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NUMERICAL STUDY ON THE NON-UNIQUENESS OF SOLUTIONS TO THE SHOWALTER–SIDOROV PROBLEM FOR ONE DEGENERATE MATHEMATICAL MODEL OF AN AUTOCATALYTIC REACTION WITH DIFFUSION

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The article is devoted to a numerical study of the phase space of a mathematical model of an autocatalytic reaction with diffusion, based on a degenerate system of equations of the "distributed" brusselator. In this mathematical model, the rate of change in one of the components of the system can significantly exceed the other, which leads to a degenerate system of equations. This model belongs to a wide class of semilinear Sobolev type equations. We will identify the conditions for the existence, uniqueness or multiplicity of solutions to the Showalter–Sidorov problem, depending on the parameters of the system. The theoretical results obtained made it possible to develop an algorithm for the numerical solution of the problem based on the modified Galerkin method. The results of computational experiments are presented.

Keywords: Sobolev type equations; Showalter–Sidorov problem; non-uniqueness of solutions; distributed brusselator.

Introduction

Currently, research is conducted on the phenomena of self-organization in various nonequilibrium systems consisting in the emergence and evolution of ordered spatio-temporal structures. An example of the latter is autowaves, which are formed in excitable media in response to external disturbance. There are many examples of excitable media: nerve and muscle tissues, colonies of microorganisms, a number of chemical solutions and gels, magnetic superconductors with current, some solid-state systems [1–4].

Studying the reaction mechanism, as a result of which ordered temporal and (or) spatial structures arise, mathematical models were obtained, one of which is an autocatalytic reaction with the diffusion

$$\left\{ \begin{array}{r} \varepsilon_1 v_t = \alpha_1 v_{ss} + \gamma - (\delta+1)v + v^2 w, \\ \varepsilon_2 w_t = \alpha_2 w_{ss} + \delta v - v^2 w. \end{array} \right. \tag{1}$$

Here $v = v(s,t)$ and $w = w(s,t)$ are functions characterizing the concentration of reagents, the terms $\alpha_1 v_{ss}$, $\alpha_2 w_{ss}$ characterize the diffusion of reagents according to Fickey's law, ($\alpha_1, \alpha_2 \in \mathbb{R}_+$ are diffusion coefficients), the parameters $\gamma, \delta \in \mathbb{R}_+$ characterize the concentration starting reagents that are supposed to be constant.

The system of equations (1) was investigated in various aspects, and in many reseaches, along with the case $\varepsilon_1 > 0$ or $\varepsilon_2 > 0$, the case $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ is also discussed [5, 6]. The need to study cases when one of these parameters is equal to zero is associated with the fact that the rate of change in one of the components significantly exceeds the other. In this article, we will be interested in both cases. In the case of $\varepsilon_1 = 0$, the phase space of the system of equations (1) may contain singularities of the Whitney type [7], which leads to the non-uniqueness of the solution. In this case, the system of equations (1) takes the form

$$\begin{cases} 0 = \alpha_1 v_{ss} + \gamma - (\delta + 1)v + v^2 w, \\ w_t = \alpha_2 w_{ss} + \delta v - v^2 w. \end{cases} \tag{2}$$

In the case of $\varepsilon_2 = 0$, the equations (1) take the form

$$\begin{cases} v_t = \alpha_1 v_{ss} + \gamma - (\delta + 1)v + v^2 w, \\ 0 = \alpha_2 w_{ss} + \delta v - v^2 w. \end{cases} \tag{3}$$

Consider degenerate systems of equations (2) or (3) in the cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary of the class C^∞ with the boundary conditions

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}_+, \tag{4}$$

and the initial condition

$$w(0) = w_0 \text{ for case } \varepsilon_1 = 0, \tag{5}$$

$$v(0) = v_0 \text{ for case } \varepsilon_2 = 0. \tag{6}$$

Problems (2), (4), (5) and (3), (4), (6) are reduced to the initial Showalter–Sidorov problem

$$L(u(0) - u_0) = 0 \tag{7}$$

for a semilinear Sobolev type equation

$$L\dot{u} = Mu + N(u). \tag{8}$$

Here $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$, N is a nonlinear operator, $\mathfrak{U}, \mathfrak{F}$ are Banach spaces. G.A. Sviridyuk and his followers found the conditions for the unique solvability of problem (7), (8) [9, 11]. Namely, when the operator M is (L,p)-sectorial (bounded) and the phase space of the equation (8) is a simple Banach C^∞-manifold, there exists the unique quasistationary (semi)trajectory of problem (7), (8) passing through the point u_0, which belongs pointwise to the phase space [8]. Recall that a Banach C^∞-manifold is called simple if any its atlas is equivalent to an atlas containing a single map. In particular, if the operator M is $(L,0)$-sectorial (bounded), then any solution (7), (8) is a quasistationary (semi)trajectory [9–11]. If the phase space of equation (8) belongs to a smooth Banach manifold having singularities such as Whitney assemblies, the Showalter–Sidorov problem (7) for equation (8) can have several solutions. It was shown in [7, 12, 13] that the phase space of equation (8) contains singularities of the type of 1-assemblies or 2-assemblies Whitney. Therefore, the Showalter–Sidorov problem for such equations can have one or multiple solutions or a solution may not exist. In the course of this research, we identify the conditions for the existence and uniqueness or multiplicity of solutions to problem (2), (4), (5), or (3), (4), (6), depending on system parameters.

