the framework of the theory of degenerate families of solving operators [7]. In order to study the solvability of initial problems for stationary Leontief type systems, the papers [9, 10, 11] use the theory of degenerate groups. In the numerical study of problems for such systems, the Showalter – Sidorov initial condition [12]

$$\left(L(\nu L - M)^{-1}\right)^{p+1}(u(0) - u_0) = 0, \qquad \nu \in \mathbb{C}: \quad \det(\nu L - M) \neq 0$$

allows to remove the constraints of the initial data matching, for example, when using the classical Cauchy initial condition [9]. In addition, in modern studies in the field of Sobolev type equations, the initial Showalter – Sidorov condition is considered to be more natural in order to study various applied problems [12]. Note that the solution to the optimal control problem for stationary Leontief type systems is used not only to describe economic systems, but also to simulate technical systems [13, 14, 15]. For the first time, non-stationary Sobolev type equations were considered in [16], and the proposed methods [17] were used to study various problems (see, for example, [18, 19, 20]).

In order to solve applied problems, sometimes it is necessary to consider situations, when some of the conditions on the desired vector function are given at the initial moment of time, while the remaining conditions, due to the features of the simulated process, are given at the final moment of time. In this case, it is adequate to consider the initial-final value conditions [21] for Leontief type systems. Let us consider the initial-final value problem in the following form:

$$P_{in}(u(0) - u_0) = 0, \quad P_{fin}(u(T) - u_T) = 0 \tag{2}$$

where P_{in}, P_{fin} are matrices to set conditions at the initial and final moments of time. Note that earlier the solvability of initial-final value problems for non-stationary Sobolev type equations was investigated when solving an optimal control problem of solutions to such problems (see, for example, [22, 23]). The main purpose of this article is to solve initial-final value problem (2) for equation (1), and apply the obtained theoretical results to the study of the initial-final value problem for the non-stationary Leontief model [2].

The article besides the introduction and bibliography contains three parts. In the first part, we give an information on the flows of solving matrices, as well as the solution to the Showalter – Sidorov and Cauchy problems for a non-stationary Leontief type system. In the second part, the initial-final value problem is described and its solution is constructed using flows of matrices. In the third part, dynamic balance Leontief model is investigated in the non-stationary case with the initial-final value condition.

1. Solvability of Initial Problems for Non-Stationary Leontief Type Systems

Here and below we denote a set of matrices of order $n \times m$ by a symbol $\mathbb{M}_{n\times m}$. Let $L, M \in \mathbb{M}_{n\times n}$ be square matrices of order n. Following by [7, 13], we call sets $\rho^L(M) = \{\mu \in \mathbb{C} : \det(\mu L - M) \neq 0\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ an *L-resolvent set* and an *L-spectrum* of the matrix M correspondingly. It is easy to see [7, 13] that is $\rho^L(M) = \emptyset$ or an L-spectrum of matrix M consists a finite number of points. Additionally we note that sets $\rho^L(M)$ and $\sigma^L(M)$ don't change with the transition to the other basis.

For a complex variable $\mu \in \mathbb{C}$ we define matrix-valued functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ with a domain set $\rho^L(M)$. These matrix-valued functions we call an *L-resolvent*, a *right L-resolvent* and a *left L-resolvent* of the matrix M correspondingly. Also by the results of [7, 13] the L-resolvent, the left L-resolvent and the right L-resolvent of the matrix M are holomorphic in domain $\rho^L(M)$.

Definition 1. A matrix M is called an *L-regular* if $\rho^L(M) \neq \emptyset$ and it's called (L,p)*-regular* if, in addition, ∞ is a polar of order p of the function $(\mu L - M)^{-1}$.

Remark 1. If infinity is a removable singular point of the L-resolvent of the matrix M then we set $p = 0$. Also we note that for square matrices the parameter p can't be more than the dimension of space n.

