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ON THE OPTIMAL CONTROL PROBLEM TO SOLUTIONS OF ONE GRANBERG MODEL

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The active development of methods for solving inhomogeneous systems of differential equations with a degenerate matrix with a derivative is primarily associated with a wide range of applied problems. Optimal control to solutions of these problems is also an important area of research. The article considers the problem on optimal control to solutions of the non-stationary Granberg model. The main methods of the study are methods of the theory of degenerate (semi) groups and optimal control for Sobolev type equations. The given example of solving the problem from the monograph written by A. G. Granberg illustrates the advantages of the applied methods for solving. Namely, the methods do not require the introduction of assumptions that were applied earlier and do not correspond to real situations when solving such problems. Also, as an example, we give an exact solution to the optimal control problem in which the planned values of economic indicators are taken in the form of a second-order polynomial with a control action in the form of a third-order polynomial. In addition, we propose an algorithm for numerically solving the optimal control problem under consideration.

Keywords: Leontief type equations; algorithm for the numerical solution; Granberg model; optimal control; non-stationary model.

Introduction

Let L and M be square matrices of order n, $\det L = 0$, then the system

$$L\dot{x} = Mx + f \tag{1}$$

represents the degenerate balance model of W. W. Leontief in the monetary form [1]. Here $x = col\,(x_1, \ldots, x_n)$ and $\dot{x} = col\,(\dot{x}_1, \ldots, \dot{x}_n)$ are the vector functions of gross output volume and its growth, respectively; L is the matrix of unit capital costs, $M = \mathbb{I} - A$, and A is the matrix of unit direct costs; f is the vector function that determines the final demand.

Systems of the form (1) under the condition $\det L = 0$ take place in various fields of knowledge, for example, in problems of hydrodynamics [2], metrology [3], etc., see also [4]. Note that the condition of degeneracy of the system, $\det L = 0$, is one of the distinguishing features of balance models of the economy, since resources of a certain type cannot be stored [1]. Moreover, note that balance models often have a non-stationary form, i.e. the matrices included in system (1) depend on time (see, for example, [5]). W. W. Leontief was the first to study systems of the form (1) unresolved with respect to the derivative. At the same time, such systems represent a special case of Sobolev type equations [6].

In the work of any economic system, control is necessary for the effective achievement of goals. In the 60s of the XX century, A. G. Granberg proposed to construct optimization intersectoral interregional models while considering models of a regional and national economy. However, even W. W. Leontief spoke about the possibility of applying balance models to an enterprise. The work of A. V. Keller [7] provides an example of a balance model for an enterprise.

Methods for solving inhomogeneous systems of differential equations with a degenerate matrix in the derivative are most actively and successfully developed in the framework of the scientific school headed by G. A. Sviridyuk [8, 9].

The papers [10, 11] consider various types of optimal control problems for the degenerate balance dynamic Leontief-type models

$$L\dot{x} = Mx + f + Bu, \tag{2}$$

$$\left((\mu L - M)^{-1} L\right)^{p+1} (x(0) - x_0) = 0, \tag{3}$$

including the optimal control problem with the quality functional

$$J(u) = \sum_{q=0}^{1} \int_{0}^{\tau} \left\| Cx^{(q)}(u,t) - y_0^{(q)}(t) \right\|^2 dt + \sum_{q=0}^{\theta} \int_{0}^{\tau} \langle N_q u^{(q)}(t), u^{(q)}(t) \rangle dt, \tag{4}$$

where the matrix M is (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, $\theta = \overline{0, p+1}$, $Cx(u,t)$ are the actual values of economic indicators, $y_0(t) = (y_{01}(t), \dots, y_{0n}(t))$ are the planned values of the same indicators without spasmodic changes, $\|\cdot\|$ and $\langle\cdot\rangle$ are the norm and scalar product in $\mathbb{R}^n$, respectively, C is a square matrix of order n, and N_q are symmetric positive defined matrices. System (2) takes into account the control action on the system (the vector function Bu), i.e., as a result of solving the problem, we determine the control value u and the gross output volume x required to achieve the planned indicators $y_0 = Cx_0(t)$.

