MSC 34K27, 34K29, 47D06, 58D25

PERTURBATION METHODS FOR INVERSE PROBLEMS RELATED TO DEGENERATE DIFFERENTIAL EQUATIONS

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Identification problem for possibly degenerate evolution equations on Banach spaces are considered. Such inverse problems are changed to direct differential problem, whose regular solvability has had recently large development. Some applications to concrete PDEs are given.

Keywords: inverse problem, degenerate differential equation, linear relation, perturbation method.

Introduction

In this preliminary section, motivation for development shall be described. Let us consider the simplest case of evolution problem described by

$$\frac{dy}{dt} = Ay + f(t)z, \quad 0 \le t \le \tau, \tag{1}$$

$$y(0) = y_0 \in D(A). \tag{2}$$

Our goal is to identify the solution-pair (y, f), where y is the solution to (1), (2) and $f \in C([0,\tau],\mathbb{C})$, under the additional information

$$\Phi[y(t)] = g(t), \quad 0 \le t \le \tau, \tag{3}$$

where $\Phi \in X^*$, the dual space of X, and $g \in C([0,\tau],\mathbb{C})$.

Usually, such a problem is reduced to a fixed point problem in a natural way.

If A generates an analytic semigroup on X, let $T(t)$ be a such semigroup. Then necessarily

$$y(t) = T(t)y_0 + \int_0^t T(t-s)zf(s)ds,$$

so that formally in a first step,

$$\Phi[y(t)] = \Phi[T(t)y_0] + \Phi[\int_0^t T(t-s)zf(s)ds]$$

implies, taking into account information (3),

$$\frac{d}{dt}\Phi[y(t)] = g'(t) = \frac{dg}{dt}(t) = \frac{d}{dt}\Phi[T(t)y_0] + \frac{d}{dt}\Phi[\int_0^t T(t-s)zf(s)ds] =$$

$$= \Phi[AT(t)y_0] + f(t)\Phi[z] + \Phi\int_0^t \frac{\partial}{\partial t}T(t-s)zf(s)ds. \tag{4}$$

Suppose $\Phi[z] \neq 0$. Then (4) implies that $f(t)$ must satisfy the integral equation

$$f(t) = \frac{1}{\Phi[z]}\{g'(t) - \Phi[AT(t)y_0] - \Phi\int_0^t \frac{\partial}{\partial t}T(t-s)zf(s)ds\}. \tag{5}$$

Assuming that z belongs to the real interpolation space $(X, D(A))_{\theta,\infty}$ it is not a difficult task to show that the integral equation (5) admits a unique solution $f \in C([0,\tau];\mathbb{C})$. After this step, well known results on solvability of Cauchy problem related to

$$y' = Ay + F(t), 0 \le t \le \tau,$$

apply, guaranteeing existence and uniqueness of the solution pair (y, f). In last times, a different strategy was proposed by AL Horani and Favini [1]–[3]. It is developed on the ground of the perturbation of generators. In fact, apply Φ to equation (1). Taking into account information (3), we get

$$g'(t) = \Phi[Ay(t)] + f(t)\Phi[z]$$

provided that $g \in C^1([0,\tau];\mathbb{C})$. Assumption $\Phi[z] \neq 0$ allows to deduce that necessarily

$$f(t) = \Phi[z]^{-1}\{g'(t) - \Phi[Ay(t)]\}.$$

Substituting such function in (1), we get that y satisfies the direct problem

$$y'(t) = Ay(t) - \frac{1}{\Phi[z]}\Phi[Ay(t)]z + \frac{g'}{\Phi[z]}z$$

i.e.

$$y'(t) = (A + B)y(t) + g'(t)\Phi[z]^{-1}z$$

where B is the linear operator from $D(B) := D(A)$ into X

$$By = -\frac{1}{\Phi[z]}\Phi[Ay]z$$

One observes that B is compact from $D(B)$ into X and thus a well known result from Desh and Schappeber guarantees that $A+B$ with $D(A + B) = D(A)$ generates an analytic semigroup on X.

More precisely, we use the methods of Favini, Lorenzi, Tanabe [8] and observe that a change of variable $y = e^{kt}v$, where $k > 0$, transforms the Cauchy problem

$$y' = (A + B)y + \frac{g'(t)}{\Phi[z]}z, \quad 0 \le t \le \tau,$$

$$y(0) = y_0$$

into the problem

$$v' = (A + B - k)v + e^{-kt}\frac{g'(t)}{\Phi[z]}z, \tag{6}$$

$$v(0) = y_0$$

and we know that for k large enough $A + B - k$ is closed and it has a bounded inverse. It follows that if $z \in (X, D(A))_{\theta,\infty} = (X, D(A + B))_{\theta,\infty}$, $g \in C^1([0,\tau];\mathbb{C})$, $(A + B)y_0 = Ay_0 - \frac{\Phi[Ay_0]}{\Phi[z]}z \in (X, D(A)_{\theta,\infty})$, i.e, $Ay_0 \in (X, D(A))_{\theta,\infty}$, then there exists a unique solution v to (6) such that

$$v' - e^{-kt}\frac{g'(t)}{\Phi[z]}z = (A + B - k)v \in \mathbb{C}^{\theta}([0,\tau];X) \cap B([0,\tau];(X, D(A))_{\theta,\infty}).$$

Here $B([0,\tau];X)$ denotes the space of all bounded functions from $[0,\tau]$ into the Banach space X.