Remark 2. The term the "L-regular matrix M" is equivalent to a "regular band of matrices $\mu L - M$" by K. Weierstrass [24]. Also the relatively p-regular matrices ($p \in \mathbb{N}_0$) is a particular case of relatively p-bounded operators from Sobolev type theory [7, 13].

We choose in complex plane $\mathbb{C}$ a closed loop. This loop bound a part of the plane, which consist the relatively spectrum $\sigma^L(M)$ of the matrix M. The next integrals

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)d\mu \quad \text{and} \quad Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)d\mu \tag{3}$$

have meaning as integrals of holomorphic functions by a closed loop. If the matrix M is an (L,p)-regular ($p \in \mathbb{N}_0$) then the matrices P and Q are projectors [7, 13].

Let the restriction of the matrix L to $\ker P$ and $\operatorname{im} P$ be denoted by L_0 and L_1 correspondingly. By the same way we denote the restriction of the matrix M to $\ker P$ and $\operatorname{im} P$ by M_0 and M_1.

Lemma 1. *If a matrix M is an (L,p)-regular ($p \in \mathbb{N}_0$) then there exists inverse matrices L_1^{-1} and M_0^{-1}.*

There is the existence of matrices $H = M_0^{-1}L_0$ and $S = L_1^{-1}M_1$ by Lemma 1.

On the interval $\mathfrak{J} \subset \mathbb{R}$ we consider the Cauchy problem

$$u(t_0) = u_0 \qquad (t_0 \in \mathfrak{J}) \tag{4}$$

for the homogeneous non-stationary matrix equation

$$L\dot{u}(t) = a(t)Mu(t), \tag{5}$$

where a scalar function $a : \mathfrak{J} \to \mathbb{R}_+$ will be defined further.

Definition 2. A vector-function $u \in C^1(\mathfrak{J}; \mathbb{R}^n)$ is called a *solution of equation* (5) if it satisfies to this equation on $\mathfrak{J}$. The solution of equation (5) is called a *solution of a Cauchy problem* (4), (5) if, in addition, it satisfies to the condition (4).

Definition 3. A two-parameter family $U(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{M}_{n\times n}$ is called a *degenerate flow of solving matrices* (or shortly a*degenerate resolving flow*) for equation (5) if it satisfies to the next conditions

(i) $U(t,t) = P$;

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$;

(iii) if for an arbitrary $u_0 \in \mathbb{R}^n$ a vector-function $u(t) = U(t,t_0)u_0$ is a solution of equation (5) (by the Definition 2).

Degenerate flow of matrices is called *analytic* if its matrices admit analytic extension to a hole complex plane $\mathbb{C}$ with retaining the properties (i) and (ii) from Definition 3.

Since the matrices are a finite dimensional analogue of the bounded operators (see Remark 2) then by virtue [17] the next statement is true.

Theorem 1. *If M is an (L,p)-regular matrix ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R};\mathbb{R}_+)$ then the family $\{U(t,s) \in \mathbb{M}_{n\times n} : \ t,s \in \mathbb{R}\}$ with the matrices*

$$U(t,s) = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)\exp\left(\mu\int\limits_s^t a(\zeta)d\zeta\right)d\mu, \qquad s<t, \tag{6}$$

is a analytical degenerate resolving flow of matrices for equation (5)*.*

Remark 3. The matrices (6) can be construct by using of a Hille – Wider – Post approximation [25].

Consider the Showalter – Sidorov problem

$$P(u(t_0) - u_0) = 0 \tag{7}$$

for the non-stationary nonhomogeneous matrix equation

$$L\dot{u}(t) = a(t)Mu(t) + f(t) \tag{8}$$

with function $f : \mathfrak{J} \to \mathbb{R}^n$. Here and below we denote $(E_n - Q)f(t) = f^0(t)$, where E_n is an identity matrix of order n.

Definition 4. The solution of equation (8) is called a *solution of the Showalter – Sidorov problem* (7), (8) if in addition it satisfies to the condition (7).