The main purpose of this article is the numerical study to the optimal control problem with functional (4) to solutions of one non-stationary Granberg model [4]. In order for this we use the results of the paper [12].

1. Statement of the Problem

In the problem on optimal control to the solutions of the non-stationary Granberg model, we consider the degenerate dynamic balance model

$$L\dot{x}(t) = a(t)Mx(t) + u(t), \tag{5}$$

where L and M are square matrices of order n, and, perhaps, $\det L = 0$, the matrix M is (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, $a : [0,\tau] \to \mathbb{R}_+$, $u : [0,\tau] \to \mathbb{R}^n$. $u = (u_1, \dots, u_n)$ is the vector function that characterizes the quantitative indicators associated with the behavior of end consumers, and u is such that

$$\sum_{q=0}^{p+1} \int_{0}^{\tau} \left\| u^{(q)}(t) \right\|^2 dt \le d, \tag{6}$$

$d = const$, n is the number of branches of the economic system; $x(u) \in \chi$, $u \in \mathfrak{U}_\partial$, χ is the space of solutions, $\mathfrak{U}$ is the space of controls, $\mathfrak{U}_\partial \subset \mathfrak{U}$ is a closed convex subset, which is the set of admissible controls satisfying (6). Moreover, *the space of controls*

$$\mathfrak{U} = \{u \in L_2\left((0,\tau);\mathbb{R}^n\right) : u^{(p+1)} \in L_2\left((0,\tau);\mathbb{R}^n\right), \quad p \in \{0\} \cup \mathbb{N}\},$$

the space of solutions

$$\mathcal{X} = \{x \in L_2((0,\tau);\mathbb{R}^n) : \dot{x} \in L_2((0,\tau);\mathbb{R}^n)\}.$$

In the space $\mathfrak{U}$, we consider the compact convex set $\mathfrak{U}_\partial$, which is the set of *admissible controls.*

Therefore, it is necessary to determine the optimal value $v = col\,(v_1, \ldots, v_n) \in \mathfrak{U}_\partial$, which is necessary to achieve the minimum difference between the actual and the desired values of economic indicators, i.e. such that

$$J(v) = \min_{u \in \mathfrak{U}_\partial} J(u) =$$
$$= \min_{u \in \mathfrak{U}_\partial} \left(\sum_{q=0}^{1} \int_0^\tau \left\| Cx^{(q)}(u,t) - y_0^{(q)}(t) \right\|^2 dt + \sum_{q=0}^{\theta} \int_0^\tau \langle N_q u^{(q)}(t), u^{(q)}(t) \rangle dt \right), \tag{7}$$

moreover, $x(v)$ must satisfy both the degenerate balance model (5) and the Showalter-Sidorov condition (3).

According to the economic sense of the corresponding terms in the balance model, we add the conditions

$$x_i(t) \geq w_i \geq 0, \quad i = \overline{1,n}, \tag{8}$$

where w_i is the minimum required amount of the volume of products or services of the i-th type of activity;

$$(u)_i \leq 0. \tag{9}$$

2. Granberg Model

Dynamic intersectoral models are detailed analogs of the models of reproduction of the social product and national income, generalize statistical balance and optimization intersectoral models, and are used as a theoretical and methodological basis for applied dynamic models with intersectoral balance matrices [4]. In the general case, economical dynamic models describe the development paths of a set of indicators characterizing the state of an economic object (enterprise, industry, ...) depending on time. The initial state of an economic system, i.e. the input state $u = u(t)$, is transformed to the output state $y = y(t)$, while the transformation operator can be the transfer function $W(s) = \dfrac{y(s)}{u(s)}$ of the complex variable s under some initial conditions.

The dynamic model of V. V. Leontief is given by the system of linear differential equations of the first order with constant coefficients, which is unresolved with respect to derivatives and has the form [4]

$$X(t) = AX(t) + B\dot{X}(t) + C(t), \tag{10}$$

where $X(t)$ is the column vector of production volumes; $\dot{X}(t)$ is the column vector of absolute production increases; $C(t)$ is the column vector of consumption including non-productive accumulation; A is the matrix of coefficients of direct material cost, including the costs of reimbursement of disposal and capital repairs of fixed production assets; B is the matrix of coefficients of capital intensity of production increases, i.e. the cost of production accumulation per unit of increase in the corresponding types of products.