This implies that

$$Av = A(A+B-k)^{-1}(A+B-k)v \in C^{\theta}([0,\tau];X),$$

$$v' \in B\left([0,\tau];(X,D(A))_{\theta,\infty}\right).$$

It is also easy to verify that this implies $y \in C^1([0,\tau];X)$, $\mu' \in B([0,r];(X,D(A))_{\theta,\infty})$, $Ay \in C^{\theta}([0,\tau];X)$, cfr. Al Horani, Favini [1]–[2]. More general problems are treated in Favini, Lorenzi, Tanabe [8], [9], [10].

This perturbation approach applies also when A generates a C_0-semigroup in X. If $B \in \mathcal{L}(D(A);X)$ and range B is contained in the Favard space

$$F_1 = \{x \in X : \sup_{t>t_0} \|tA(t-A)^{-1}x\| < \infty\}$$

for some $t_0 > 0$ (in particular, if X is reflexive, it is shown that $F_1 = D(A)$, see Engel-Nagel monograph) then $A+B$ generates a C_0-semigroup in X, cfr. Engel-Nagel [Semigroup theory, p.204]. As a consequence, if A generates a C_0-semigroup in X, problem (1)-(3) admits a unique strict solution

$$(y,f) \in \{C([0,\tau];D(A)) \cap C^1([0,\tau];X)\} \times C([0,\tau];\mathbb{C})$$

provided that $z \in F_1$, $\Phi \in X^*, \Phi[z] \neq 0, y_0 \in D(A)$, $g \in W^{2,1}([0,\tau];\mathbb{C})$, $g(0) = \Phi[y_0]$. See Al Horani – Favini[3].

To conclude, one takes into account the direct results from Sinestra. These are the main results on the regular case and many examples of application to PDE's could be described. See the final section in a related paper.

In this paper the case with possible degenerations in which the term $\frac{dy}{dt}$ is multiplied by an operator M, possibly noninvertible, is considered and the additional interaction includes functionals $\Phi_i, i = 1..N$, i.e.

$$\frac{d}{dt}(My)(t) = L_1 y(t) + \sum_{i=1}^{N} h_i(t) z_i + F(t), \quad 0 \leq t \leq \tau, \tag{7}$$

$$(My)(0) = x_0 = My_0 \tag{8}$$

$$\Phi_i[y(t)] = g_i(t), i = 1..N, (\text{see [7]}), \tag{9}$$

where the closed linear operators L, M satisfy the weak parabolic estimate

$$\|M(zM - L_1)^{-1}\|_{\mathcal{L}(X)} \leq C(1+|z|)^{-\beta}, \tag{10}$$

for all z in the region

$$\Sigma_\alpha := \{z \in \mathbb{C}; Rez \geq -C(1+|lnz|)^{\alpha}\}, C > 0, 0 < \beta \leq \alpha \leq 1. \tag{11}$$

Examples of operators L satisfying resolvent estimates like

$$\|(z-L)^{-1}\|_{\mathcal{L}(X)} \leq C(1+|z|)^{-\beta},$$

$\forall z, Rez \geq -C(1 + |lnz|)^{\alpha}$, $0 < \beta \leq \alpha \leq 1$, can be found, for instance in Krein's monograph, Chapter 1, section 8 (Evolution equations well posed according to Shilov). As a simple example, whose main idea comes from professor Yuli Eidelman (Tel Aviv), take $X = L^2(\mathbb{R})$ and $L : X \to X$, $D(L) = H^3(\mathbb{R})$, $Lu = u'' - u''' + \gamma u, u \in H^3(\mathbb{R}), \gamma < 0$. By using Fourier transform, it is easy to verify that the previous estimates hold with $\alpha = \beta = 1/2$. As another simple example, take $m(x) \geq 0$ a measurable function on $[0, 1]$, M the multiplication operator by $m(x)$ in $L^2(0, 1), Lu = u''$, $D(L) = H_0^1(0, 1) \cap H^2(0, 1)$. Then one recognizes, see Favini and Yagi [16] that

$$\|(z - L)^{-1}\|_{\mathcal{L}(L^2(0,1))} \leq C(1 + |z|)^{-1/2}$$

holds in a sector $Rez \geq -C(1 + |lnz|)^{1/2}$, whit C a suitable positive number. Therefore (10) holds with $\alpha = \beta = 1/2$.