Theorem 2. *Let $t_0, T \in \mathfrak{J}$, M is a (L,p)-regular matrix ($p \in \mathbb{N}_0$) and function $a \in C^{p+1}([t_0,T];\mathbb{R}_+)$. Then for an arbitrary function $f : [t_0,T] \to \mathbb{R}^n$, such that $Qf \in C^1([t_0,T];\operatorname{im} Q)$ and $f^0 \in C^{p+1}([t_0,T];\ker Q)$, and for an arbitrary initial value $u_0 \in \mathbb{R}^n$ there is a unique solution $u \in C^1([t_0,T];\mathbb{R}^n)$ of the Showalter – Sidorov problem* (7) *for equation* (8)*, witch has the form*

$$u(t) = U(t,t_0)Pu_0 + \int\limits_{t_0}^t U(t,s)L_1^{-1}Qf(s)ds - \sum_{q=0}^p H^q M_0^{-1}(E_n - Q)\left(\frac{1}{a(t)}\frac{d}{dt}\right)^q \frac{f(t)}{a(t)}, \tag{9}$$

where a symbol $\left(\dfrac{1}{a(t)}\dfrac{d}{dt}\right)^q$ in the last summand means applying q times of this operator.

If additionally the initial data satisfies to the condition

$$(E_n - P)\,u_0 = -\sum_{q=0}^p H^q M_0^{-1}(E_n - Q)\left(\frac{1}{a(t)}\frac{d}{dt}\right)^q \frac{f(t)}{a(t)}\Bigg|_{t=t_0}$$

then the function (13) *is a unique solution of the Cauchy problem* (4) *for equation* (8)*.*

2. Solvability of the Initial-Final Value Problems for the Non-Stationary Leontief Type Systems

Let $L, M \in \mathbb{M}_{n\times n}$ be square matrices od order n, such us $\det L = 0$ and M is an (L,p)-regular matrix ($p \in \mathbb{N}_0$). In order to state of the initial-final problem consider a condition

$$\sigma^L(M) = \sigma^L_{in}(M) \cup \sigma^L_{fin}(M), \quad \text{где} \quad \sigma^L_{in}(M) \cap \sigma^L_{fin}(M) = \emptyset,$$

witch automatically satisfies for Leontief type system. Then there are exist the closed loops $\gamma_{in}, \gamma_{fin} \subset \mathbb{C}$, witch bound these components of relatively spectrum, i.e. there are exist $D_{in}, D_{fin} \subset \mathbb{C}$, such that

$$\gamma_{in} = \partial D_{in},\ \gamma_{fin} = \partial D_{fin}, \quad D_{in} \supset \sigma^L_{in}(M),\ D_{fin} \supset \sigma^L_{fin}(M).$$

By analogy with (3) we define the next spectral projectors

$$P_{in} = \frac{1}{2\pi i}\int\limits_{\gamma_{in}} R^L_\mu(M)d\mu \quad \text{and} \quad P_{fin} = \frac{1}{2\pi i}\int\limits_{\gamma_{fin}} R^L_\mu(M)d\mu. \tag{10}$$

Finally, consider the initial-final value problem

$$P_{in}(u(0) - u_0) = 0, \qquad P_{fin}(u(T) - u_T) = 0 \tag{11}$$

for the non-stationary Leontief type system

$$L\dot{u}(t) = a(t)Mu(t) + f(t) \tag{12}$$

with function $f : [0,T] \to \mathbb{R}^n$, witch will be define later.

Definition 5. The solution of equation (12) is called a *solution of initial-final value problem* (11), (12) if in addition it satisfies to the condition (11).

