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A NUMERICAL ALGORITHM FOR SOLVING INVERSE FILTRATION PROBLEMS WITH THE POINTWISE OVERDETERMINATION

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The inverse problems of recovering the right-hand side in a pseudoparabolic equations of filtration with the use of the pointwise overdetermination are studied. We expose some existence and uniqueness theorems which are the base of an numerical algorithm of recovering the right-hand side (the source function) and a solution. The problem is well-posed and the stability estimates hold. It can be reduced to a Volterra-type integral equation, where the operator has a small norm for small time segments. The finite element method is used to reduce the problem to a system of ordinary differential equations which is solved by the finite difference method. The idea of the predictor-corrector method is employed in the algorithm. The results of numerical experiments are presented. They show a good convergence of an approximate solutions to a solution.

Keywords: inverse problem; pseudoparabolic equation; filtration; fissured rock; numerical solution.

Introduction

In the present article we consider an inverse problem of recovering the right-hand side in a Sobolev-type equation of the third order. These equations belong to the class of equation unsolvable with respect to higher derivatives. The systematical study of equations of this class began with the S. L. Sobolev articles (see [1]). Afterwards, S. L. Sobolev's results were generalized by many authors. We can refer to the known results by S. A. Galpern, S. G. Krein, M. I. Vishik, R. Schowalter, T. I. Zelenyak, G. V. Demidenko, S. I. Uspenskii, and many other authors (see the bibliography in [2]). The most known third order Sobolev-type models are the equation of Rossby waves [3] proposed by C. G. Rossby in 1939 and the filtration theory equations derived by G. I. Barenblatt, Iu. P. Zheltov and I. N. Kochina [4] in 1960. The latter model is written as

$$u_{1t} - \eta\Delta u_{1t} - k\Delta u_1 = 0, \tag{1}$$

where the parameter k corresponds to the piezo-conductivity of fissured rock and u is the pressure. The dimensionless coefficient α characterizes the intensity of the liquid transfer between the blocks and fissures. More general models can include nonlinearities arising from fluid type (a liquid or a gas), concentration (porosity, absorption or saturation) and the exchange rate [5].

General equations of the form (1) can be written as follows:

$$L(t,x,D)u_t - M(t,x,D)u = f, (x,t) \in Q = G \times (0,T), \tag{2}$$

where L, M are second order operators and G is a bounded domain in $\mathbb{R}^n$. The equation (2) is furnished with initial and boundary conditions of the form

$$u(0,x) = u_0(x), \quad Ru|_S = g(t,x), \tag{3}$$

with $Ru = u$ or $Ru = \sum_{i=1}^{n} \gamma_i(t,x)u_{x_i} + \sigma(t,x)u$ (other boundary conditions are also possible). We look for the right-hand side f of the form

$$f = \sum_{i=1}^{r} q_i(t) f_i(t,x) + f_0(t,x), \quad f_i \in L_\infty(0,T; L_p(G)), \tag{4}$$

Our problem is stated as follows: find the functions $\{q_i(t)\}_{i=1}^{r}$ and a solution u to the problem (2)-(3) such that

$$u(t, y_i) = \psi_i(t), \qquad (i = 1, 2, .., r), \tag{5}$$

where y_i are arbitrary points lying in G and $f_i(t,x)$ are given functions.

Sobolev-type equations of the form (2) with various differential operators L_1 and L_2 of even order in the spacial variables also arise in the mathematical models of the heat conduction, wave processes, quasistationary processes in semiconductors and magnetics, in the models for filtration of the two-phase flow in porous media with the dynamic capillary pressure (see [7, 36], [6] and the bibliography therein). Detailed bibliography and the results concerning the solvability of direct problems for Sobolev-type equations and their abstract analogs can be found, for instance, in [9, 8, 10, 38, 37]. The first results devoted to inverse problems for pseudoparabolic equations were obtained in [11], where an inverse problems of recovering an unknown source f of a special form in (2) is considered. Large number of results is exposed in the monographs [13], [12]). The problems of recovering coefficients, in particular, the coefficients $k(t)$ and η are studied in [16, 17, 18], where integral overdetermination conditions are used. The problem (1)-(5) is considered in [19, 20, 21] and it is proven that this problem is uniquely solvable under natural conditions for the data. Closed results on recovering the right-hand side of the form $f(t)g(x)$ (the function $f(t)$ is not known) are exposed in [14, 15] even for more general classes of the equations. Exposition of numerical methods for solving inverse problem can be found, for instance, in [22, 23]. We can refer also to the articles [25, 28, 27, 30, 31, 32, 29, 39] devoted to different numerical methods of solving boundary value problems for Sobolev-type equations. At the same time, the number of articles devoted to numerical solving inverse problems for Sobolev-type equations is rather limited (see, for instance, [33, 32, 34]. Most of the articles are devoted to different model problems. Some numerical methods for solving filtration problems of the form (2)-(5) but for simpler models are presented in [24]. Here the Sobolev-type equation for the pressure is replaced with a parabolic one.