1. Preliminaries

We need linear relations or multivalued linear operators A. Such operator acts from X into $\mathcal{P}(X)$ with domain $D(A) = \{x \in X, Ax \neq \emptyset\}$ which is a linear subspace of X. Recall that

$$Ax + Ay \subset A(x + y), \lambda Ax \subseteq A(\lambda x), \lambda \in \mathbb{C}, x, y \in D(A)$$

The graph of A is $G(A) = \{(x, y); x \in D(A), y \in Ax\}$ if $U \in P(X)$, $U \cap G(A) \neq \emptyset$, the restriction $A|U$ of A to U is defined by $D(A|U) = U \cap D(A)$ and $(A|U)x = Ax, x \in D(A|U)$.

The inverse A^{-1} of A is a m.l. operator given by

$$D(A^{-1}) = R(A) = imm\ A$$

and

$$A^{-1}y = \{x \in D(A) : y \in Ax\}.$$

The kernel $N(A)$ of A is $A^{-1}0 = \{x \in D(A), 0 \in Ax\}$. If $N(A) = \{0\}$, A is said to be injective.

If $\lambda \in \mathbb{C}$, A and B are m.l. operators, then λA and AB, BA are defined in a natural way. If $U \in \mathcal{P}(A) \setminus \emptyset$, I_U $(I_X = I)$ denotes the identity operator in U. If A, B are m.l. operators, $B \subset A$ if $D(B) \subseteq D(A)$ and $Bx = Ax$ for all $x \in D(B)$.

If $B \subset A$ and $Bx = Ax$ $\forall x \in D(B)$, one says that A is an extension of B. If A is an extension of B, then $G(B) \subseteq G(A)$, but the inverse does not hold. A single-valued linear operator S is called a linear section of A if $D(A) = D(S)$ and $S \subset A$, i.e., $Sx \in Ax$ for all $x \in D(A)$.

If $U, V \in \mathcal{P}(X) \setminus \emptyset$, the distance $d(U, V)$ is defined by

$$d(U, V) = \inf_{u \in U, v \in V} \|u - v\|_X.$$

$d(x, V) = d(V, x)$ is the distance from $\{x\}$ to V.

If A is a m.l. in X, we define

$$\|Ax\| = d(Ax, 0)$$

and it is easily seen that $\|Ax\| = d(z, 0)$ $\forall z \in Ax$, so that $\|Ax\| = d(Ax, 0)$.

Norm $\|A\|$ of a linear relation A is defined by

$$\|A\| = \sup_{x\in D(A),\|x\|\leq 1} \|Ax\|.$$

If $D(A) = X$ and $\|A\| < \infty$, A is said to be bounded.

Notice that $\|A\|$ is not a true norm if A is not single-valued. A linear relation A is closed if its graph is closed in $X \times X$. The resolvent set of A is

$$\rho(A) = \{z \in \mathbb{C}; (zI - A)^{-1} \in \mathcal{L}(X)\}.$$

If $\rho(A) \neq \emptyset$, then A is closed.

If A is a linear relation, $A^0(zI - A)^{-1}$ is the linear section of $A(zI - A)^{-1}$ given by $z(zI - A)^{-1} - I,\ z \in \rho(A)$. A^0 is in fact only a symbol. We shall say that A satisfies (H_A) if $\rho(A)$ contains $\Sigma_\alpha = \{z \in \mathbb{C}; Re\ z \geq -C(1 + |lnz|)^\alpha\}$ for some $\alpha \in (0, 1],\ C > 0$ and $\exists\beta \in (0, \alpha]$, $\tilde{C} > 0$ such that

$$\|(zI - A)^{-1}\|_{\mathcal{L}(X)} \leq \tilde{C}(1 + |z|)^{-\beta},$$

$\forall z \in \Sigma_\alpha$.

If M, L are closed linear operators in X, $D(L) \subseteq D(M)$, let us introduce the linear relation $A = LM^{-1}$ by

$$D(A) = M(D(L)), Ax = \{Ly; y \in D(L), x = My\}.$$

Then it is seen that $(zM - L)M^{-1} = zI - A, z \in \mathbb{C}$ and $M(zM - L)^{-1} = (zI - A)^{-1}$.

Denote by $\rho_M(L)$ the M-modified resolvent set of L:

$$\rho_M(L) = \{z \in \mathbb{C} : (zM - L)^{-1} \in \mathcal{L}(X)\}.$$

If M is also closed, then $\rho_M(L) \subseteq \rho(A)$. Therefore if L and M are closed linear operators in X and

$$\Sigma_\alpha \in \rho_M(L)\ and\ \|M(zM - L)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |z|)^{-\beta}\forall z \in \Sigma_\alpha, \qquad (H_{M,L})$$

then m.l. operator $A = LM^{-1}$ satisfies (H_A).

