SHORT NOTES

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NUMERICAL SOLUTION OF A LINEAR SYSTEM OF NAVIER – STOKES EQUATIONS IN AN AXISYMMETRIC DOMAIN

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The system of Navier – Stokes equations simulates the dynamics of a viscous incompressible fluid. The problem on the existence of solutions to the Cauchy – Dirichlet problem for this system is one of the most difficult mathematical problems of the present century. However, the question on the existence of solutions to the Cauchy – Dirichlet problem for the system of Navier – Stokes equations still remains unsolved. This article shows how to obtain eigenvalues for the system in the case of an axisymmetric domain.

Keywords: system of Navier – Stokes equations; Galerkin method; multipoint initial-final value condition.

Introduction

The system of Navier – Stokes equations

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \nabla p, \nabla \cdot u = 0 \tag{1}$$

with the Dirichlet condition

$$\vec{u}(x,t) \;\; = \;\; 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R} \tag{2}$$

simulates the dynamics of a simple viscous incompressible fluid. Here $u = (u_1, u_2, \cdots, u_n)$, $u_i = u_i(x,t)$, $n = 2,3$ is the vector function corresponding to the fluid velocity; the scalar function $\nu \in \mathbb{R}_+$ characterizes the viscosity of the fluid, respectively. In various aspects, equations (1) were studied in [1], [4], [5]. We present the Galerkin method for the system of the Navier – Stokes equations in the case of an axisymmetric domain with the multipoint initial-final value condition

$$P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{0,n}. \tag{3}$$

1. Derivation of System of Navier – Stokes Equations

Consider the equation of motion of a continuous medium in the Cauchy form:

$$\rho\frac{d\vec{v}}{dt} = \operatorname{div}\vec{\Pi} + \vec{f}. \tag{4}$$

Equation (4) can be obtained from the second Newton's law in the d'Alembert form:

$$\int_V (\vec{F} - \vec{a})\rho dV + \int_S \vec{p}_n dS = 0. \tag{5}$$

Here V is a bounded three-dimensional volume of a continuous medium, S is its sufficiently smooth surface, ρ is a density of elementary volume dV. The vectors $\vec{F}$ and $\vec{a}$ denote external force and total acceleration per unit mass of volume V, respectively. The vector $\vec{p}_n$ corresponds to the normal component of the surface force acting on the surface element dS.

Represent the vector $\vec{p}_n$ in the form

$$\vec{p}_n = \vec{p}_1\cos(\vec{n}, x_1) + \vec{p}_2\cos(\vec{n}, x_2) + \vec{p}_3\cos(\vec{n}, x_3), \tag{6}$$

where $\vec{p}_k$ is the stress vector on the elementary area, which is perpendicular to the axis Ox_k, and $k = 1, 2, 3$. In turn, each vector $\vec{p}_k$ can be represented as

$$\vec{p}_k = (p_{k_1}, p_{k_2}, p_{k_3}), \quad k = 1, 2, 3. \tag{7}$$

Here p_{k_k} is the normal component of the vector $\vec{p}_k$, and the other two components correspond to the tangent components. Substitute (7) into (6) and obtain

$$\vec{p}_n = \vec{\Pi}\vec{n}, \tag{8}$$

where the matrix $\vec{\Pi} = ||p_{k_l}||, k, l = 1, 2, 3$ is called a *tensor of elastic stresses*, and the vector $\vec{n} = (n_1, n_2, n_3)$, $n_k = \cos(\vec{n}, x_k)$ corresponds to the unit normal.

Consider (5) and use (8) in order to represent the second term as

$$\int_S \vec{p}_n dS = \int_S \vec{\Pi}\vec{n} dS. \tag{9}$$

Apply the Gauss-Ostrogradsky formula to (9) and obtain

$$\int_S \vec{\Pi}\vec{n} dS = \int_V \operatorname{div}\vec{\Pi} dV. \tag{10}$$

Here the vector

$$\operatorname{div}\vec{\Pi} = \left(\sum_{k=1}^{3}\frac{\partial p_{k1}}{\partial x_k}, \sum_{k=1}^{3}\frac{\partial p_{k2}}{\partial x_k}, \sum_{k=1}^{3}\frac{\partial p_{k3}}{\partial x_k}\right).$$

Finally, substitute (10) into (5) and obtain

$$\int_V (\vec{f} + \operatorname{div}\vec{\Pi} - \rho\frac{d\vec{v}}{dt}) dV = 0. \tag{11}$$

Here $\vec{f} = \rho\vec{F}$ is the vector of external mass forces, and the vector

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + v_k\frac{\partial\vec{v}}{\partial x_k} = \vec{a}$$

corresponds to the dynamic full acceleration. Since the volume V is arbitrary, (4) immediately follows from (11).