In this paper, we performed a numerical study of problem (2), (4), (5), or (3), (4), (6) and developed an algorithm based on the modified Galerkin method for the case of degenerate equations. Based on this method, we present the desired functions in the form of a Galerkin sum

$$v_m(s,t) = \sum_{k=1}^{m} a_k(t)\varphi_k(s), \quad w_m(s,t) = \sum_{k=1}^{m} b_k(t)\varphi_k(s),$$

where $\{\varphi_k(s)\}$ are the eigenfunctions of the homogeneous Dirichlet problem of the Laplace operator Δ in the domain Ω, $a_k(t)$, $b_k(t)$ satisfy the system of algebraic-differential equations and the corresponding initial conditions. For the first time, the Galerkin method for semilinear Sobolev type equations was considered by G.A. Sviridyuk and T.G. Sukacheva [14]. This method is especially effective for degenerate equations or systems of equations. In the works [15–17], the Galerkin method was used to find approximate solutions in the case of degenerate semilinear Sobolev type equations.

1. Morphology of Phase Space of Mathematical Model of Autocatalytic Reaction with Diffusion

Consider the degenerate system of equations of the distributed brusselator (2) in the cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary of class C^∞ with boundary conditions (4) and the initial Showalter–Sidorov condition (5) in the case $\varepsilon_1 = 0$. Put $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2 = (\overset{\circ}{W}{}^1_2(\Omega))^2$, $\mathfrak{U} = (L_2(\Omega))^2$. The space $\mathfrak{U}$ is Hilbert with the scalar product $[u, \zeta] = \langle v, \xi\rangle + \langle w, \eta\rangle$, where $u = (v, w), \zeta = (\xi, \eta)$, and by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $L_2(\Omega)$. Denote $\mathfrak{F}$ the space adjoint to $\mathfrak{U}$ with respect to the duality $[\cdot,\cdot]$. Construct the linear operators $L, M : \mathfrak{U} \to \mathfrak{F}$

$$[Lu, \zeta] = \langle w, \xi\rangle, \qquad u, \zeta \in \mathfrak{U},$$

$$[Mu, \zeta] = -\alpha_1\langle v_{s_i}, \xi_{s_i}\rangle - \alpha_2\langle w_{s_i}, \eta_{s_i}\rangle, \quad u, \zeta \in \mathfrak{U}, \text{ where } \operatorname{dom} M = \mathfrak{H},$$

the nonlinear operator

$$[N(u), \zeta] = \langle \gamma - (\delta + 1)v + v^2 w, \xi\rangle + \langle \delta v - v^2 w, \eta\rangle$$

and $\operatorname{dom} N = L_4(\Omega) \times L_4(\Omega) = \mathfrak{U}_N$. By construction, the operator $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$.

Consider the sectorial operator $A \in \mathcal{C}l(\mathfrak{U})$ and $\operatorname{Re}\sigma(A) < 0$, $\operatorname{dom} A = W_2^1(\Omega)$ was obtained in the work [18]. Let $A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int\limits_0^\infty t^{-\alpha-1}e^{-At}dt$ for any $\alpha > 0$. Define A^α as the operator inverse to $A^{-\alpha}$, $\operatorname{dom} A^\alpha = \operatorname{im} A^{-\alpha}$, $A^0 = I$. Next, for each $\alpha \geq 0$ we assume that $\mathfrak{U}^\alpha = \operatorname{dom} A^\alpha$, and endow the space $\mathfrak{U}^\alpha$ with the norm of the graph $\|u\|_\alpha = \|A^\alpha u\|$, $u \in \mathfrak{U}$. Construct the auxiliary interpolation space $\mathfrak{U}_\alpha$. According to the Sobolev embedding theorem the space $\mathfrak{U} \hookrightarrow L_\infty(\Omega)$ for $\alpha \in (\frac{1}{2}, 1]$. Take $\mathfrak{U}_\alpha = \mathfrak{U}_1^0 \oplus \mathfrak{U}_\alpha^1$, where $\mathfrak{U}_\alpha^1 = \{0\} \times \mathfrak{U}^\alpha$, $\mathfrak{U}_1^0 = W_2^1(\Omega) \times \{0\}$. There exist dense and continuous embeddings $\mathfrak{H} \hookrightarrow \mathfrak{U}_\alpha \hookrightarrow \mathfrak{U}_N \hookrightarrow \mathfrak{U}$. The operator $N \in C^\infty(\mathfrak{U}_\alpha; \mathfrak{F})$.

Therefore, we reduced the problem (2), (4) to semilinear Sobolev type equation (8). Note that condition (5) takes the form (7). We are interested in the solvability of problem (2), (4) (5) for any $u_0 = col(v_0, w_0) \in \mathfrak{H}$.