If A is a m.l. operator in X, $0 \in \rho(A)$, $D(A)$ becomes a Banach space with the norm

$$\|x\|_{D(A)} = \|Ax\|,\ x \in D(A).$$

We recall that if $(Y, \|\cdot\|_Y)$ is a Banach space, $C^0(0, +\infty; Y)$ denotes the set of all continuous Y-valued functions on $(0, +\infty)$.

Let us introduce the mean space of Lions – Peetre. Let g be a strongly measurable function from $(0, +\infty)$ into Y and define

$$\|g\|_{L^{+\infty}_*(Y)} = \sup_{\xi\in(0,\infty)} \|g(\xi)\|_Y.$$

If $\gamma \in (0, 1)$, $S(\gamma, \infty, X, \gamma - 1, \infty, D(A)) = \{x \in X, x = v_0(t) + v_1(t)$ for all $t > 0$, $v_0 \in C^0(0, +\infty; X)$, $v_1 \in C^0(0, +\infty; D(A))\}$ such that

$$\|\xi^\gamma v_0\|_{L^\infty(X)} < +\infty,\ \|\xi^{\gamma-1}v_1\|_{L^\infty(D(A))} < +\infty.$$

The Banach space $(X, D(A))_{\gamma,\infty}$ is defined by

$$(X, D(A))_{\gamma,\infty} = S(\gamma, \infty, X, \gamma - 1, \infty, D(A)),$$

$$\|x\|_{(X,D(A))_{\gamma,\infty}} = \inf\{\|\xi^{\gamma} v_0\|_{L^\infty(X)} + \|\xi^{\gamma-1} v_1\|_{L^\infty(D(A))}\},$$

where the infimum is taken on all possible representatives as indicated above.

If $[0, \infty) \subseteq \rho(A)$ and $\gamma \in (0, 1)$, A being a linear relation in X, introduce the Banach space X_A^γ by

$$X_A^\gamma = \{x \in X : |x|_{X_A^\gamma} := \sup_{t>0} \|t^\gamma A^0 (t - A)^{-1} x\|_X < \infty\},$$

$$\|x\|_{X_A^\gamma} := \|x\|_X + |x|_{X_A^\gamma}$$

and observe that if $\alpha = \beta = 1$, $X_A^\gamma = (X, D(A))_{\gamma,\infty}$ in the single-valued case. We have (see Favaron – Favini [5], Prop. 4.3).

Proposition 1. *If A is a linea relation in X satisfying (H_A), then for any $\gamma \in (0, 1)$ $X_A^\gamma \hookrightarrow (X, D(A))_{\gamma,\infty}$. Moreover, if $\gamma \in (1 - \beta, 1)$, $(X, D(A))_{\gamma,\infty} \hookrightarrow X_A^{\gamma+\beta-1}$.*

Proposition 2. *Let M, L_1, L_2 be closed linear operators in X. Let $D(L_1) \subseteq D(L_2)$, $D(L_1) \subseteq D(M)$, $0 \in \rho(L_1) \cap \rho(L_1 + L_2)$ and (H_{M,L_1}), (H_{M,L_1+L_2}) hold. Put*

$$A_0 = (L_1 + L_2)M^{-1}, \ A_1 = L_1 M^{-1}$$

and assume $L_2|_{D(L_1)} \in \mathcal{L}(D(L_1); X)$. Then

$$X_{A_1}^\gamma \hookrightarrow X_{A_0}^{\gamma+\beta-1}, \ \gamma \in (1 - \beta, 1).$$

2. Perturbation results

The following result extends a result from Lunardi [8], Proposition 2.4.1 (ii) for $\alpha = \beta = 1$ and $M = I$.

Theorem 1. *Let M, L_1, L_2 be single-valued closed linear operators in X, $D(L_1) \subseteq D(L_2) \cap D(M)$, $0 \in \rho(L_1)$ and (H_{M,L_1}) hold, $\beta \in (0, \alpha], \alpha \in (0, 1]$. Moreover, $L_2|_{D(L_1)} \in \mathcal{L}(D(L_1), Y)$, where Y is a Banach space such that $Y \hookrightarrow \{(X, D(A_1))_{\gamma,\infty}, X_{A_1}^\gamma\}$, $\gamma \in (1 - \beta, 1)$, $A_1 = L_1 M^{-1}$. Then there exist $C_1, C_2 > 0$ such that*

$$\|M(zM - (L_1 + L_2)^{-1})\|_{\mathcal{L}(X)} \leq C_1(1 + |z|)^{-\beta}$$

for all $z \in \Sigma_\alpha, |z| \geq C_2$.

A basic role is played from lemma as follows taking into account that $L_1 + L_2$ may not having a bounded inverse even if L_1 has.

