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NUMERICAL RESEARCH OF THE BARENBLATT–ZHELTOV–KOCHINA MODEL ON THE INTERVAL WITH WENTZELL BOUNDARY CONDITIONS

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In terms of numerical research, we study the Barenblatt–Zheltov–Kochina model, which describes dynamics of pressure of a filtered fluid in a fractured-porous medium with the general Wentzell boundary conditions. Based on the theoretical results associated with Galerkin method, we develop an algorithm and implement the numerical solution of the Cauchy–Wentzell problem on the interval $[0,1]$. In particular, we examine the asymptotic approximation of the spectrum of the one-dimensional Laplace operator and present result of a computational experiment. In the paper, these problems are solved under the assumption that the initial space is a contraction of the space $L^2(0,1)$.

Keywords: Barenblatt–Zheltov–Kochina equation; Wentzell boundary conditions; numerical research; Galerkin method.

Introduction

Let us consider the Cauchy–Wentzell problem

$$\begin{aligned} &u(x,0)=v_0(x),\ x\in[0,1]\\ &u_{xx}(0,t)+\alpha_0u_x(0,t)+\alpha_1u(0,t)=0,\\ &u_{xx}(1,t)+\beta_0u_x(1,t)+\beta_1u(1,t)=0 \end{aligned} \tag{1}$$

for the Barenblatt–Zheltov–Kochina equation on the interval $[0,1]$

$$\lambda u_t(x,t)-u_{txx}(x,t)=\alpha u_{xx}(x,t)+f(x,t),\quad (x,t)\in[0,1]\times\mathbb{R}_+, \tag{2}$$

which describes dynamics of pressure of a filtered fluid in a fractured-porous medium. Here α and λ are the material parameters characterizing the environment; the parameter $\alpha\in\mathbb{R}_+$; the function $f=f(x,t)$ plays the role of external loading.

The purpose of this work is to show new approach to solve of problem (1)–(2) with the Wentzell boundary conditions. Namely, according to the modified Galerkin method, describe the solution to the Cauchy–Wentzel problem. Except Introduction, Conclusion and the References, the article contains four sections. Analytical research of the Barenblatt-Zheltov–Kochina model is given in Section 1. The algorithm for the numerical solution to the this model is presented in Section 2. The ideas of computational implementation are described in Section 3. The result of a computational experiment on resolvability of the Cauchy–Wentzell problem in the Barenblatt–Zheltova–Kochina model are given in Section 4.

1. Analytical Research of the Barenblatt–Zheltov–Kochina Model

In this section, we recall the main results proved in the paper [1] necessary for further numerical solution of problem (1)–(2). Let us consider the differential operator

$$Au(x) = u''(x), \quad x \in [0,1] \tag{3}$$

with the general Wentzell boundary conditions

$$Au(0) + \alpha_0 u'(0) + \alpha_1 u(0) = 0, \tag{4}$$

$$Au(1) + \beta_0 u'(1) + \beta_1 u(1) = 0. \tag{5}$$

By formulas (3)–(5) we define the linear operator $A : dom\ A \subset \mathfrak{F} \to \mathfrak{F}$. Here $\mathfrak{F}$ is the space $(L^2[0,1], dx\Big|_{(0,1)} \oplus \eta ds\big|_{\{0,1\}})$ with the norm

$$\|u\|^2_{\mathfrak{F}} = \int\limits_0^1 |u(x)|^2 dx + \eta_0 |u(0)|^2 + \eta_1 |u(1)|^2,$$

where dx is the Lebesque measure on the interval $(0,1)$; ds is the point measure at the boundary; $\eta_0 = \frac{1}{-\alpha_1}, \eta_1 = \frac{1}{\beta_1}$, where $\alpha_1 < 0 < \beta_1$ are positive weights. The full construction of the space $\mathfrak{F}$ is given in [2]. Also, we consider the linear manifold $dom\ A = \{u \in C^2[0,1]$: conditions $(4),(5)$ are fulfilled$\}$ as the domain of the operator A.

Then the operator A has the following properties.