Denote by $\{\nu_k\}$ the sequence of eigenvalues of the following spectral problem:

$$\begin{aligned} -\Delta\varphi = \nu\varphi,\ s \in \Omega, \\ \varphi(s) = 0,\ s \in \partial\Omega, \end{aligned} \tag{9}$$

where the eigenvalues are numbered in non-decreasing order, taking into account their multiplicity. Denote by $\{\varphi_k\}$ the corresponding eigenfunctions orthonormalized in the sense of the scalar product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$.

Definition 1. *A vector function $u \in C^1((0, \tau); \mathfrak{U}) \cap C((0, \tau); \mathfrak{U}_N)$ satisfying equation (8) we is called a solution of the equation. A solution $u = u(t)$ to equation (8) is called a solution to problem (7), (8), if* $\lim\limits_{t\to 0+} ||L(u(t) - u_0)||_{\mathfrak{F}} = 0$.

Construct

$$\mathfrak{M} = \{u \in \mathfrak{U}_\alpha : \langle \alpha_1 v_{s_i}, \xi_{s_i} \rangle = \langle \gamma - (\delta + 1)v - v^2 w, \xi \rangle \} \tag{10}$$

and note that all solutions to system of equations (2) satisfying the boundary conditions (4) belongs to this set as trajectories. The following theorem was obtained and proved in [7].

Theorem 1. [7] *Let $\gamma, \delta \in \mathbb{R}_+$, $\delta \in (0, \alpha_1\nu_1 - 1)$, then for any vector $w \in \mathfrak{H}_2$ there exists a single vector $v \in \mathfrak{H}_1$ such that $u = col(v, w) \in \mathfrak{M}$.*

Consider the case $\delta = \alpha_1\nu_1 - 1$. Let φ_1 be the eigenfunction of problem (9) corresponding to the eigenvalue ν_1 and normalized in the sense of $L_2(\Omega)$ and

$$\mathfrak{H}_1^\perp = \{v^\perp \in \mathfrak{H}_1 : \langle v^\perp, \varphi_1 \rangle = 0\}, \ \mathfrak{H}_2^\perp = \{w^\perp \in \mathfrak{H}_2 : \langle w^\perp, \varphi_1 \rangle = 0\}.$$

Then $v = v^\perp + r\varphi_1$ and $w = w^\perp + q\varphi_1$, where $r, q \in \mathbb{R}$, $v \in \mathfrak{H}_1$ and $w \in \mathfrak{H}_2$. The set $\mathfrak{M}$ (10) takes the following form:

$$\mathfrak{M} = \left\{ u = (v, w) \in \mathfrak{H} : \left\{ \begin{array}{l} \langle \alpha_1 v_{s_i}^\perp, \xi_{s_i}^\perp \rangle = \langle \gamma - (\delta+1)v^\perp - (v^\perp + r\varphi_1)^2 (w^\perp + q\varphi_1), \xi^\perp \rangle, \\ \langle \gamma, \varphi_1 \rangle = \langle (v^\perp + r\varphi_1)^2 (w^\perp + q\varphi_1), \varphi_1 \rangle \end{array} \right. \right\}. \tag{11}$$

Set (11) may contain features such as Whitney assemblies, as was shown in [7].

Theorem 2. [7] *For any $u_0 = (v_0, w_0) \in \mathfrak{H}$, $\gamma, \delta \in \mathbb{R}_+$ the set $\mathfrak{M}$ has 1-assemblies Whitney.*

Transform the second equation of the system (11) to the form:

$$\begin{gathered} r^2 \int_\Omega (w^\perp + q\varphi_1)\varphi_1^3 ds + 2r \int_\Omega (v^\perp)^2 (w^\perp + q\varphi_1)\varphi_1^2 ds + \\ + \int_\Omega (v^\perp)^2 (w^\perp + q\varphi_1)\varphi_1 ds + \int_\Omega \gamma\varphi_1 ds = 0. \end{gathered} \tag{12}$$

Note that equation (12) is a quadratic equation of type $ar^2 + br + c = 0$ with respect to r, where

$$\begin{gathered} a = \int_\Omega (w^\perp + q\varphi_1)\varphi_1^3 ds, \qquad b = 2\int_\Omega (v^\perp)^2 (w^\perp + q\varphi_1)\varphi_1^2 ds, \\ c = \int_\Omega (v^\perp)^2 (w^\perp + q\varphi_1)\varphi_1 ds + \int_\Omega \gamma\varphi_1 ds, \\ D = b^2 - 4ac. \end{gathered} \tag{13}$$

By virtue of the theorem on the existence of solution to problem (8), (7), which is described in detail in [9, 19], formulas (13), and Theorem 1, we have the following.

Theorem 3. *For any $u_0 = (v_0, w_0) \in \mathfrak{H}$, $\gamma, \delta \in \mathbb{R}_+$ and*
(i) if $\delta \in (0, \alpha_1\nu_1 - 1)$, then there exists a unique solution to problem (2), (4), (5);
(ii) if $\delta = \alpha_1\nu_1 - 1$, $D = 0$, then there exists unique solution to problem (2), (4), (5);
(iii) if $\delta = \alpha_1\nu_1 - 1$, $D > 0$, then there exists two solutions to problem (2), (4), (5).