Lemma 1. *Suppose $k > 0$ suitable large so that $0 \in \rho(kM + L_1)$ and $0 \in \rho(kM + L_1 + L_2)$. Let $A_1 = L_1 M^{-1}$, $A_2 = (L_1 + L_2)M^{-1}$ and assume $L_1 \in \mathcal{L}(D(L_1), X_{A_1}^\theta)$, $\theta > 1 - \beta$. Then for all $\sigma \in (0, 1)$*

$$X_{A_1}^\sigma = X_{A_2}^\sigma = X_{(kM+L_1+L_2)M^{-1}}^\delta.$$

See Favaron, Favini, Tanabe [7]. The second basic lemma concerns existence, uniqueness and regularity of solutions to the problem

$$\frac{d}{dt}(My)(t) = Ly(t) + \sum_{i=1}^{N} h_i(t) z_i + F(t), \quad 0 \le t \le \tau, \tag{12}$$

$$(My)(0) = My_0, \tag{13}$$

where the pair (M,L) satsfies

$$\|M(z+L)^{-1}\|_{\mathcal{L}(x)} \le C(1+|z|)^{-\beta}, z \in \Sigma_\alpha, \quad 0 < \beta \le \alpha \le 1,$$

cfr. Favaron, Favini, Tanabe [7] and Favaron, Favini [6].

Lemma 2. *Let L, M be two closed linear single-valued operators in X such that $0 \in \rho(L), (H_{M,L})$ holds and $5\alpha+2\beta > 6$. Assume $y_0 \in D(L)$, $(h_1..h_N) \in \prod_{i=1}^N C^{\sigma_i}([0,\tau];\mathbb{C})$, $z_0 = Ly_0 + F(0)$, $(z_0, z_1..z_N) \in \prod_{j=0}^N Y_{\gamma_j}$, where $\sigma_i \in ((3-2\alpha-\beta)/\alpha, 1)$, $i = 1, ..., N$, $\gamma_j \in (5-3\alpha-2\beta, 1), j = 0, 1, ..., N$, $\prod_{j=0}^N Y_{\gamma_j} \in (\prod_{j=0}^N (X, D(A))_{\gamma_j\infty}, \prod_{j=1}^N X_A^{\gamma_j})$, $A = LM^{-1}$. Let $\gamma = \min_{j=0,1..N} \gamma_j, \tau = \min_{i=1..N}\{\sigma_i, \chi_{\alpha,\beta,\gamma}\}$, where*

$$\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha.$$

Then for every fixed $\delta \in I_{\alpha,\beta,\tau}$, with

$$I_{\alpha,\beta,\gamma} = \begin{cases} ((3-2\alpha-\beta)/\alpha, \gamma], if \ \gamma \in ((3-2\alpha-\beta)/\alpha, 1/2), \\ ((3-2\alpha-\beta)/\alpha, 1/2), if \ \gamma \in [1/2, 1) \end{cases}$$

problem (12) has a unique solution $y \in C^\delta([0,\tau]; D(L))$ satisfying $y(0) = y_0$ and $My \in C^{1+\delta}([0,\tau]; X)$ provided that $F \in C^\theta([0,\tau]; X)$, $\theta \in [\delta + (3-2\alpha-\beta)/\alpha, 1)$.

3. Solution of the inverse problem

Now we handle the inverse problem (6)-(8). Applying the linear functionals $\Phi_1..\Phi_N$ to both the members in (6) and taking into account information (8), we obtain the following linear system for the N unknowns $h_1..h_N$:

$$\sum_{j=1}^{N} \Phi_i[z_j] h_j(t) = \frac{d}{dt} g_i(t) - \Phi_i[L_1 y(t)] - \Phi_i[F(t)], i = 1..N, t \in [0,\tau].$$

If the square matrix of order N

$$U = \begin{pmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_N] \\ \\ \Phi_N[z_1] & \dots & \Phi_N[z_N] \end{pmatrix}$$

is invertible, denoting with $det(U)$ the determinant of U, the solution $(h_1(t), .., h_N(t))$ to the previous system is given by

$$h_i(t) = [det(U)]^{-1} \sum_{k=1}^{N} U(k,i) \{\frac{d}{dt} g_k(t) - \Phi_k[F(t)] - \Phi_k[L_1 y(t)]\}, \quad i = 1, .., N,$$

where $U_{(k,i)} = (-1)^{k+i} detU(k,i) \in \mathbb{C},\ k,i = 1,...,N$, is the cofactor of the element $\Phi_k[y_i]$ of U, $U(k,i)$ being the square matrix of order $N-1$ obtained from U cancelling the k-th row and the i-th column (by convention $U(1,1) = 1$ if $N = 1$).