Lemma 1. *Let the operator A be defined by formulas (3)–(5). Then*

(i) dom $A = \{u \in C^2[0,1]$: conditions (4), (5) are fulfilled$\}$ is a Banach space with regard to the norm $\|u\|_{C^2[0,1]}$;

(ii) dom A is densely embedded in $\mathfrak{F}$;

(iii) $A \in \mathcal{L}(domA; \mathfrak{F})$.

Moreover, the following Theorem 1 confirms the existence of solution to problem (1)–(2).

Theorem 1. *Suppose that the linear operator A satisfies the conditions of Lemma 1, and $f \in \mathfrak{F}$ is a fixed vector. Then*

(i) if $\lambda \notin \sigma(A)$, then for any $v_0 \in domA$ and $f \in \mathfrak{F}$ there exists the unique solution $u \in C^2(\mathbb{R}; domA)$ to problem (1)–(2), which has the form

$$u(x,t) = \sum_{k=1}^{\infty} e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} < v_0, \varphi_k >_{\mathfrak{F}} \varphi_k(x) + \sum_{k=1}^{\infty} \left(e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} - 1 \right) \frac{< f, \varphi_k >_{\mathfrak{F}}}{\alpha\lambda_k} \varphi_k(x);$$

(ii) if $\lambda \in \sigma(A)$, then for any $f \in \mathfrak{F}$ and

$$v_0 \in \mathfrak{P}_f = \left\{ u \in domA : \alpha\lambda < u, \varphi_k >_{\mathfrak{F}} = - < f, \varphi_k >_{\mathfrak{F}}, \lambda_k = \lambda \right\}$$

there exists the unique solution $u \in C^2(\mathbb{R};\mathfrak{P}_f)$ to problem (1)–(2), and the solution has the form

$$u(x,t) = -\frac{1}{\alpha\lambda}\sum_{\lambda=\lambda_k} < f,\varphi_k >_{\mathfrak{F}} \varphi_k(x) + \sum_{\lambda\neq\lambda_k} e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} < v_0(x),\varphi_k >_{\mathfrak{F}} \varphi_k(x) +$$

$$\sum_{\lambda\neq\lambda_k}\left(e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t}-1\right)\frac{< f,\varphi_k >_{\mathfrak{F}}}{\alpha\lambda_k}\varphi_k(x).$$

2. The Algorithm for the Numerical Solution of the Barenblatt–Zheltova–Kochina Model

It is necessary to find an approximate solution using the modify Galerkin method, since the Barenblatt-Zheltov–Kochina model may be degenerate. Let us construct Galerkin approximations solution to the Cauchy–Wentzell problem in the following form:

$$\widetilde{u}(x,t) = u_N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x), \tag{6}$$

where $\{\varphi_k : k \in \mathbb{N}\}$ are eigenfunctions of the one-dimensional Laplace operator A and correspond to its eigenvalues, orthonormal by the norm $< \cdot,\cdot >_{\mathfrak{F}}$, which are numbered in non-increasing order taking into account the multiplicity.

Substitute approximate solution (6) to equation (2) and take the scalar product of equation (2) and eigenfunctions $\varphi_k(x)$ with respect to $< \cdot,\cdot >_{\mathfrak{F}}$. We obtain the following system:

$$\begin{cases} (\lambda-\lambda_1)u_1'(t) = \alpha u_1(t) + f_1(t), \\ (\lambda-\lambda_2)u_2'(t) = \alpha u_2(t) + f_2(t), \\ \dots \\ (\lambda-\lambda_N)u_N'(t) = \alpha u_N(t) + f_N(t). \end{cases} \tag{7}$$

Depending on the parameters λ , we have algebraic or first-order differential equations in the system (7). Let us consider these conditions in more details.

(i) $\lambda \notin \sigma(A)$. Due to this fact, the mathematical model is non-degenerate, and all the equations in the resulting system are ordinary differential equations of the first order. For the solvability this system with respect to $u_k(t)$, we take the scalar product of initial condition (1) and eigenfunctions $\varphi_k(x)$ with respect to $< \cdot,\cdot >_{\mathfrak{F}}$. Then, we solve the system (7) with appropriate initial conditions and find the coefficients $u_k(t)$ in the approximate solution $\widetilde{u}(x,t)$.