Consider the degenerate system of equations of the distributed brusselator (3) in the cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary of class C^∞ with boundary conditions (4) and the initial Showalter–Sidorov condition (6) in the case $\varepsilon_2 = 0$. We construct the linear operator $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$

$$[Lu, \zeta] = \langle v, \xi \rangle, \; u, \zeta \in \mathfrak{U}.$$

Construct the auxiliary interpolation space $\mathfrak{U}_\alpha$. According to the Sobolev embedding theorem the space $\mathfrak{U} \hookrightarrow L_\infty$ for $\alpha \in (\frac{1}{2}, 1]$. Take $\mathfrak{U}_\alpha = \mathfrak{U}_1^0 \oplus \mathfrak{U}_\alpha^1$, where $\mathfrak{U}_1^0 = \{0\} \times W_2^1(\Omega)$, $\mathfrak{U}_\alpha^1 = \mathfrak{U}^\alpha \times \{0\}$. There exist dense and continuous embeddings $\mathfrak{H} \hookrightarrow \mathfrak{U}_\alpha \hookrightarrow \mathfrak{U}_N \hookrightarrow \mathfrak{U}$ so the operator $N \in C^\infty(\mathfrak{U}_\alpha; \mathfrak{F})$.

Construct

$$\mathfrak{M} = \{u \in \mathfrak{U}_\alpha : \langle \beta w_{s_i}, \eta_{s_i} \rangle + \langle \delta(v - v^2 w), \eta \rangle = 0\}$$

and note that all solutions to system of equations (3) satisfying the boundary conditions (4) belong to this set as trajectories.

Theorem 4. [20] *Let $\gamma, \delta \in \mathbb{R}_+$ and $\langle \alpha_2 w_{s_i}, \eta_{s_i} \rangle + \langle \delta v_0^2 w, \eta \rangle \neq 0$ hold for all $w \in \mathfrak{H}_2$ and all $t \in (0, \tau)$. Then the phase space of the equation (3) is the set $\mathfrak{M}$, which is a simple Banach C^∞-manifold.*

Theorem 5. [20] *Let the point $u_0 = (v_0, w_0) \in \mathfrak{M}$, $\alpha_1, \alpha_2, \delta, \gamma \in \mathbb{R}_+$, with $\langle \alpha_2 w_{s_i}, \eta_{s_i} \rangle + \langle \delta v_0^2 w, \eta \rangle \neq 0$, for all $w \in \mathfrak{H}_2$ and $t \in (0, \tau)$. Then there exists the unique solution (v, w) to problem (3), (4), (6).*

2. Numerical Experiment

Based on the theoretical results and the modified Galerkin method, we develop an algorithm for the numerical method for solving the Showalter–Sidorov problem for the distributed brussellator model, which allows one to find approximate solutions in the interval for the given initial values and coefficients $\alpha_1, \alpha_2, \beta, \gamma$, and also construct a graph of an approximate solution. We present an algorithm for finding an approximate solution to problem (2), (4), (5) in the case $\varepsilon_1 = 0$ or (3), (4), (6) in the case $\varepsilon_2 = 0$:

Step 1. Find the eigenvalues $\{\nu_k\}$ and the eigenfunctions $\{\varphi_k(s)\}$ of the homogeneous Dirichlet problem of the Laplace operator Δ in the domain Ω.

Step 2. Represent the desired functions in the form of the Galerkin sum

$$v_m(s,t) = \sum_{k=1}^{m} a_k(t)\varphi_k(s), \qquad w_m(s,t) = \sum_{k=1}^{m} b_k(t)\varphi_k(s),$$

where $a_k(t)$, $b_k(t)$ satisfy the system of algebraic-differential equations and the corresponding initial conditions, which have formed in *Step 4*.

Step 3. Taking

$$r = a_1(0), \quad q = b_1(0), \quad v^\perp = \sum_{k=2}^{m} a_k(t)\varphi_k, \quad w^\perp = \sum_{k=2}^{m} b_k(t)\varphi_k,$$

and substituting the obtained values in formulas (13), we verify the uniqueness or multiplicity of the solution to the Showalter–Sidorov problem for the given initial conditions and the resulting $a_k^j(0)$, $b_k^j(0)$, $j = 1, 2$. In the case when $D > 0$, the considered problem has two solutions $v_{01}(s,t)$ and $v_{02}(s,t)$. Therefore, the system of algebraic-differential equations have two solutions and two sets $a_k(t)$ and $b_k(t)$ for each of the solutions, respectively. In this case, all subsequent steps must be done twice for each of the sets $a_k(t)$ and $b_k(t)$.

In the case of $\varepsilon_1 = 0$, we proceed sequentially to *Steps 4.1, 5.1, 6.1*, in the case of $\varepsilon_2 = 0$ we proceed sequentially to *Steps 4.2, 5.2, 6.2*, respectively.