Replacing these values in (6), we obtain the following initial-value problem in which only the unknown y appears:

$$\frac{d}{dt}(My(t)) = (L_1 + L_2)y(t) + \sum_{i=1}^{N} f_i(t)z_i + F(t), t \in [0,\tau], \tag{14}$$

where L_2 and $f_i,\ i = 1,...,N$ denote respectively,

$$D(L_2) := D(L_1),\ L_2x := -[detU]^{-1} \sum_{k,i=1}^{N} U(k,i)\Phi_k[L_1x]z_i, \quad t \in [0,\tau],$$

$$f_i(t) = [det(U)]^{-1} \sum_{k=1}^{N} U(k,i)\{\frac{d}{dt}g_k(t) - \Phi_k[F(t)]\}, \quad i = 1,..,N.$$

Now, $L_1 + L_2$ might not to be closed, as it is easy to verify from counterexamples. To overcome this difficulty, we operate the change of variable $y = e^{kt}w,\ k > 0$ large, so that our system reads

$$\frac{d}{dt}(Mw(t)) = (L_1 + L_2)w(t) - kMw(t) + \sum_{i=1}^{N} \widetilde{f}_i(t)z_i + \widetilde{F}(t), \tag{15}$$

where $(\tilde{F}(t), \tilde{f}_i(t)) = (e^{-kt}F(t), e^{-kt}f_i(t))$.

Observe that $(L_1 + L_2 - kM)M^{-1} = (L_1 + L_2)M^{-1} - kI = A_0 - kI$.

Moreover (cfr. Favini and Yagi [16])

$$\begin{aligned} kM - L_1 - L_2 &= (kM - L)[I - (kM + L_1)^{-1}L_2] \\ &= (kM - L)(I - L_1^{-1}(kML_1^{-1}) - 1)^{-1}L_2) \\ &= (kM - L_1)(I - L_1^{-1}A_1^0(k - A_1)^{-1}L_2). \end{aligned}$$

On the other hand, $L_2 \in \mathcal{L}(D(L_1), Y_\gamma)$, see Lemma 2, and we have from Favini and Yagi [16], p.49, that

$$\|A_1^0(k - A_1)^{-1}x\|_X \le Ck^{1-\beta-\gamma}\|x\|_{X^\gamma_{A_1}}$$

This implies that operator

$$L_1^{-1}A_1^0(k - A_1)^{-1}L_2$$

is bounded from $D(L_1)$ into itself with norm less than 1 for k large enough. Thus for k large $kM - L_1 - L_2$ has an inverse

$$(I - L_1^{-1}A_1^0(k - A_1)^{-1}L_2)^{-1}(kM - L_1)^{-1}.$$

The first operator is viewed in $\mathcal{L}(D(L_1))$, while the second one acts from X into $D(L_1)$. Thus, in particular $kM - L_1 - L_2$ has a bounded inverse in X. Therefore, it is closed, too.

Observe also that in view of Lemma 1,

$$X^{\theta}_{L_1M^{-1}} = X^{\theta}_{(L_1+L2-kM)M^{-1}},$$

provided that $\gamma > 1-\beta$.

Taking into account these preliminaries, we can solve the given inverse problem as following.

We must solve (14) together with initial condition

$$(Mu)(0) = ML_1^{-1}L_1y_0.$$

Applying Lemma 2 with $L = L_1 + L_2 - kM$, (H_{M,L_1}) holds with $5\alpha + 2\beta > 6$,

$$y_0 \in D(L_1), (\widetilde{f}_1, .., \widetilde{f}_N) \in \prod_{i=1}^{N} C^{\sigma_i}([0,\tau] : \mathbb{C}), \sigma_i \in ((3-2\alpha-\beta)/\alpha_1, 1),$$

$$z_0 = (L_1 + L_2 - kM)y_0 + F(0),$$

$(z_0, z_1, .., z_N) \in \prod_{j=0}^{N} X^{\gamma_j}_{L_1M^{-1}}$, $\gamma_j \in (5-3\alpha-2\beta, 1)$, $j = 0, .., N$ (notice that $5-3\alpha-2\beta \geq 1-\beta$) and let

$$\gamma = \min_{j=0...N} \gamma_j, \tau = \min_{i=1..N} \{\sigma_i, \chi_{\alpha,\beta,\gamma}\},$$

where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$.

Recall that

$$I_{\alpha,\beta,\gamma} = \begin{cases} ((3-2\alpha-\beta)/\alpha, \gamma), & \text{if } \gamma \in ((3-2\alpha-\beta)/\alpha, 1/2), \\ ((3-2\alpha-\beta)/\alpha, 1/2), & \text{if } \gamma \in [1/2, 1). \end{cases}$$

Then for any fixed $\delta \in I_{\alpha,\beta,\tau}$ and $\widetilde{F} \in C^{\theta}([0,\tau]; X)$, $\theta \in [\delta + (3-2\alpha-\beta)/\alpha, 1)$ problem (12), (13) admits a unique strict solution w satisfying $w(0) = y_0$, $Mw \in C^{1+\delta}([0,\tau]; X)$,

$$L_1w = L_1(L_1 + L_2 - kM)^{-1}(L_1 + L_2 - kM)w \in C^{\delta}([0,\tau]; X).$$

Condition $(L_1 + L2 - kM)y_0 + F(0) \in X^{\gamma^0}_{L_1M^{-1}}, \gamma_0 > 5-3\alpha-2\beta$ is guaranteed by $L_1y_0 + F(0) \in X^{\gamma_0}_{L_1M^{-1}}$ if $\gamma_j \in (5-3\alpha-2\beta, \beta)$, $j = 0, 1, .., N$, $\alpha + \beta > 5/3$.