(ii) $\lambda \in \sigma(A)$. Without loss of generality suppose that $\lambda = \lambda_{m_1} = \dots = \lambda_{m_r}$, where r is the multiplicity of the root. Then, some of equations are algebraic, and some equations are ordinary differential equations of the first order. Let us consider separately systems composed of algebraic equations and differential equations of the first order. Note that the solution to the original problem exists, according to Theorem 1, if the initial function $v_0(x)$ belongs to the phase space

$$\mathfrak{P}_f = \left\{u \in domA : \alpha\lambda < u,\varphi_k >_{\mathfrak{F}} = - < f,\varphi_k >_{\mathfrak{F}}, \lambda_k = \lambda\right\}.$$

3. The Computational Implementation of the Numerical Solution to the Barenblatt–Zheltova–Kochina Model

Since the Galerkin method is not of most interest, we describe the main ideas, in the author' view, associated with implementation of a numerical solution.The block diagram of the algorithm is shown in Figure 1 .

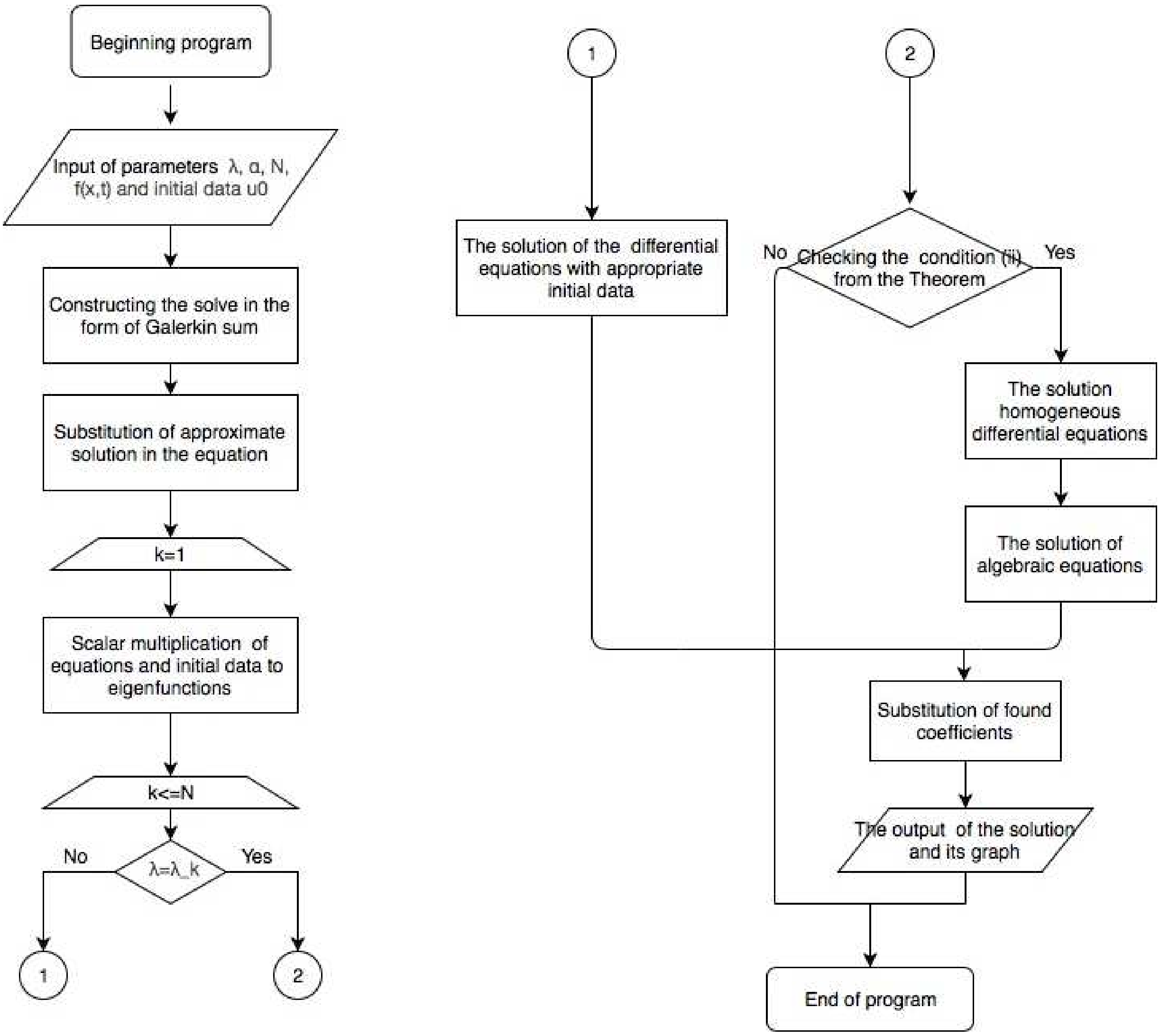

Fig. 1. Block diagram of the algorithm

Remark 1. Due to the fact that we are the first to consider this problem, we describe the asymptotic approximation of eigenvalues. The paper [1] shows that the operator A has a discrete, finite multiplicity spectrum with the unique limit point at $-\infty$. Consider the case of $\lambda < 0$. The transcendental equation has the following form

$$\frac{\lambda_n + \alpha_1}{\alpha_0\sqrt{-\lambda_n}} = \frac{cot(\sqrt{-\lambda_n}) - \frac{\beta_0\sqrt{-\lambda_n}}{\lambda_n} + \frac{\beta_1}{\lambda_n}cot(\sqrt{-\lambda_n})}{1 + \frac{\beta_0\sqrt{-\lambda_n}}{\lambda_n}cot(\sqrt{-\lambda_n}) + \frac{\beta_1}{\lambda_n}}$$

Substitute $x = \sqrt{-\lambda_n}$, for $\lambda_n < 0$ and have the alternative form

$$cot(x) = xK(x),$$

where