We use the theoretical results exposed in [19, 20, 21], where the existence and uniqueness theory as well as the stability estimates for solutions can be found, describe numerical methods applicable to a wide class of inverse problems with the pointwise ovedetermination of the form (2)-(5), and present the results of numerical experiments.

1. Preliminaries

We consider a general inverse problem on recovering functions occurring into the right-hand side of the equation. We assume that

$$L = \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{n} a_i(t,x)\partial_{x_i} + a_0(t,x)$$

and the operator M is representable as

$$Mu = \sum_{i,j=1}^{n} b_{ij}(t,x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + b_0(t,x)u,$$

Put $(u,v) = \int_G u(x)v(x)\,dx$. All function spaces as well as coefficients of the equations are assumed to be real.

We employ the Sobolev spaces $W_p^s(G)$ and Hölder spaces $C^{\alpha}(\overline{G})$ (see the definitions in [35]). The symbol $L_p(0,T;H)$ (H is a Banach space) stands for the space of strongly measurable functions defined on $[0,T]$ with values in H. Given an interval $J=(0,T)$ and a domain $G \in \mathbb{R}^n$, put $Q=(0,T)\times G$ and $W_p^{r,s}(Q) = W_p^r(J;L_p(G)) \cap L_p(J;W_p^s(G))$. Respectively, $W_p^{r,s}(S) = W_p^r(J;L_p(\Gamma)) \cap L_p(J;W_p^s(\Gamma))$ $(S=(0,T)\times\partial G)$. Similarly, we can define the Hölder spaces $C^{r,s}(\overline{Q})$.

Next, we describe the condition on the data of the problem. We assume that the operator L is elliptic, i. e., there exists a constant $\delta > 0$ such that

$$\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \ge \delta_0|\xi|^2 \quad \forall\ \xi \in R^n, \quad \forall\ (t,x) \in \overline{Q}. \tag{6}$$

Fix a parameter $p > n$ and assume that

$$b_{ij} \in L_\infty(Q),\ \ b_i, b_0 \in L_\infty(0,T;L_p(G)), \tag{7}$$

$$a_{ij} \in C(\overline{Q}), \quad a_i, a_0 \in C([0,T];L_p(G)) \quad (i,j=1,2,..,n); \tag{8}$$

a) $a_0(t,x) \le 0$ a.e. (almost everywhere) in Q in the case of the Dirichlet boundary conditions and $a_0^0 \le 0$ a.e. in Q and $a_0^0 < 0$ a.e. in some neighborhood about S in the case of the oblique derivative problem;

$$\gamma_i, \gamma_{it}, \sigma, \sigma_t \in C^{1/2,1}(\overline{S}),\ \ i=1,2,\ldots,n. \tag{9}$$

$$\psi_i(0) = u_0(y_i) \quad (i=1,2,..,r),\ \ R(0,x,D)u_0|_\Gamma = g(0,x). \tag{10}$$

Let $s_0 = 2-1/p$ in the case of the Dirichlet boundary conditions and $s_0 = 1-1/p$ otherwise. Construct a matrix B with the rows

$$L^{-1}f_1(t,y_j), .., L^{-1}f_r(t,y_j)$$

where $j=1,2,...,r$ and assume that the there exists a constant $\delta_0 > 0$ such that

$$|\det B| \ge \delta_0 \quad \forall t \in [0,T]. \tag{11}$$

Here $L^{-1}f_i$ is a solution U_i to the problem $LU_i = f_i$, $U_i|_{t=0} = 0$, $RU_i|_S = 0$.