Indeed, $L_2y_0 \in X^{\overline{\delta}}_{L_1M^{-1}}, \overline{\delta} = \min \gamma_1..\gamma_N$ and $My_0 \in X^{\omega}_{L_1M^{-1}}$ for all $\omega < \beta$, since $y_0 \in D(L_1)$. Thus we have what required provided that $\gamma_1..\gamma_N \in (0, \beta)$, too.

In our application to inverse problems, we have seen that in fact $f_i(t)$ is a linear combination of $g^1_k(t)$ and $\Phi_k[F(t)]$. So we can establish the main identification problem (6)–(8) as follows.

Theorem 2. *Suppose*

$$L_1y_0 + F(0) \in X^{\gamma_0}_{A_1}, \gamma_0 \in (5-3\alpha-2\beta, \beta), \alpha \geq \beta > 0, \alpha + \beta > 5/3$$

H_{A_1} *holds with* $5\alpha + 2\beta > 6$

$$(z_1..z_N) \in \prod_{j=1}^{N} X^{\gamma_j}_{A_1}, \gamma_j \in (5-3\alpha-2\beta, \beta),\ \beta > \gamma_j > \gamma_0,\ j = 1, 2, \ldots, N,$$

$$\tau = \min\{\mu, (\alpha+\beta+\gamma-2)/\alpha\},$$

where $\mu = \min_{i=0,1} \mu_i$, $F \in C^{H_0}([0,\tau];X)$, $\mu_0 - 1/2 \in [(3-2\alpha-\beta)/\alpha, 1/2)$ $(g_1..g_N) \in \prod_{i=1}^{N} C^{1+\mu_1}([0,\tau],\mathbb{C})$, $\mu_1 \in [(3-2\alpha-\beta)/\alpha, 1)$. *Then for every fixed* $\delta \in I_{\alpha,\beta,\tau}$ *the identification problem (6)-(8) admits a unique* $(y, h_1..h_N)$ *such that* $y \in C^{\delta}([0;\tau], D(L_1), My \in C^{1+\delta}C[0,\tau];X)$, $y(0) = y_0$ *and* $f_i \in C^{\delta}([0,\tau];\mathbb{C}), i = 1..N$.

4. Examples

Example 1. If L, M are matrices $n \times n$, M singular, L invertible and 0 is a simple pole for $(\lambda I - M)^{-1}$, then LM^{-1} satisfies our condition with $\alpha = \beta = 1$ and $(\mathbb{C}^{\mathbb{M}}; D(A))_{\theta,\infty} = (C^M, M(\mathbb{R}^n))_{\theta,\infty} \equiv rangeM = X_A^{\theta}$.

Therefor our condition for the inverse problems became very simple.

Example 2. Let $\Omega \subset \mathbb{R}^n$, $n \in N$ be a bounded open domain with C^2 boundary and let $(X, \|\cdot\|_X) = (L^p(\Omega), \|\cdot\|_{L^p(\Omega)}), p \in (1,\infty)$. Let $m \in L^{\infty}(\Omega)$ and nonnegative. Let M be the multiplication operator by m so that $M \in \mathcal{L}(X)$. Let us consider operator L_1

$$L_1 u = -\sum_{|\overline{\eta_1}|,|\overline{\eta_2}|=1} D^{\overline{\eta_1}}(a_{\overline{\eta_1},\overline{\eta_2}} D^{\overline{\eta_2}} u) - a_0 u,$$

$$u \in D(L_1) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega).$$

More generally, one can take Robin B.C.

Here $a_0, a_{\overline{\eta_1},\overline{\eta_2}} : \Omega \to R$ $|\overline{\eta_i}| = 1, \overline{\eta_i} = (\eta_1..\eta_n), \eta_i \in (0,1), i = 1,2,\ j = 1..n$, $a_0 \in C(\overline{\Omega})$, $a_{\overline{\eta_1},\overline{\eta_2}} = a_{\overline{\eta_2},\overline{\eta_1}} \in C^1(\overline{\Omega})$, $a_0(x) \geq v_0$, $v_1|y|^2_{R^n} \leq \sum_{|\eta_1|,|\eta_2|=1} a_{\overline{\eta_1},\overline{\eta_2}}(x) y^{\overline{\eta_1}} y^{\overline{\eta_2}} \leq v_0|y|^2$ $\forall (x,y) \in \overline{\Omega} \times R^n$, $v_0, v_1, v_2 > 0$. Then it is shown in Favini and Yagi [16]

$$\|M(zM - L)^{-1}\|_{\mathcal{L}(x)} \leq C(1+|z|)^{-1/p}, \forall z \in \Sigma_1.$$