$$K(x) = \frac{1 - \frac{\alpha_1}{x^2} - \frac{\beta_1}{x^2} + \frac{\alpha_1\beta_1}{x^4} + \frac{\beta_0\alpha_0}{x^2}}{-\alpha_0 + \frac{\alpha_0\beta_1}{x^2} + \beta_0 - \frac{\alpha_1\beta_0}{x^2}}$$

Due to the fact that $tan(x) \cdot cot(x) = 1$, we have $tan(x) = \frac{1}{K(x)}$. Therefore,

$$x_n = \pi n + arctan\left(\frac{1}{K(x)}\right), n \in \mathbf{Z}.$$

Applying the method of asymptotic iterations we have

$$x_n \sim \pi n + \left(\frac{-\alpha_0 + \beta_0}{\pi n}\right) + O\left(\frac{1}{n^3}\right).$$

Consequently,

$$\lambda_n \sim -\left(\pi n + \left(\frac{-\alpha_0 + \beta_0}{\pi n}\right) + O\left(\frac{1}{n^3}\right)\right)^2.$$

Consider the case of $\lambda > 0$. In this case the existence of solution depends on the Wentzell boundary conditions. If the transcendental equation has the form

$$\frac{\lambda + \alpha_0\sqrt{\lambda} + \alpha_1}{\lambda - \alpha_0\sqrt{\lambda} + \alpha_1} = \frac{e^{2\sqrt{\lambda}}(\lambda + \beta_0\sqrt{\lambda} + \beta_1)}{\lambda - \beta_0\sqrt{\lambda} + \beta_1},$$

and is solvable, we add λ in set of eigenvalues.

Remark 2. In order to find the solutions to the transcendental equations we use, for example, the method of moving chords.

```
double f(double x)
{
    return F(x); //here we substitute the function
}

double method_chord(double x_prev, double x_curr, double e)
{
    double x_next;
    double i=0;
    x_next=x_curr-f(x_curr)*(x_prev-x_curr)/(f(x_prev)-f(x_curr));
    while(fabs(x_next-x_curr)>e)
    {
        i++;
        x_prev=x_curr;
        x_curr=x_next;
        x_next=x_curr-f(x_curr)*(x_prev-x_curr)/(f(x_prev)-f(x_curr));
    }
    cout<<"N="<<i<<endl;
    return x_next;
}

int main()
{
    double x0=6.1;
    double x1=3.2;
    double e=0.5E-7;
    double x=method_chord(x0,x1,e);
    cout<<"X="<<x<<endl;
    return 0;
}
```