The following theorem follows from the results in [20], [21].

Theorem 1. *Let the conditions* (6)-(11) *be fulfilled and let*

$$f_0 \in L_p(Q),\ \ f_i \in L_\infty(0,T;L_p(G)), \quad u_0(x) \in W_p^2(G),$$

$$g_t \in L_p(0,T;W_p^{s_0}(G)),\ \ \psi_i \in W_p^1(0,T), i=1,2,\ldots,r, \quad p>n.$$

Then there exists a unique solution to $(u, q_1, \ldots, q_r)$ *the problem* (2)-(5) *such that*

$$u \in W_p^1(0,T;W_p^2(G)), \quad q_i(t) \in L_p(0,T) \ (i = 1,2,\ldots,r).$$

A solution satisfies the estimate

$$\|u\|_{W_p^1(0,T;W_p^2(G))} + \sum_{i=1}^{r} \|q_i(t)\|_{L_p(0,T)} \leq$$

$$c(\|f_0\|_{L_p(Q)} + \|g_t\|_{L_p(0,T;W_p^{s_0}(G))} + \sum_{i=1}^{r} \|\psi_i\|_{W_p^1(0,T)}).$$

This theorem actually justifies the numerical algorithm presented below and the scheme of the algorithm is taken from its proof.

2. Description of the Algorithm

To simplify the presentation, we describe the idea of the algorithm in the model case. We rely on some integral identities. Consider the problem

$$L_0 u_t + k(t) L_1 u = f = f_0 + \sum_{i=1}^{r} q_i(t) f_i(x,t), \tag{12}$$

$$\frac{\partial u}{\partial n}|_\Gamma = g, \quad u(0,x) = u_0(x), \tag{13}$$

$$u(y_i,t) = \psi_i(t), \quad i = 1,2,\ldots,r, \tag{14}$$

where

$$L_0 u = -div(a_0(x,t)\nabla u_t) + b_0(x,t) \cdot \nabla u + c_0(x,t) u,$$

$$L_1 u = -div(a_1(x,t)\nabla u) + b_1(x,t) \cdot \nabla u + c_1(x,t) u,$$

and a_0, a_1, c_0, c_1 are scalar functions and b_0, b_1 are vector-function of length n. The functions u and $q_i(t)$ are unknown. We assume that all conditions of Theorem 1 for the data are fulfilled. Let $\varphi \in L_q(0,T;W_q^1(G))$ $(1/q + 1/p = 1)$ be a test function and let a function u be a solution to the problem (12), (13) from the class pointed out in Theorem 1. Integrating by parts in the identity

$$(L_0 u_t, \varphi) + k(t)(L_1 u, \varphi) = (f, \varphi), \ \varphi \in L_q(0,T;W_q^1(G)), \tag{15}$$

we arrive at the equality

$$a(u_t, \varphi) + k(t) b(u, \varphi) = l(\varphi) + k(t) l_1(\varphi) + \sum_{k=1}^{r} q_k(t)(f_k, \varphi)), \tag{16}$$

where $a(u_t, \varphi) = (a_0 \nabla u_t, \nabla\varphi) + (b_0 \cdot \nabla u_t + c_0 u_t, \varphi)$, $b(u, \varphi) = (a_1 \nabla u, \nabla\varphi) + (b_1 \cdot \nabla u + c_1 u, \varphi)$, $l(\varphi) = (f_0, \varphi) + l_0(\varphi)$, $l_0(\varphi) = \int\limits_\Gamma a_0 g_t \varphi d\Gamma$, $l_1(\varphi) = \int\limits_\Gamma a_1 g \varphi dt$.