Theorem 3. *Let M is a (L,p)-regular matrix ($p \in \mathbb{N}_0$) and function $a \in C^{p+1}([0,T];\mathbb{R}_+)$. Then for any function $f : [0,T] \to \mathbb{R}^n$, such that $Qf \in C^1([0,T];\operatorname{im} Q)$ and $f^0 \in C^{p+1}([0,T];\ker Q)$, and for an arbitrary values $u_0, u_T \in \mathbb{R}^n$ there is exist a unique solution $u \in C^1([0,T];\mathbb{R}^n)$ of the initial-final value problem (11) for equation (12), which has the form*

$$u(t) = -\sum_{q=0}^{p} H^q M_0^{-1}(E_n - Q)\left(\frac{1}{a(t)}\frac{d}{dt}\right)^q \frac{f(t)}{a(t)} + U(t,t_0)P_{in}u_0 +$$

$$+\int\limits_{t_0}^{t} U(t,s)L_1^{-1}Q_{in}f(s)ds + U(t,T)P_{fin}u_T - \int\limits_{t}^{T} U(t,s)L_1^{-1}Q_{fin}f(s)ds, \tag{13}$$

where the symbol $\left(\frac{1}{a(t)}\frac{d}{dt}\right)^q$ *in the first summand means applying q times of this operator and by the analogy with* (10) *the matrices Q_{in} and Q_{fin} are given by the formulas*

$$Q_{in} = \frac{1}{2\pi i}\int\limits_{\gamma_{in}} L^L_\mu(M)d\mu, \quad Q_{fin} = \frac{1}{2\pi i}\int\limits_{\gamma_{fin}} L^L_\mu(M)d\mu.$$

Proof. Let us act on the system (12) sequentially with matrices $E_n - Q$, Q_{in} and Q_{fin} and reduce it to a system of the form

$$H\dot{u}^0 = a(t)u^0 + M_0^{-1}f^0(t), \tag{14}$$

$$\dot{u}_{in} = a(t)SP_{in}u_{in} + L_1^{-1}Q_{in}f(t), \tag{15}$$

$$\dot{u}_{fin} = a(t)SP_{fin}u_{fin} + L_1^{-1}Q_{fin}f(t). \tag{16}$$

After that, it remains to be noted that by virtue of the classical results on the solvability of systems the first term of the solution (13) is the solution (14), the second and third terms of this solution constitute the solution (15) and the last two terms from solution (13) resolve (16).

□

Remark 4. When considering a multipoint initial-final value problem [26, 22] for Leontief type systems (see Remark 1), a natural restriction on the number of conditions m in the multipoint initial-final value problem is satisfied. Namely, because of the finiteness of the space must be satisfied the condition

$$m \leq n - \dim\ker P.$$

3. Non-Stationary Leontief Balance Model with the Initial-Final Value Condition

A dynamic balance model of the form (12) is considered in [2] with matrices L and M of the form

$$L = \begin{pmatrix} \frac{7}{20} & \frac{1}{20} & \frac{21}{20} \\ \frac{1}{100} & \frac{103}{200} & \frac{8}{25} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \frac{3}{4} & \frac{-1}{5} & \frac{-11}{20} \\ \frac{-7}{25} & \frac{22}{35} & \frac{-3}{5} \\ \frac{-4}{15} & \frac{-2}{15} & \frac{13}{15} \end{pmatrix}.$$

Using the equation (12) with these matrices, the relationships between the three branches of the economy are described: agriculture, industry and household. The third line in the matrix L is zero, since the labor can not be stored. Thus, we consider a non-stationary system of the form

$$\begin{cases} \frac{7}{20}\dot{u}_1(t) + \frac{1}{20}\dot{u}_2(t) + \frac{21}{20}\dot{u}_3(t) = a(t)\left(\frac{3}{4}u_1(t) - \frac{1}{5}u_2(t) - \frac{11}{20}u_3(t)\right) + f_1(t), \\ \frac{1}{100}\dot{u}_1(t) + \frac{103}{200}\dot{u}_2(t) + \frac{8}{25}\dot{u}_3(t) = a(t)\left(-\frac{7}{25}u_1(t) + \frac{22}{35}u_2(t) - \frac{3}{5}u_3(t)\right) + f_2(t), \\ 0 = a(t)\left(-\frac{4}{15}u_1(t) - \frac{2}{15}u_2(t) + \frac{13}{15}u_3(t)\right) + f_3(t). \end{cases} \tag{17}$$