Remark 3. Since the required eigenfunctions are not orthonormal, we find them applying the Gram-Schmidt orthogonalization method using the scalar product with respect to $< \cdot, \cdot >_{\mathfrak{F}}$. The implementation of scalar product is presented below.

```
float f1(float x)
{
    return 0.2082614333*(cos(5.928333984*x)+6.434378310*sin(5.928333984*x));
}

float f2(float x)
{
    return 0.2082614333*(cos(5.928333984*x)+6.434378310*sin(5.928333984*x));
}
float g(float x){
    return f1(x)*f2(x);
}

float integrate(float a, float b, int n){
    float h, tempp;

    float trap_int = (g(a)+g(b))/2.0;
    tempp = a;
    h = (b - a) / n;

    for (tempp= a+h; tempp < b; tempp+=h){
        cout<<tempp<<endl;
        trap_int += g(tempp);
    }

    return trap_int * h;
}

float scal()
{
    return integrate(0,1 , 10)+f1(0)*f2(0)/3.0+f2(1)*f1(1)/6.0;
}

int main()
{
    double rezscal=scal();
    cout<<"scal="<<rezscal<<endl;
    return 0;
}
```

4. The Result of a Computational Experiment on Resolvability of the Cauchy–Wentzell Problem in the Barenblatt–Zheltov–Kochina Model

Example 1. Let us consider the Cauchy–Wentzell problem for the equation

$$\lambda u_t(x,t) - u_{txx}(x,t) = \alpha u_{xx}(x,t) + f(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}_+, \tag{8}$$

where $\lambda = 1$, $\alpha = 1$, $f(x,t) = sin(x) + cos(x)$,

$$\begin{aligned} &u(x,0) = sin(x),\\ &u_{xx}(0,t) + u_x(0,t) - 3u(0,t) = 0,\\ &u_{xx}(1,t) - u_x(1,t) + 6u(1,t) = 0. \end{aligned}$$

Let $N = 6$, then the approximate solution have the following form:

$$\widetilde{u}(x,t) = u_6(x,t) = \sum_{k=1}^{6} u_k(t)\varphi_k(x). \tag{9}$$

Solve the Sturm-Liouville problem and find the basis functions $\varphi_k(x)$ in decomposition (9). Using the method of moving chords for the transcendental equations of the corresponding form

$$cot(x) = x \cdot \frac{1+\frac{3}{x^2}-\frac{6}{x^2}-\frac{18}{x^4}-\frac{1}{x^2}}{\frac{6}{x^2}-2-\frac{3}{x^2}}, x=\sqrt{-\lambda_n}, \lambda_n<0$$
$$\frac{\lambda+\sqrt{\lambda}-3}{\lambda-\sqrt{\lambda}-3} = \frac{e^{2\sqrt{\lambda}}(\lambda-\sqrt{\lambda}+6)}{\lambda+\sqrt{\lambda}+6)}, \lambda>0$$

find and write the eigenfunctions of the one-dimensional Laplace operator.

We have the eigenvalues

$$\begin{aligned}
\lambda_1 &= -x_1^2 = -35.14514947,\\
\lambda_2 &= -x_2^2 = -84.71034130,\\
\lambda_3 &= -x_3^2 = -153.8532547,\\
\lambda_4 &= -x_4^2 = -242.7027758,\\
\lambda_5 &= -x_5^2 = -351.2803151,\\
\lambda_6 &= 5.39027.
\end{aligned}$$

Let us find $\varphi_k(x)$ and construct an orthonormal basis using the Gram-Schmidt method. Substitute approximate solution (9) to equation (8) and take the scalar product of equation (8) and eigenfunctions $\varphi_k(x)$ with respect to $<\cdot,\cdot>_{\mathfrak{F}}$. We obtain the following system:

$$\begin{cases}
36.14514947u_1'(t)+35.14514947u_1(t)-0.8269934837=0,\\
85.71034130u_2'(t)+84.71034130u_2(t)+0.4628025921=0,\\
154.8532547u_3'(t)+153.8532547u_3(t)-0.1585809892=0,\\
243.7027758u_4'(t)+242.7027758u_4(t)+0.2327184742=0,\\
352.2803151u_5'(t)+351.2803151u_5(t)-0.1438824946=0,\\
-4.39027u_6'(t)-5.39027u_6(t)+1.100046897=0.
\end{cases} \tag{10}$$