Next, we look for a solution $\varphi_j(x,t)$ $(j = 1,2,\ldots,r)$ to the problem

$$L_0^* \varphi_j = \delta(x - y_j), \quad a_0 \tfrac{\partial \varphi_j}{\partial n} + b \cdot n \varphi_j|_\Gamma = 0, \tag{17}$$

where L_0^* is a formally adjoint to L_0 and δ is the Dirac delta-function. Inserting φ_j in (16), we obtain that

$$\psi_{jt} + k(t)(b(u,\varphi_j) - l_1(\varphi_j)) = l(\varphi_j) + \textstyle\sum_{k=1}^{r} q_k(t)(f_k,\varphi_j). \tag{18}$$

Hence, we conclude that

$$\textstyle\sum_{k=1}^{r} q_k(t)(f_k,\varphi_j) = \psi_{jt} + k(t)(b(u,\varphi_j) - l_1(\varphi_j)) - l(\varphi_j). \tag{19}$$

Note that the definition of φ_j implies that

$$\sum_{k=1}^{r} q_k(t)(f_k,\varphi_j) = \sum_{k=1}^{r} q_k(t)L^{-1}f_k(y_j,t).$$

Thus, this expression can be written as $R\vec{q}$ and in view of the condition (11) the determinant of the matrix R does not vanish. The above integral identities allow us to construct the iteration procedure realized in the proof of Theorem 1 (see [19]-[21]). Let $\vec{q}_0 = R^{-1}F_0$, with $F_{0j} = \psi_{jt} + k(t)(b(u_0,\varphi_j) - l_1(\varphi_j)) - l(\varphi_j)$. Given a vector-function $\vec{q}_i$, we can construct u^{i+1} as a solution to the problem (12), (13) with $\vec{q} = \vec{q}_i$ and to determine the next iteration $\vec{q}_{i+1}$ from the equalities

$$\begin{gathered} \vec{q}_{i+1} = R^{-1}F_i, \quad F_i = (F_{i1},\dots,F_{ir}), \\ F_{ij} = \psi_{jt} + k(t)(b(u_{i+1},\varphi_j) - l_1(\varphi_j)) - l(\varphi_j). \end{gathered} \tag{20}$$

The latter formula almost corresponds to the iteration procedure in the proof of the fixed point theorem for the operator S constructed in the proof of Theorem 1 in [20], [21], where it is proven that the process converges.

3. Numerical Algorithm

The algorithm is iterative and relies on the finite element method. We define a triangulation of G, the mesh nodes, $x_1, x_2, ..., x_N$, and the corresponding piecewise linear functions $\{\varphi_i(x)\}$ (thus, $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Without loss of generality, we can assume that the observation points y_j are mesh node x_{m_j} ($j = 1,2,\dots,r$). An approximate solution to (12), (13) is sought in the form $u^N = \sum_{i=1}^{N} q_i(t)\varphi_i(x)$. Assume that $a(u,\varphi) = (a_0\nabla u, \nabla\varphi) + (b_0 \cdot \nabla u + c_0 u, \varphi)$, $b(u,\varphi) = (a_1\nabla u, \nabla\varphi) + (b_1\cdot\nabla u + c_1 u, \varphi)$, $l(\varphi) = (f_0,\varphi) + l_0(\varphi)$, $l_0(\varphi) = \int_\Gamma a_0 g_t \varphi d\Gamma$, $l_1(\varphi) = \int_\Gamma a_1 g\varphi dt$.

The vector-function $C(t) = (c_1(t), c_2(t), ..., c_N(t))^T$ is a solution to the system of ordinary differential equations

$$AC_t + k(t)BC = F_0 + F_1,\ C(0) = (u_0(x_1), u_0(x_2), \dots, u_0(x_N))^T, \tag{21}$$

where A, B are matrices with the entries $a_{ij} = a(\varphi_j,\varphi_i)$, $b_{ij} = b(\varphi_j,\varphi_i)$, and $F_0 = (F_{01},\dots,F_{0N})^T$, $F_{0i}(t) = l(\varphi_i) + k(t)l_1(\varphi_i)$, $F_1 = (F_{11},\dots,F_{1N})^T$, where $F_{1j} = \sum_{k=1}^{r} q_k(t)(f_k,\varphi_j)$. Thus, the vector F_1 is representable as $F_1 = R\vec{q}$, where the matrix $R = \{r_{ij}\}$ has the entries $r_{ij} = (f_j,\varphi_i)$. We thus have that

$$AC_t + k(t)BC = F_0 + R\vec{q},\ C(0) = (u_0(x_1), u_0(x_2), \dots, u_0(x_N))^T, \tag{22}$$