Next, consider the matrix

$$\vec{D} = \left|\left|\frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right)\right|\right|, \quad k, l = 1, 2, 3,$$

which is called a *strain rate tensor*. The physical meaning of the component

$$\frac{\partial v_k}{\partial x_k}$$

of the tensor $\vec{D}$ is the change in the velocity of translational motion of the area element perpendicular to the axis $Ox_k,\ k = 1, 2, 3$. If the tensors $\vec{D}$ and $\vec{\Pi}$ are such that

$$\vec{\Pi} \;=\; -pI + 2\nu\vec{D},$$

then the continuous medium is called *Newtonian fluid.*

Assume that the fluid is incompressible, i.e.

$$\operatorname{div}\vec{v} = 0,$$

then we have

$$\operatorname{div}\vec{\Pi} = -\operatorname{grad}\vec{p} +$$

$$+2\nu\left(\frac{1}{2}\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\partial v_k}{\partial x_1} + \frac{\partial v_1}{\partial x_k}\right), \frac{1}{2}\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\partial v_k}{\partial x_2} + \frac{\partial v_2}{\partial x_k}\right), \frac{1}{2}\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\partial v_k}{\partial x_3} + \frac{\partial v_3}{\partial x_k}\right)\right) =$$

$$= -\operatorname{grad}\vec{p} + \nu\Delta\vec{v}. \qquad (12)$$

Assume that the density $\rho = 1$, substitute (12) into (4) and obtain the *system of Navier – Stokes equations* [2]

$$\begin{aligned} \vec{v_t} &= \nu\nabla^2\vec{v} - (\vec{v}\cdot\nabla)\vec{v} - \nabla p + \vec{f}, \\ 0 &= \nabla\cdot\vec{v}, \end{aligned} \qquad (13)$$

which describes the dynamics of a viscous incompressible fluid.

Here $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\right)$ is Hamilton operator, $\vec{v_t} = \frac{\partial v}{\partial t}$.

2. Galerkin Method for System of Navier – Stokes Equations

Following E.N. Lorentz [3], we consider system (1) to be invariant under a shift along one of the horizontal coordinates, i.e. we consider the values $u_i = u_i(x,t), i = 1,2,3, p = p(x,t)$ to be constant along the coordinate x_2, and consider the value $u_2(x,t)$ to be constant. In this case, the incompressibility equation $\nabla \cdot u = 0$ takes the form

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} = 0, \tag{14}$$

therefore, we can define a stream function up to an additive constant by the equations

$$\frac{\partial \Psi}{\partial x_1} = -u_3, \frac{\partial \Psi}{\partial x_3} = u_1. \tag{15}$$

Replace the notation of the coordinates x_1 and x_3 with the more standard $x_1 = x$ and $x_3 = z$ and represent system (1) as

$$\begin{cases} \frac{\partial u_1}{\partial t} = \nu \nabla^2 u_1 + \nabla p, \\ \frac{\partial u_2}{\partial t} = \nu \nabla^2 u_2 + \nabla p, \\ \frac{\partial u_3}{\partial t} = \nu \nabla^2 u_3 + \nabla p, \end{cases} \tag{16}$$

$$\begin{cases} \frac{\partial^2 \Psi}{\partial t \partial x_3} = \nu(\frac{\partial^2 \Psi}{\partial x_1 \partial x_3} + \frac{\partial^2 \Psi}{\partial x_3^2}) - \nabla p, \\ \frac{\partial^2 \Psi}{\partial t \partial x_1} = \nu(\frac{\partial^2 \Psi}{\partial x_1 \partial x_3} + \frac{\partial^2 \Psi}{\partial x_1^2}) + \nabla p, \end{cases} \tag{17}$$

$$\nabla^2 \frac{\partial \Psi}{\partial t} = -\nu \nabla^4 \Psi. \tag{18}$$

Consider the Dirichlet problem

$$\begin{cases} \Psi(x,0,t) = \Psi(x,H,t), \\ \Psi(0,z,t) = \Psi(L,z,t), \\ \nabla^2 \Psi(0,z,t) = \nabla^2 \Psi(L,z,t) \end{cases} \tag{19}$$

for equation (18) in the rectangle $[0,L] \times [0,H]$.