Step 4.1. In the case of $\varepsilon_1 = 0$, scalarly multiply in $L_2(\Omega)$ the system of equations (1) by the eigenfunctions $\varphi_i(s)$, $i = 1, ..., m$, and form the system of algebraic-differential equations:

$$\begin{gathered}-\alpha_1 \sum_{k=1}^{m} a_k(t)\nu_k\langle\nu_k, \varphi_i\rangle + \gamma - (\delta+1)\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle + \\ +(\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle)^2 \sum_{k=1}^{m} b_k(t)\langle\varphi_k, \varphi_i\rangle = 0,\end{gathered} \tag{14}$$

$$\begin{gathered}\sum_{k=1}^{m} \frac{d}{dt} b_k(t)\langle\varphi_k, \varphi\rangle - \alpha_2 \sum_{k=1}^{m} b_k(t)\nu_k\langle\varphi_k, \varphi_i\rangle - \\ +\delta \sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle - (\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle)^2 \sum_{k=1}^{m} b_k(t)\langle\varphi_k, \varphi_i\rangle = 0, \\ i = 1, \ldots, m,\end{gathered} \tag{15}$$

with the condition

$$\langle w(0) - w_0, \varphi_i\rangle = 0. \tag{16}$$

Step 4.2. In the case of $\varepsilon_2 = 0$, scalarly multiply in $L_2(\Omega)$ the system of equations (1) by the eigenfunctions $\varphi_i(s)$, $i = 1, ..., m$, and form the system of algebraic-differential equations:

$$\begin{gathered}\sum_{k=1}^{m} \tfrac{d}{dt} a_k(t)\langle\varphi_k, \varphi_i\rangle - \alpha_1 \sum_{k=1}^{m} a_k(t)\nu_k\langle\varphi_k, \varphi_i\rangle + \gamma - (\delta+1)\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle + \\ +(\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle)^2 \sum_{k=1}^{m} b_k(t)\langle\varphi_k, \varphi_i\rangle = 0,\end{gathered} \tag{17}$$

$$\begin{gathered}\alpha_2 \sum_{k=1}^{m} b_k(t)\nu_k\langle\varphi_k, \varphi_i\rangle - \delta \sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle - \\ -(\sum_{k=1}^{m} a_k(t)\langle\varphi_k, \varphi_i\rangle)^2, \qquad i = 1, \ldots, m,\end{gathered} \tag{18}$$

with the condition

$$\langle v(0) - v_0, \varphi_i\rangle = 0. \tag{19}$$

Step 5.1. Find $a_k(0)$ by scalarly multiplying in $L_2(\Omega)$ the initial condition (16) by the eigenfunctions $\varphi_i(s)$, $k = 1, ..., m$.

Step 5.2. Find $b_k(0)$ by scalarly multiplying in $L_2(\Omega)$ the initial condition (19) by the eigenfunctions $\varphi_i(s)$, $k = 1, ..., m$.

Step 6.1. Having solved the system of algebraic equations (14) with respect to $a_k(0)$, obtain the values of $b_k(0)$.

Step 6.2. Having solved the system of algebraic equations (18) with respect to $b_k(0)$, obtain the values $a_k(0)$.

Step 7. Using the Runge – Kutta method of order 4–5, find a solution to the system of differential equations (15) or (17) with the initial conditions (16) or (19).

The program "Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the autocatalytic reaction model with diffusion" is intended to find an approximate solution of the Showalter–Sidorov problem for a distributed brussellator uniqueness model or multiplicity of solutions and implements the algorithm described above. The program is written in Maple and implemented in the Maple 2017 computer mathematics system. The program implements a modified Galerkin method and phase space method. The program constructs an approximate solution to the problem in the form of a Galerkin sum over several first eigenfunctions of the homogeneous Dirichlet problem of the Laplace operator Δ in the domain Ω, a system of algebraic-differential equations and corresponding initial conditions are generated. The program allows to find numerically a solution to the Showalter–Sidorov problem for a distributed brussellator model, and to plot an approximate solution. Fig. 1 shows a diagram of an algorithm for solving the Showalter–Sidorov problem for a distributed brussellator model.