To solve (22), we involve the finite difference method (FDM) (the implicit scheme) and replace (22) with the finite difference equation

$$A_n \frac{C_n - C_{n-1}}{\tau} + k_n B_n C_n = F_{0,n} + R_n \vec{q}_n, \ C_0 = C(0), \tag{23}$$

where $n = 1, 2, \ldots, M$, $\tau = T/M$, and F_{0n}, A_n, B_n, R_n are the values of the right-hand side in (22), and the matrices A, B at $n\tau$. We assume here that the approximation $\vec{q}_n$ is a piecewise constant vector-function taking the value $\vec{q}_n$ on $((n-1)\tau, n\tau]$. Respectively, a piecewise constant approximation of a solution $C(t)$ to (21) is a piecewise constant function equal to the vector C_n on the set $((n-1)\tau, n\tau]$. An analog of the overdetermination condition is as follows:

$$\frac{(C_n)_{m_j} - (C_{n-1})_{m_j}}{\tau} = \psi_{jt}((n-1)\tau), \quad j = 1, 2, \ldots, r, \tag{24}$$

where $(C_n)_{m_j}$ is the m_j-th coordinate of the vector C_n. From (24) we have that

$$\psi_{jt}((n-1)\tau) + k_n((A_n)^{-1} B_n C_n)_{m_j} - ((A_n)^{-1} F_{0,n})_{m_j} = ((A_n)^{-1} R_n \vec{q}_n)_{m_j}, \tag{25}$$

where $j = 1, 2, \ldots, r$. Denote by α_{ij}^n the entries of the matrix $(A_n)^{-1} R_n$. We have that the right-hand side in (25) can be written as $S_n \vec{q}_n$ and S_n is the matrix with entries $\beta_{j,l}^n = \alpha_{m_j l}^n$. The left-hand side is the vector G_n with the coordinates $G_j^n = \psi_{jt}((n-1)\tau) + k_n((A_n)^{-1} B_n C_n)_{m_j} - ((A_n)^{-1} F_{0,n})_{m_j}$, $j = 1, 2, \ldots, r$. Thus, we can consider the equation

$$G_n = S_n \vec{q}_n. \tag{26}$$

Next, describe the numerical algorithm. First, we find the vector $\vec{q}_0$ from the equality

$$G_0 = S_0 \vec{q}_0, \tag{27}$$

where the vector G_0 the coordinates $G_j^0 = \psi_{jt}(0) + k_0((A_0)^{-1} B_0 C_0)_{m_j} - ((A_0)^{-1} F_{0,0})_{m_j}$ $(j = 1, 2, \ldots, r)$. Next, we put $\vec{q}_1^{\,1} = \vec{q}_0$ and find the vector C_1^1 from the equality

$$A_n \frac{C_n^i - C_{n-1}}{\tau} + k_n B_n C_n = F_{0,n} + R_n \vec{q}_n^{\,i}, \ C_0 = C(0), \tag{28}$$

where $i = 1$ and $n = 1$. Let G_n^i be the vector with the coordinates $G_j = \psi_{jt}((n-1)\tau) + k_n((A_n)^{-1} B_n C_n^i)_{m_j} - ((A_n)^{-1} F_{0,n})_{m_j}$, $j = 1, 2, \ldots, r$. Next we find the vector $\vec{q}_1^{\,2}$ from the equality