Following E.N. Lorentz [3], we find the three-dimensional Galerkin approximation to problem (19) for equation (18). To this end, as the basis functions of the Galerkin method, we take the eigenfunctions of the following problem:

$$\begin{cases} -\nabla^2 \phi = \lambda \phi, \quad [0,L] \times [0,H], \\ \phi(x,0) = \phi(x,H) = 0, \\ \phi(0,z) = \phi(L,z), \\ \phi'(0,z) = \phi'(L,z). \end{cases} \tag{20}$$

All non-trivial solutions to problem (20) can be divided into three families:

$$\begin{cases} \alpha_{lk} = \{\sin \frac{\pi l}{H} z \cdot \sin \frac{2\pi k}{L} x\}, \quad l,k \in \mathbb{N}, \\ \beta_{lk} = \{\sin \frac{\pi l}{H} z \cdot \cos \frac{2\pi k}{L} x\}, \quad l,k \in \mathbb{N}, \\ \gamma_l = \{\sin \frac{\pi l}{H} z\}, \quad l \in \mathbb{N}. \end{cases} \tag{21}$$

We take the Galerkin approximation to the solution to problem (19) for system (20) in the form

$$\Psi = X(t)\alpha_{11}. \tag{22}$$

It is easy to see that (22) satisfies boundary conditions (19).

Substitute (22) into (18) and obtain

$$\pi^2\left(\frac{1}{H^2}+\frac{4}{L^2}\right)\dot{X}\alpha_{11} = -\nu\pi^4\left(\frac{1}{H^2}+\frac{4}{L^2}\right)^2 X\alpha_{11}. \tag{23}$$

The first equation (23) immediately implies

$$\dot{X} = -aX, \tag{24}$$

where

$$a = \nu\pi^2\left(\frac{1}{H^2}+\frac{4}{L^2}\right). \tag{25}$$

3. Numerical Experiments

In the first experiment, we take the following values of the auxiliary parameters: $\lambda = 1; \beta = 2; \alpha = -1; \chi = 1; \nu = 2$ (see Fig. 1).

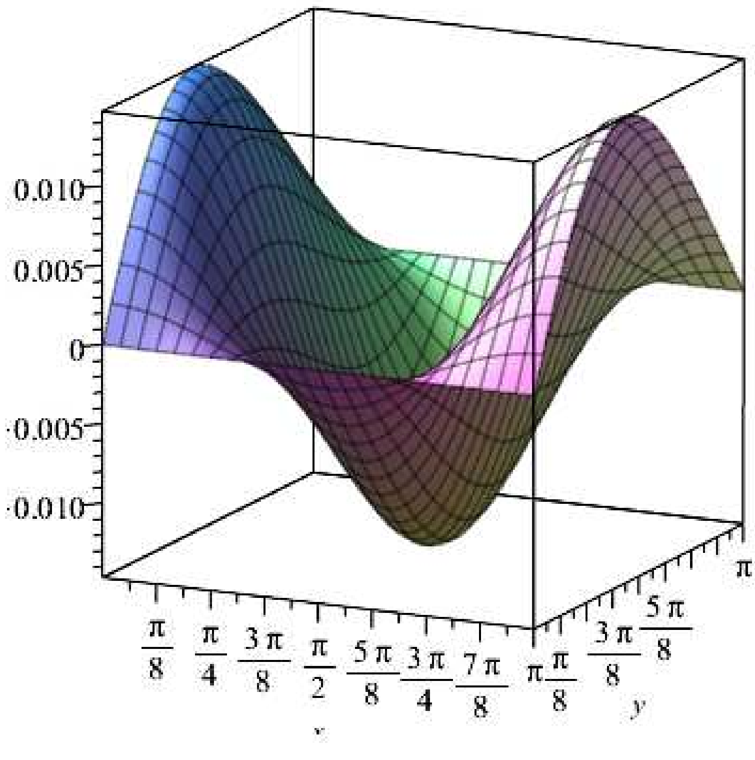

Fig. 1. Graphic of the solution to problem (1) – (3) at the time $t = 1$

In the second experiment, we take the following values of the auxiliary parameters: $\lambda = 1; \beta = 4; \alpha = -2; \chi = 1; \nu = 2$ (see Fig. 2).