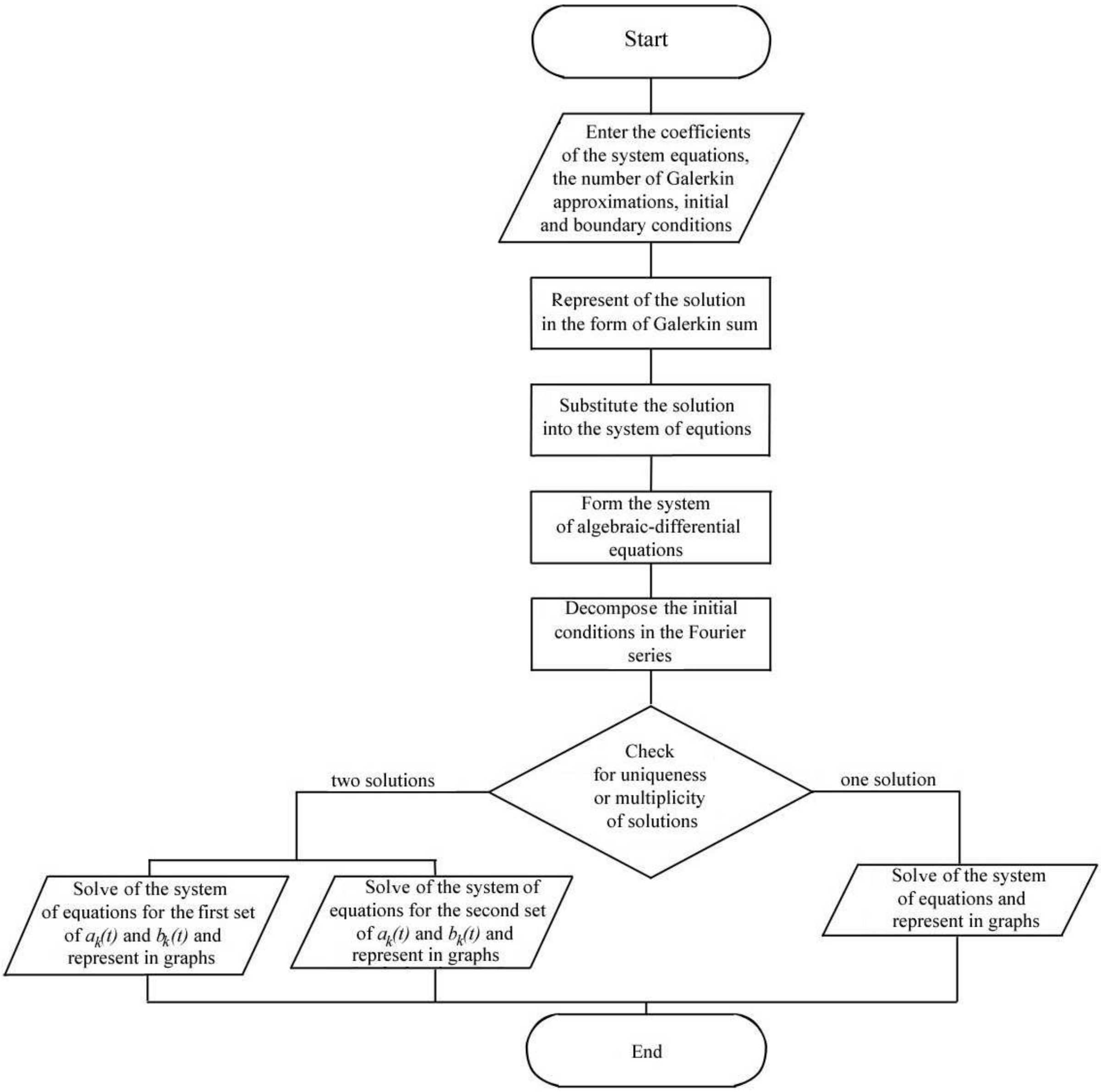

Fig. 1. Block diagram of the algorithm

To implement the developed algorithm, built-in functions and standard operators of the Maple 2017 software package were used. Graphic images were obtained using the **plots** package. The program runs on a personal computer of the Intel platform and on the Microsoft Windows operating system. The output is the display on the screen of graphs of two components of the solution at certain intervals.

Example 1. It is required to find a solution to problem (1), (4), (5) under the following conditions:

$$\Omega = (0,\pi), \quad w_0(s) = \sqrt{\frac{2}{\pi}}\sin(s) + \sqrt{\frac{2}{\pi}}\sin(2s),$$

$$\alpha_2 = 1, \quad \alpha_2 = 2, \quad \gamma = 2, \quad \delta = 1, \quad m = 2, \quad \varepsilon_1 = 0.$$

We write the problem (1), (4), (5) under the following conditions:

$$\begin{cases} 0 = 2v_{ss} + 2 - 2v + v^2 w, & s \in (0,\pi), \\ w_t = 2w_{ss} + v - v^2 w, & s \in (0,\pi),\ t \in (0,1), \\ w(0,t) = w(\pi,t) = 0, & t \in [0,1], \\ w(s,0) = \sqrt{\frac{2}{\pi}}\sin(s) + \sqrt{\frac{2}{\pi}}\sin(2s), & s \in [0,\pi]. \end{cases} \tag{20}$$

On the interval $(0,\pi)$ the eigenfunctions of the homogeneous Dirichlet problem of the Laplace operator Δ have the form: $\varphi_k(s) = \sqrt{\frac{2}{\pi}}\sin(ks)$, $k = 1, 2, \ldots$ We represent the desired functions in the form of a Galerkin sum:

$$v(s,t) = \sqrt{\tfrac{2}{\pi}}(a_1(t)\sin(s) + a_2(t)\sin(2s)),$$
$$w(s,t) = \sqrt{\tfrac{2}{\pi}}(b_1(t)\sin(s) + b_2(t)\sin(2s)).$$

Taking $r = a_1(0)$, $q = b_1(0)$, $v^{\perp} = a_2(t)\varphi_2$, $w^{\perp} = b_2(t)\varphi_2$, and substituting the obtained values in formulas (13), we get $D = 12,19081162$. As follows from Theorem 3, Showalter–Sidorov problem (5) for the system of equations (2) for the given system parameters and initial data have two solutions $(v_{01}(s,t), w_{01}(s,t))$ and $(v_{02}(s,t), w_{02}(s,t))$. Tables 1, 2 and in Fig. 2 present the result of the numerical solution of the system of algebraic-differential equations taking into account the initial conditions.