$$G_n^i = S_n \vec{q}_n^{\,i+1}, \tag{29}$$

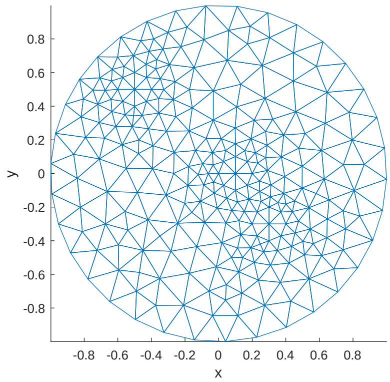
where $n = 1$ and $i = 1$. Using the vector $\vec{q}_1^{\,2}$ in (28) with $n = 1$ and $i = 2$ we can find the vector C_1^2 and so on. The process is going on until $\|\vec{q}_1^{\,i+1} - \vec{q}_1^{\,i}\| < \varepsilon$, with $\varepsilon > 0$ is a given number. Next, we take $C_1 = C_1^{i+1}$, $\vec{q}_1 = \vec{q}^{\,i+1}$. Assume that we have found the vectors $C_{n-1}, \vec{q}_{n-1}$. We take $\vec{q}_n^{\,1} = \vec{q}_{n-1}$ and calculate the vector C_n^1 from (28) with $i = 1$. Define the vector G_n^1 and find the vector $\vec{q}_n^{\,2}$ from (29) with $i = 1$. We repeat the arguments until $\|\vec{q}_n^{\,i+1} - \vec{q}_n^{\,i}\| < \varepsilon$. In this case we put $\vec{q}_n = \vec{q}_n^{\,i+1}$, $C_n = C_n^{i+1}$. Repeating the arguments we can calculate all quantities $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_M$, $C_1, C_2, \ldots, C_M$.

4. The Results of Numerical Experiments

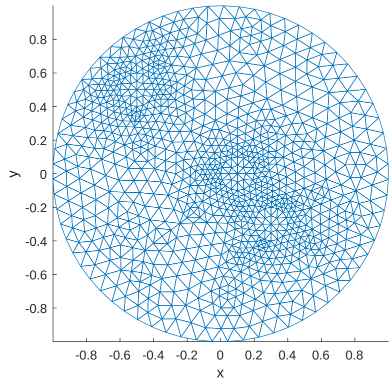
In this section we analyze the results of numerical experiments. The characteristics of the computer are as follows: the processor Intel(R) Core(TM) i3-8100 CPU @ 3.60GHz 3.60GHz, 8.00 GB RAM, the 64-digit operating system Windows 10 Pro.

As a result of calculations, we obtain approximate values of a solution $(u(x, y, t),\ \vec{q}(t))$ of the problem (12)-(13) at points $(t_1, t_2, \dots t_N)$. Here the point (x, y) belongs to the unit circle centered at $(0, 0)$. We present the results of calculations only for the vector-function $\vec{q}$.

To solve the problem numerically, we use two meshes for this domain with the number of nodes $N_1 = 263$ and $N_2 = 1015$ (Fig. 1).



a)



b)

Fig. 1. Meshes: a) $N_1 = 263$; b) $N_2 = 1015$

We consider the equation (12), where $r = 3$. We use the following data:
the solution: $u(x, y, t) = (x^2 + 1) \cdot (y^2 + 1) \cdot (1 + t)$;
the initial data: $u|_{t=0} = (x^2 + 1) \cdot (y^2 + 1)$;
the Neumann boundary conditions: $g = 2(t + 1)(y(x^2 + 1) + x(y^2 + 1))$;
the additional information: $\psi_i(t) = (x_i^2 + 1) \cdot (y_i^2 + 1) \cdot (1 + t)$;
the unknown function: $q_1 = 1, q_2 = t^2 + 2, q_3 = (t - 2)^3$,
the coefficients: $a_0 = (t+1)(x^2+1)$, $a_1 = (t^2+y+4)/(x^2+1)$, $b_{0,1} = x^3 \cdot t$, $b_{0,2} = (x+y) \cdot t$, $b_{1,1} = y^2/(t + 1)$, $b_{1,2} = xt/(y + 1)$, $c_0 = 1/(x^2 + y^2 + 1)$, $c_1 = 1/(1 + t)$;
the right-hand side: $f = y^2 - 12x^6(y^2 + 1)(t + 1) + x(x^2 + 1)(y^2 + 1) - 2(x^2 + 1)(x^6 + 1)(t + 1) - 2(x^6 + 1)(y^2 + 1)(t + 1) + 2tx(y^2 + 1)(x + x^3) - 2tx^3(x^2 + 1)(t + 1) - 8tx^3(y^2 + 1)(t + 1) + 2ty(x^2 + 1)(x + y) + 2txy(x^2 + 1)(t + 1)/(y + 1) - x(t^2 + 2) - 1 - y(t - 2)^3$.
We will use the Neumann boundary conditions from (16) which are represented as

$$\frac{\partial u}{\partial n}|_\Gamma = (u_x x + u_y y) = g.$$

The additional information (14) is given at the observation point $x_{m_1} = (x_1,\ y_1) = (0.3, -0.3)$, $x_{m_2} = (x_2,\ y_2) = (0.1, 0)$, $x_{m_3} = (x_3,\ y_3) = (-0.5, 0.5)$.