Therefore (H_{M,L_1}) is satisfied with $(\alpha,\beta) = (1;1/p)$. We must take $p \in (1,4/3)$ and our assumptions read $L_1 y_0 + f(0) \in X_{A_1}^{\varphi_0}$, $\varphi_0 \in (3/p', 1/p)$, $(z_1..z_N) \in \prod_{i=1}^{N} X_{A_1}^{\gamma_i}$, $\gamma_i \in (2/p', 1/p)$, $i = 1,\dots,N$, $F \in C^{M_0}([0,\tau];X)$, $M_0 - 1/2 \in (1/p', 1/2)$, $(g_1..g_N) \in \prod_{i=1}^{N} C^{1+\mu_1}([0,\tau];\mathbb{C})$, $\mu_1 \in (1/p', 1)$, $i = 1,\dots,N$.

Functionals Φ_i are defined by $\Psi_i \in L^{p'}(\Omega), \Phi_i[u] = \int_{\Omega} \Psi_i(x) u(x) dx$.

Values of p larger than 4/3 can be admitted if m is more regular, precisely, ρ-regular, i.e., $m \in C^1(\overline{\Omega})$ and $|\bigtriangledown m(x)| \leq C m(x)^{\rho}$, where $0 < \rho \leq 1$, $x \in \overline{\Omega}$. We refer to Favini, Lorenzi, Tanabe [8], [9], [10].

Example 3. Consider the problem

$$\frac{\partial u}{\partial t} = \triangle(a(x)u) + f(x,t), (x,t) \in \Omega \times (0,\tau),$$

$$a(x)u(x,t) = 0, (x,t) \in \partial\Omega \times (0,\tau),$$

$$u(x,0)=u_0(x),$$

where $a(x)\geq 0, a(x)>0$ a.e. in Ω.

Changing the unknown u to $v=a(x)u$ we get

$$\frac{\partial}{\partial t}\left(\frac{v}{a(x)}\right)=\triangle v+f(x,t),$$

$$v(x,t)=0,(x,t)\in\partial\Omega\times(0,\tau),$$

$$\frac{v}{a(x)}(x,0)=u_0(x).$$

This problem can be handled provided that

$$a^{-1}\in L^1(\Omega), if\ n=1$$

$$a^{-1}\in L^r(\Omega) forsome\ r>1, if\ n=2$$

$$a^{-1}\in L^{n/2}(\Omega), if\ n\geq 3$$

We obtain the resolvent estimate

$$\|M(\lambda M-L)^{-1}\|_{\mathcal{L}(H^{-1}(\Omega))}\leq C(1+|\lambda|)^{-1},$$

$\lambda\in\Sigma_1$. Therefore, the previous results apply to

$$\frac{\partial y}{\partial t}=\triangle(a(x)v)+\sum_{i=1}^{N}f_i(t)z_i(x)+F(t,x)$$

in the space $X=H^{-1}(\Omega)$. If one wants to consider $X=L^2(\Omega)$, other conditions on $a(x)$ are necessary. See Favini and Yagi, p. 83. Precisely,

$$a^{-1}\in L^r(\Omega)\begin{cases} with\ r\geq 2, when\ n=1,\\ with\ r>2, when\ n=2,\\ with\ r\geq n, when\ n\geq 3\end{cases}$$

Then one verifies that

$$\|M(\lambda M-L)^{-1}\|_{\mathcal{L}(L^2(\Omega))}\leq C|\lambda|^{-\frac{2r-n}{2r}},\lambda\in\Sigma_1.$$

Moreover, one can also consider

$$\frac{\partial u}{\partial t}=a(x)\triangle u+f(x,t),$$

$$u(x,t)=0,\ \text{on}\ \partial\Omega\times(0,\tau),$$

$$u(x,0)=u_0(x),$$

taking either $X=H_0^1(\Omega)$ or $X=L^2(\Omega)$. See Favini and Yagi [16] pp. 81-85. Of course, application to problems related to

$$\frac{\partial u}{\partial t}=a(x)\triangle u+\sum_{i=1}^{m}f_i(t)z_i+F(t,x)$$

follow easily.

Example 4. Our abstract result on the hyperbolic case apply to the concrete examples considered by A. Skubachevskii [19], [20], [21]. It should be interesting to verify if Feller semigroups that he considers are in fact weakly parabolic, too.

Example 5. Example of operators L satisfying previous resolvent estimates with $\beta < 1$, $\alpha = 1$ can be found in Tara, von Wahl [22], [23].

Example 6. Many important examples come from Russian literature. See various papers G. Sviridyuk and V. Fedorov. In particular, we quote [17] on an inverse problem for linear Sobolev type equations.

At the end, I want to thank professor Georgy Sviridyuk and Alyona Zamyshlyaeva for their competence and helpful contribution. Thanks a lot to professor H. Tanabe for his precious assistance and useful advice.

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Received November 13, 2014