Due to the fact that $\lambda \notin \sigma(A)$, the mathematical model is non-degenerate, and, according to the algorithm, all the equations in the resulting system are ordinary differential equations of the first order. Let us solve the system (10) with the initial conditions

$$\begin{aligned}
u_1(0) &= -0.2424361874,\\
u_2(0) &= 0.1553935266,\\
u_3(0) &= -0.1527659262,\\
u_4(0) &= 0.7822663760,\\
u_5(0) &= -0.1135077380,\\
u_6(0) &= 0.6048383102.
\end{aligned}$$

and find the Galerkin coefficients

$$
\begin{aligned}
u_1(t) &= 0.2353080002 - 0.2447892674 \cdot e^{-0.9723337705t},\\
u_2(t) &= -0.5463354119 + 0.1608568807 \cdot e^{-0.9883327964t},\\
u_3(t) &= 0.1030728856 - 0.1537966551 \cdot e^{-0.9935422733t},\\
u_4(t) &= -0.9588620214 + 0.7918549962 \cdot e^{-0.9958966409t},\\
u_5(t) &= 0.4095945273 - 0.1139173325 \cdot e^{-0.9971613515t},\\
u_6(t) &= 0.2040801105 + 0.4007581997 \cdot e^{-1.227776424t}.
\end{aligned}
$$

Substituting the Galerkin coefficients to representation (9) and obtain an approximate solution to the original problem. The graph of the solution is shown in Figure 2.

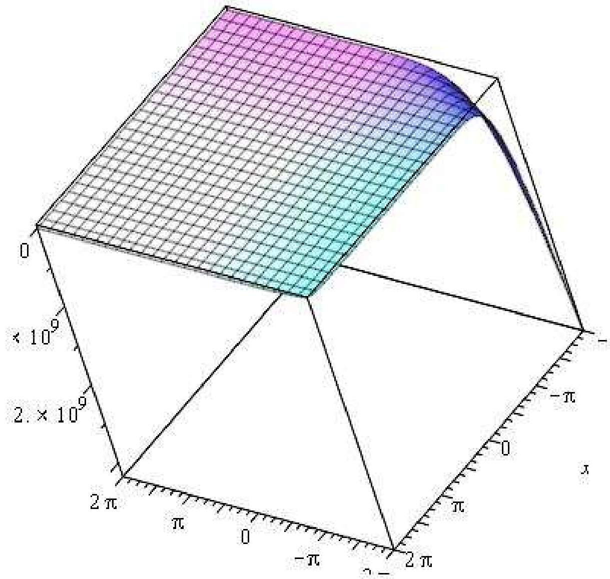

Fig. 2. Solution to the problem in Example 1

Example 2. Let us consider the Cauchy–Wentzell problem for the equation

$$\lambda u_t(x,t) - u_{txx}(x,t) = \alpha u_{xx}(x,t) + f(x,t), \quad (x,t) \in [0,1] \times \mathbb{R}_+, \tag{11}$$

where $\lambda = 0$, $\alpha = 0$, $f(x,t) = sin(2x)$,

$$
\begin{aligned}
&u(x,0) = cos(x),\\
&u_{xx}(0,t) + u_x(0,t) - 3u(0,t) = 0,\\
&u_{xx}(1,t) - u_x(1,t) + 6u(1,t) = 0.
\end{aligned}
$$

Let $N = 6$, then the approximate solution have the following form:

$$\widetilde{u}(x,t) = u_6(x,t) = \sum_{k=1}^{6} u_k(t)\varphi_k(x). \tag{12}$$

We have the same eigenvalues and the orthonormal basis $\varphi_k(x)$, since the boundary conditions are the same as in Example 1.