All numerical experiments are divided into two groups in dependence on the unknown functions u, $\vec{q}$, the boundary conditions, the noise level δ, the error between iterations ε, the coefficients $a_0, a_1, b_0(x, t) = (b_{0,1}, b_{0,2}), b_1(x, t) = (b_{1,1}, b_{1,2}), c_0, c_1$, and the right-hand sides f. If $\delta \neq 0$ then the perturbations of the overdetermination data at the moments of time $\Delta tk, \quad k = 1, 2, \ldots, N$ (Δt is a step in time) are defined as follows: $\tilde{\psi}_{it}(\Delta tk) = \psi_t(\Delta tk)(1 + \delta(2\sigma_{ik} - 1))$, where numbers $\sigma_{ik} \in [0, 1]$ are determined using the random number generator of Matlab (the function rand).

First, we compare the results of calculations (Fig. 2) for two meshes mentioned above, for data without noise, i.e., $\delta = 0$. Consider the segment $[0, T]$, $T = 1$, and $\Delta t = T/N$, $N = 100$. We take $\varepsilon = 10^{-3}$ (the error defined by the user).

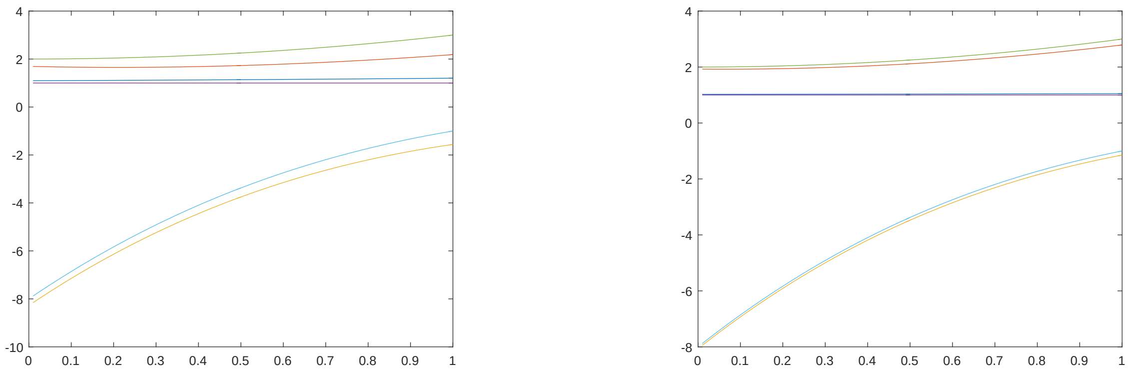


Fig. 2. Results of calculations for the meshes with N=100

Next, we take the same meshes and other parameters but $N = 400$ and we obtain the following result (Fig. 3).

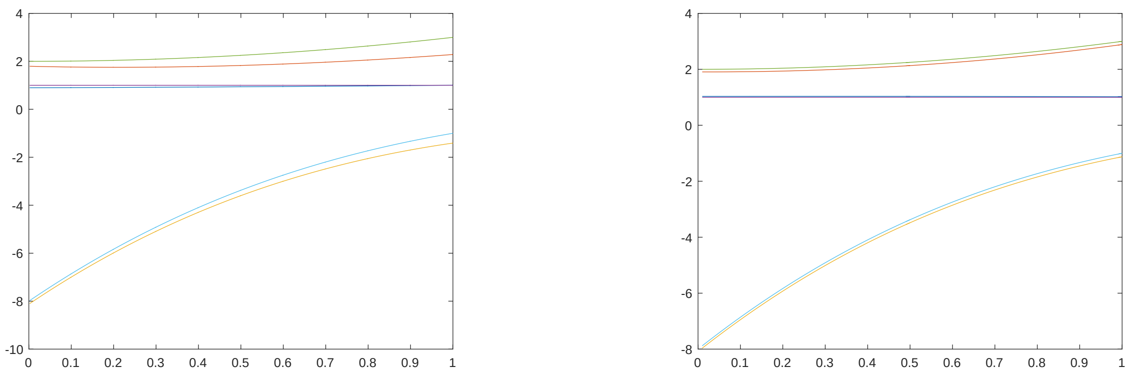


Fig. 3. Results of calculations for the meshes with N=400

In the following experiments, we add 1 percent random noise for the second mesh with $N = 100$ and $N = 400$ (Fig. 4).