In order to apply the above results, we firstly find

$$\det(\mu L - M) = \frac{-43\,295\mu^2 + 99\,225\mu - 21\,744}{140\,000}$$

and the L-spectrum of matrix M consists of two points $\sigma^L(M) = \{0,245419; 2,046416\}$. The L-resolvents of matrix M has the form

$$(\mu L - M)^{-1} = \frac{5}{3(-43\,295\mu^2 + 99\,225\mu - 21\,744)}\cdot$$

$$\cdot\begin{pmatrix} 4(9\,760 - 10\,269\mu) & 280(55\mu + 74) & 3(-14\,693\mu^2 + 13\,181\mu + 13\,040) \\ 56(141\mu + 604) & 280(151 - 175\mu) & 42(-203\mu^2 + 659\mu + 1\,208) \\ 32(538 - 357\mu) & 560(23 - 5\mu) & 3(5\,033\mu^2 - 17\,423\mu + 11\,632) \end{pmatrix}.$$

So we can conclude that $p = 0$ and the matrix M is $(L, 0)$-regular. If $p = 0$ then we get that $\ker P = \ker L = \operatorname{span}\left\{\begin{pmatrix} 0,9306 \\ 0,1800 \\ -0,3188 \end{pmatrix}\right\}$. Denote this vector by φ_0.

Since $\dim(\ker P) = 1$ then by Remark 4 for the Leontief balance model in the multipoint initial-final value condition [26, 22] we can consider only $m \leq 2$, i.e. initial-final value problem.

Denote by $\varphi_1, \varphi_2 \in \operatorname{coker} L$ vectors, which are corresponding to the points of the relative spectrum $\sigma^L(M) = \{0,245419; 2,046416\}$. We can do that because of the fact that set $\sigma^L(M)$ doesn't change with the transition to the other basis. Let's move to the new basis $\{\varphi_2, \varphi_1, \varphi_0\}$. Let the initial condition be given on the vector φ_1, and the final condition is given on φ_2. Then, in this new basis, the solution of (17) with an initial-finite condition (11) by virtue of Theorem 3 has the form

$$u(t) = -\frac{\langle f(t), \varphi_0\rangle}{a(t)}\left(M^{-1}\varphi_0\right) + U(t,t_0)\langle u_0, \varphi_1\rangle\varphi_1 + \int\limits_{t_0}^{t} U(t,s)L_1^{-1}\langle f(s), \varphi_1\rangle ds\,\varphi_1 +$$

$$+U(t,T)\langle u_T, \varphi_2\rangle\varphi_2 - \int\limits_{t}^{T} U(t,s)L_1^{-1}\langle f(s), \varphi_2\rangle ds\,\varphi_2.$$

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РЕШЕНИЕ НАЧАЛЬНО-КОНЕЧНОЙ ЗАДАЧИ ДЛЯ НЕСТАЦИОНАРНОЙ СИСТЕМЫ ЛЕОНТЬЕВСКОГО ТИПА

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Статья посвящена построению решения начально-конечной задачи для нестационарной системы леонтьевского типа. Такие системы возникают при использовании динамических балансовых моделей экономики. Отличительной чертой систем леонтьевского типа является вырожденность матрицы при производной по времени, что обусловлено тем, что некоторые виды ресурсов экономических систем невозможно запасти. К тому же, динамические балансовые системы экономики часто описываются с помощью коэффициентов зависящих от времени. В данной статье для построения решений таких систем используются разрешающие потоки матриц. Кроме того, вместо стандартного начального условия используется начально-конечное условие, которое для экономических систем может интерпретироваться как учет показателей не только в начальный момент времени, но и показателей, которые будут достигнуты в конечный момент времени.

Ключевые слова: уравнения соболевского типа; спектральный проектор; относительно регулярные матрицы; разрешающие потоки матриц.

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