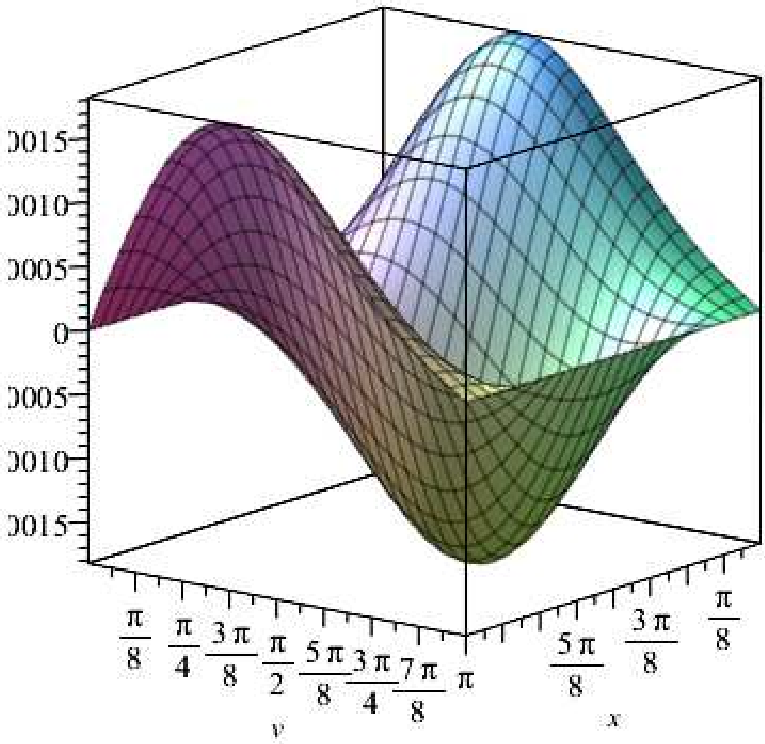

Fig. 2. Graphic of the solution to problem (1) – (3) at the time $t = 1$

References

1. Ladyzhenskaya O. A. [*Mathematical Problems of the Dynamics of a Viscous Incompressible Fluid*]. Moscow, 1970. (in Russian).
2. Landau L. D., Lifshitz E. M. Theoretical Physics. [*Hydrodynamics*]. Moscow, 1986, vol. 6. (in Russian).
3. Lorenz E. N. Deterministic Non-Periodic Flow. [*Strange Attractors*]. Moscow, 1981. (in Russian)
4. Zagrebina, S. A., Konkina A. S. The Multipoint Initial-Final Value Condition for the Navier–Stokes Linear Model. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2015, vol. 8,.no. 1. pp. 132–136. DOI: 10.14529/mmp150111
5. Zagrebina S. A., Sukacheva T. G., Sviridyuk G. A. The Multipoint Initial-Final Value Problems for Linear Sobolev-Type Equations with Relatively p-Sectorial Operator and Additive "Noise". *Global and Stochastic Analysis*, 2018, vol. 5, no. 2, pp. 129–143.

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ЧИСЛЕННОЕ РЕШЕНИЕ ЛИНЕЙНОЙ СИСТЕМЫ УРАВНЕНИЙ НАВЬЕ–СТОКСА В ОСЕСИММЕТРИЧНОЙ ОБЛАСТИ

А. С. Конкина

Система уравнений Навье – Стокса моделирует динамику вязкой несжимаемой жидкости. Проблема существования решений задачи Коши - Дирихле для этой системы вошла в список наиболее тяжелых математических проблем нынешнего века. Однако до сих пор не решен вопрос о существовании решений задачи Коши – Дирихле для системы уравнений Навье – Стокса. Проблема существования решений этой задачи оказалась настолько трудной, что она вошла в списки наиболее тяжелых математических проблем нынешнего века и за ее решение назначена награда в один миллион долларов.В данной статье показанно как получить собственные значения для системы в случае оси симметричной области.

Ключевые слова: система уравнений Навье-Стокса; метод Галеркина; многоточечное начально-конечное условие.

Литература

1. Ладыженская, О. А. Математические вопросы динамики вязкой несжимаемой жидкости / О. А. Ладыженская // М.: Наука, – 1970.
2. Ландау, Л. Д. Теоретическая физика. Т. VI / Л. Д. Ландау, Е. М. Лифшиц // Гидродинамика. – М.: Наука, 1986.
3. Лоренц, Э. Н. Детерминированное непериодическое течение / Э. Н. Лоренц // Странные аттракторы. – М.: Мир, 1981.
4. Zagrebina, S. A. The Multipoint Initial-Final Value Condition for the Navier–Stokes Linear Model / S. A. Zagrebina, A. S. Konkina // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2015. – Т. 8, № 1. – С. 132–137.
5. Zagrebina, S. A. The Multipoint Initial-Final Value Problems for Linear Sobolev-Type Equations with Relatively p-Sectorial Operator and Additive "Noise" / S. A. Zagrebina, T. G. Sukacheva, G. A. Sviridyuk // Global and Stochastic Analysis. – 2018. – V. 5, № 2. – P. 129–143.

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