Based on the results of numerical experiments for the group of data, we can conclude that the increase in the number of nodes in 4 times leads to the increase in the calculation time more than 10 times. It is lead to a significant increase in accuracy. But further splitting is already useless due to time consuming and low usability.

As we can see the decrease in the variable ε does not lead to a significant increase in the accuracy and decreasing the time of calculations.

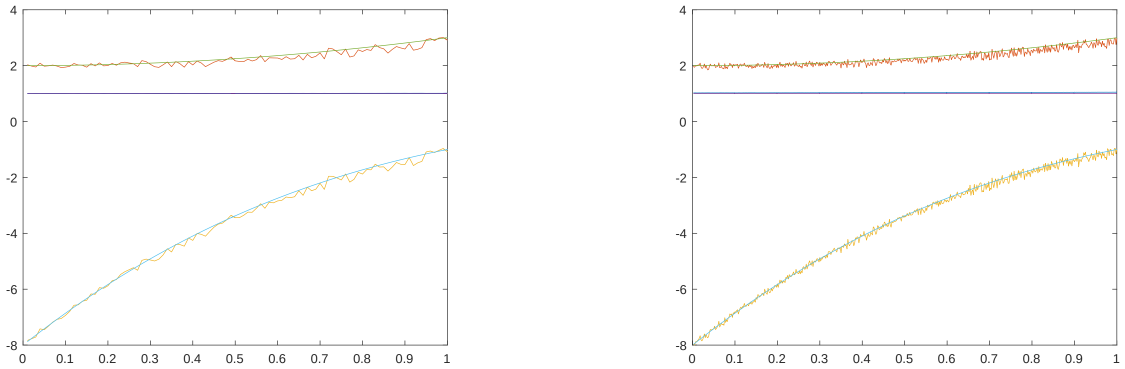


Fig. 4. Results of calculations with a random error with different N

Summing up, we can say that the use of a grid with a large number of nodes shows better accuracy, but the time of the calculation increases by an amount equal to the ratio of the number of nodes of the grids. Increasing the time step also shows a good result, especially when adding a random error. Decreasing the variable ε leads to an increase in the computation time, but does not lead to a significant increase in accuracy.

Conclusions

Under consideration is an inverse problem of recovering the right-hand side in a pseudoparabolic equation. Some theoretical results and stability estimates for solution are exposed. We propose a numerical algorithm of recovering the right-hand side with the use of the pointwise overdetermination conditions. A numerical algorithms is based on the finite element method combined with the finite difference schemes. The results of numerical experiments show a sufficiently good convergence of the algorithm.

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ЧИСЛЕННЫЙ АЛГОРИТМ ДЛЯ НЕКОТОРОЙ ОБРАТНОЙ ЗАДАЧИ ФИЛЬТРАЦИИ С ТОЧЕЧНЫМ ПЕРЕОПРЕДЕЛЕНИЕМ

С. Н. Шергин

Рассматривается обратная задача о восстановлении правой части в псевдопараболическом уравнии фильтрации с использованием точечного переопределения. Приводятся теоремы существования и единственности, которые являются основой численного алгоритма восстановления правой части (функции источника) и решения. Задача является корректной и имеет место оценка устойчивости. Задача может быть сведена к интегральному уравнению типа Вольтерра, где оператор имеет малую норму на малых промежутках времени. Используется метод конечных элементов для того, чтобы свести задачу к системе обыкновенных дифференциальных уравнений, которая решается методом конечных разностей. При построении алгоритма используется идея схемы предиктор-корректор. Представлены результаты численных экспериментов. Результаты показывают хорошую сходимость алгоритма к решению.

Ключевые слова: обратная задача; псевдопараболическое уравнение; фильтрация; трещиноватая среда; численное решение.

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