Table 1

Numerical solution $(v_{01}(s,t), w_{01}(s,t))$ to problem (2), (4), (5) (up to 10^{-5})

t	$a_1(t)$	$a_2(t)$	$b_1(t)$	$b_2(t)$
0	0,89812	0,027271	1	1
0,1	0,88959	0,017432	0,95348	0,65427
0,2	0,88371	0,011225	0,91338	0,42826
0,3	0,87925	0,0072582	0,87856	0,28042
0,4	0,87564	0,0047053	0,84820	0,18367
0,5	0,87262	0,0030559	0,82166	0,12032
0,6	0,87005	0,0019876	0,79845	0,078839
0,7	0,86785	0,012943	0,77812	0,051665
0,8	0,86594	0,00084361	0,76031	0,033862
0,9	0,86429	0,00055033	0,74470	0,022197
1	0,86285	0,00035927	0,73102	0,014551

Table 2

Numerical solution $(v_{02}(s,t), w_{02}(s,t))$
to problem (20) (up to 10^{-5})

t	$a_1(t)$	$a_2(t)$	$b_1(t)$	$b_2(t)$
0	1,1550	0,10070	1	1
0,1	1,0200	0,048011	0,85259	0,64252
0,2	0,93097	0,016139	0,73505	0,41600
0,3	0,87925	0,0072582	0,63682	0,27031
0,4	0,90708	0,0098521	0,55331	0,17604
0,5	0,88896	0,0061104	0,48170	0,11483
0,6	0,87467	0,0038317	0,41996	0,074991
0,7	0,86310	0,0024224	0,36654	0,049021
0,8	0,85359	0,0015412	0,32019	0,032069
0,9	0,84568	0,00098547	0,27991	0,020992
1	0,83903	0,00063271	0,24483	0,013748

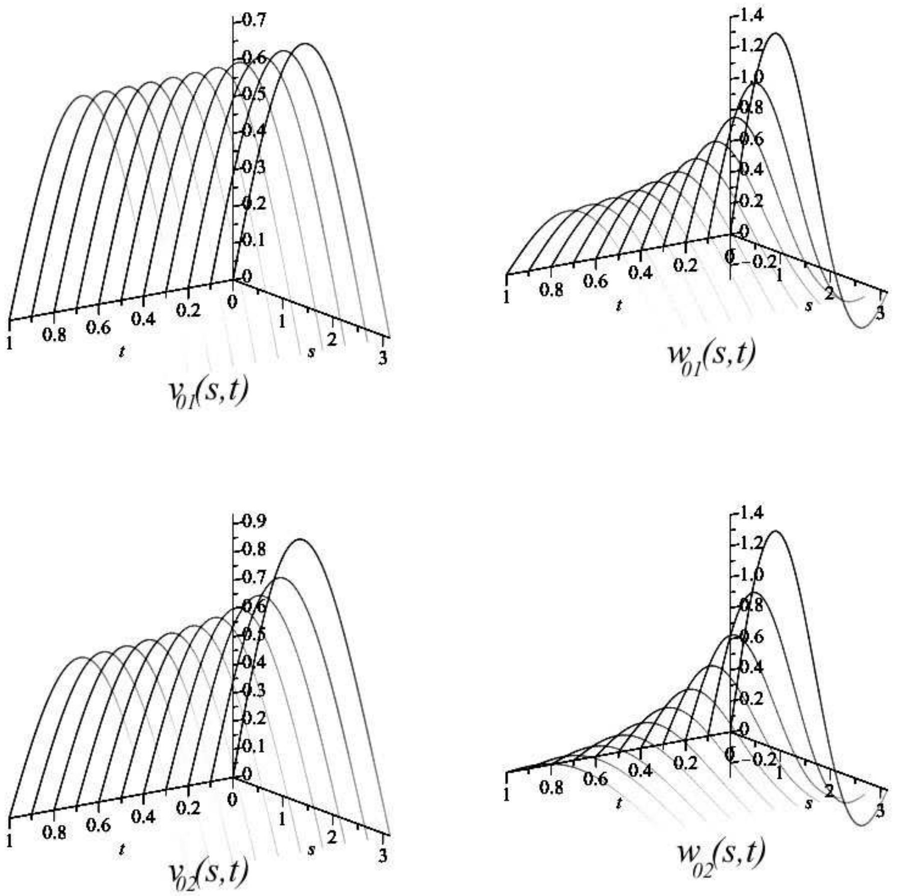

Fig. 2. Numerical solution to problem (20)

Fig. 3 shows the phase space of Showalter–Sidorov problem (5) for the system of equations (2), (4) for given system parameters and initial data.

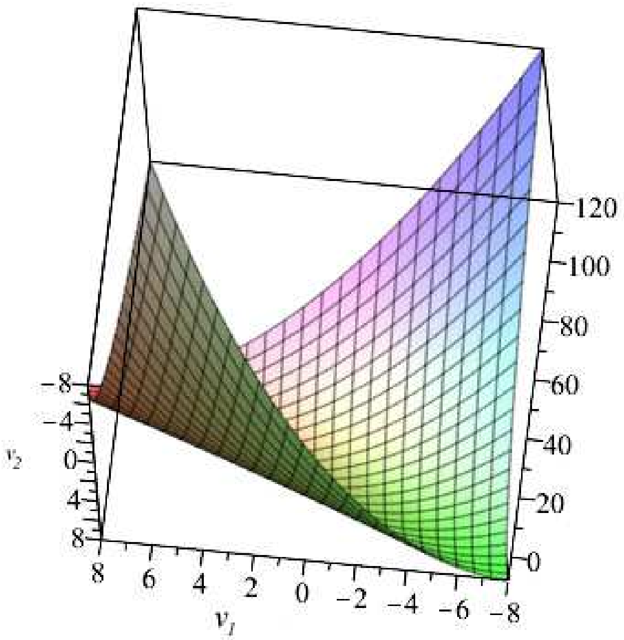

Fig. 3. Phase space of system of equation (2)