Substitute approximate solution (12) to equation (11) and take the scalar product of equation (11) and eigenfunctions $\varphi_k(x)$ with respect to $< \cdot, \cdot >_{\mathfrak{F}}$. We obtain the following system:

$$\begin{cases} 35.14514947u_1'(t) - 0.2771233330 = 0, \\ 84.71034130u_2'(t) + 0.1707707104 = 0, \\ 153.8532547u_3'(t) - 0.1808766246 = 0, \\ 242.7027758u_4'(t) + 0.08093001826 = 0, \\ 351.2803151u_5'(t) - 0.1369937226 = 0, \\ -5.39027u_6'(t) + 0.8120110804 = 0. \end{cases} \tag{13}$$

Due to the fact that $\lambda \notin \sigma(A)$, the mathematical model is non-degenerate, and, according to the algorithm, all the equations in the resulting system are ordinary differential of the first order. Let us solve the system (13) with the initial conditions

$$\begin{aligned} u_1(0) &= 0.1597368392, \\ u_2(0) &= 0.3074090655, \\ u_3(0) &= -0.581506299, \\ u_4(0) &= 0.1544918366, \\ u_5(0) &= -0.3037475666, \\ u_6(0) &= 0.4952085868. \end{aligned}$$

and find the Galerkin coefficients

$$\begin{aligned} u_1(t) &= 0.7885108960 \cdot t + .1597368392, \\ u_2(t) &= -0.2015936989 \cdot t + .3074090655, \\ u_3(t) &= 0.1175643797 \cdot t - 0.5815062990, \\ u_4(t) &= -0.3334532042 \cdot t + 0.1544918366, \\ u_5(t) &= 0.3899840575 \cdot t - 0.3037475666, \\ u_6(t) &= 0.1506438602 \cdot t + 0.4952085868. \end{aligned}$$

Substituting the Galerkin coefficients to representation (12) and obtain an approximate solution to the original problem. The graph of the solution is shown in Figure 3.

Conclusion

We constructed an algorithm and implemented the numerical solution to the Cauchy–Wentzell problem on the interval $[0, 1]$. To this end, we used the numerical methods theory, and the space, the structure of which is specified in [2]. Further, we plan to continue the results of the paper by applying the Wentzell boundary conditions in directions related to [10].

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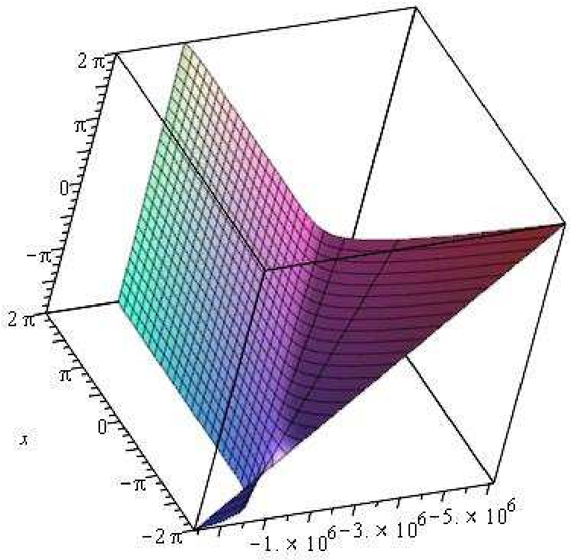

Fig. 3. Solution to the problem in Example 2

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ЗАДАЧИ КОШИ – ВЕНТЦЕЛЯ ДЛЯ МОДЕЛИ БАРЕНБЛАТТА – ЖЕЛТОВА – КОЧИНОЙ

Н. С. Гончаров

В статье рассматривается численное исследование модели Баренблатта – Желтова – Кочиной, которая описывает динамику движения жидкости в трещиновато-пористой среде. На основе теоретических результатов, связанных с методом Галеркина, разработан алгоритм и реализация численного решения задачи Коши-Вентцелля на отрезке $[0, 1]$. В частности, рассматривается асимптотическая аппроксимация спектра одномерного оператора Лапласа и приводится результат вычислительного эксперимента. В работе эти задачи решаются в предположении, что начальное пространство является сужением пространства $L^2(0, 1)$.

Ключевые слова: уравнение Баренблатта – Желтова – Кочиной; задача Коши – Вентцеля; метод Галеркина; численное моделирование.

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