Example 2. It is required to find a solution to problem (1), (4), (6) under the following conditions:

$$\Omega = (0, \pi), \quad v_0(s) = \sqrt{\frac{2}{\pi}}\sin(s) + \sqrt{\frac{2}{\pi}}\sin(2s),$$

$$\alpha_2 = 1, \quad \alpha_2 = 2, \quad \gamma = 2, \quad \delta = 1, \quad m = 2, \quad \varepsilon_2 = 0.$$

We write problem (1), (4), (5) under the following conditions:

$$\begin{cases} v_t = 2v_{ss} + 2 - 2v + v^2 w, & s \in (0, \pi), \\ 0 = 2w_{ss} + v - v^2 w, & s \in (0, \pi), \ t \in (0, 1), \\ v(0, t) = v(\pi, t) = 0, & t \in [0, 1], \\ v(s, 0) = \sqrt{\frac{2}{\pi}}\sin(s) + \sqrt{\frac{2}{\pi}}\sin(2s), & s \in [0, \pi]. \end{cases} \tag{21}$$

As follows from Theorem 5, Showalter–Sidorov problem (5) for the system of equations (3) for the given system parameters and initial data have the unique solution. Table 3 and Fig. 4 preset the result of the numerical solution of the system of algebraic-differential equations, taking into account the initial conditions.

Table 3

Numerical solution of problem (3), (4), (6) (up to 10^{-5})

t	$a_1(t)$	$a_2(t)$	$b_1(t)$	$b_2(t)$
0	3,133285	1,253314	$1,110223 \cdot 10^{-16}$	2,506628
0,1	2,351236	0,681369	0,458532	0,942530
0,2	2,215488	0,562147	0,493288	0,671034
0,3	2,137601	0,518318	0,498064	0,515260
0,4	2,085840	0,501462	0,498692	0,411737
0,5	2,050346	0,494856	0,498748	0,340750
0,6	2,025767	0,492238	0,498738	0,291592
0,7	2,008669	0,491190	0,498727	0,257396
0,8	0,025985	0,490767	0,498720	0,233544
0,9	1,988412	0,490593	0,498716	0,216882
1	1,982585	0,490520	0,498713	0,205228

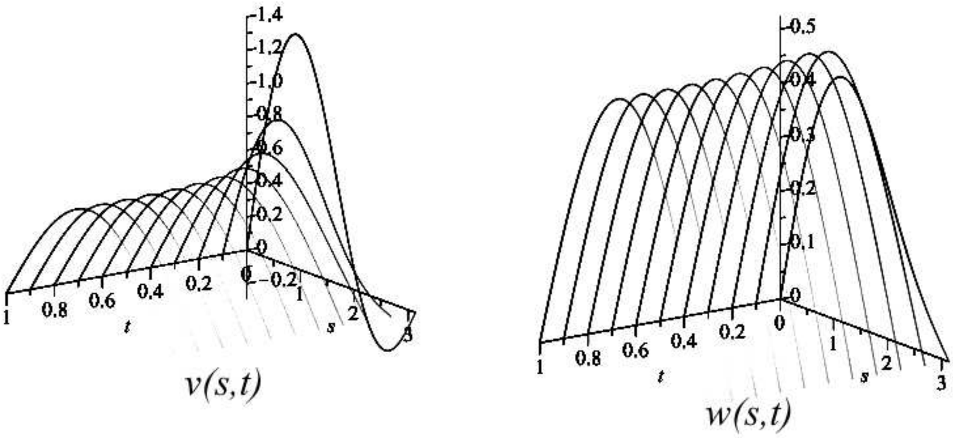

Fig. 4. Numerical solution of problem (21)

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ НЕЕДИНСТВЕННОСТИ РЕШЕНИЯ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ ОДНОЙ ВЫРОЖДЕННОЙ МАТЕМАТИЧЕСКОЙ МОДЕЛИ АВТОКАТАЛИТИЧЕСКОЙ РЕАКЦИИ С ДИФФУЗИЕЙ

О. В. Гаврилова

Статья посвящена численному исследованию фазового пространства математической модели автокаталической реакции с диффузией, основанной на вырожденной системе уравнений «распределенного» брюсселятора. В данной математической модели скорость изменения одной из компонент системы может значительно превосходить другую, что приводит к вырожденной системе уравнений. Изучаемая модель относится к широкому классу полулинейных моделей соболевского типа. Нами будут выявлены условия существования, единственности или множественности решений задачи Шоуолтера – Сидорова в зависимости от параметров системы. Полученные теоретическе результаты позволили разработать алгоритм численного решения задачи, основанный на модифицированном методе Галеркина. Приведены результаты вычислительных экспериментов.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидоров; неединственность решений; распределенный брюсселятор.

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