In further considerations, it was assumed that the matrix A is indecomposable, and the matrix B is non-degenerate. However, as the author himself points out, these assumptions are unacceptably artificial, since the real matrices A are decomposable, and the matrices B have zero rows, in particular, by industries that produce only consumer goods. Therefore, we consider equation (10) and find its solution in the general case. Let $L = B$, $M = \mathbb{I} - A$, $u(t) = -C(t)$. We obtain the Leontief type system

$$L\dot{x}(t) = Mx(t) + u(t) \tag{11}$$

under the initial conditions

$$P\left(x(0) - x_0\right) = 0. \tag{12}$$

In addition, we assume the presence of non-stationarity in the system, i.e. consider the equation of the form

$$L\dot{x}(t) = a(t)Mx(t) + u(t), \tag{13}$$

where $a(t)$ is a scalar continuous function; L and M are square matrices of order n, and, perhaps, $\det L = 0$, the matrix M is (L, p)-regular, $p \in \{0\} \cup \mathbb{N}$, $u : [0, \tau] \to \mathbb{R}^n$; $x = col\,(x_1, \ldots, x_n)$ and $\dot{x} = col\,(\dot{x}_1, \ldots, \dot{x}_n)$ are the vector functions of production volumes and their rate of change, respectively; $u = (u_1, \ldots, u_n)$ is the vector function of control; $P = \left(R^L_\mu\right)^{p+1}$.

Moreover, the economic sense of the corresponding terms in the balance model imposes the condition

$$x_i(t) \geq w_i \geq 0, \quad i = \overline{1, n}, \tag{14}$$

where w_i is the minimum required amount of production volume of the i-th industry.

Let the national economy in the context of three industries (production of tools, production of objects of labor, production of consumer goods) be characterized by the following matrices of material consumption and capital intensity:

$$A = \begin{pmatrix} 0.100 & 1,116 & 0.075 \\ 0.500 & 0.548 & 0.425 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1.5 & 1.6 & 0.9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

with the initial values of production volumes $x_1(0) = 18$, $x_2(0) = 50$, $x_3(0) = 32$.

Hereinafter, we take zero values as the minimum required amount of production volume of each industry, i.e.

$$w_i(t) = 0, \quad i = \overline{1, n}.$$

Then

$$L = \begin{pmatrix} 1.5 & 1.6 & 0.9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0.9 & -0.116 & -0.075 \\ -0.5 & 0.452 & -0.425 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

and $\det(\mu L - M) = 1.478\mu - 0.3488$. The matrix M is L-regular, and the L-resolvent of the matrix M has a removable singular point at ∞, therefore, $p = 0$ and the matrix M is

$(L, 0)$-regular. The vector function of production volumes takes the form [12]

$$x(u,t) = \lim_{k\to\infty} x_k(u,t) =$$
$$= \lim_{k\to\infty}\left[-M^{-1}\left(\mathbb{I}-Q_k\right)\frac{u(t)}{a(t)} + \left(\left(L-\frac{1}{k}M\int_0^t a(\zeta)d\zeta\right)^{-1} L\right)^k x_0 + \right. \tag{17}$$
$$\left. + \int_0^t \left(\left(L-\frac{1}{k}M\int_s^t a(\zeta)d\zeta\right)^{-1} L\right)^k \left(L-\frac{1}{k}M\right)^{-1} \left(kL_k^L(M)\right) u(s)ds\right].$$

For the given initial values of production volumes, substitute all the data in formula (17) and take the limit for $k \to \infty$. We have

$$x_1(u,t) = \frac{1}{a(t)}\left(1,083u_2(t) + 0,736u_3(t)\right),$$
$$x_2(u,t) = \frac{-1.015u_2(t) - 0.126u_3(t)}{a(t)} + 84.875\exp\left(\frac{872}{3695}\int_0^t a(\zeta)d\zeta\right) +$$
$$+\int_0^t \exp\left(\frac{872}{3695}\int_s^t a(\zeta)d\zeta\right)\left(\frac{5}{8}u_1(s) + \frac{4035}{5912}u_2(s) + \frac{2779}{5912}u_3(s)\right)ds, \tag{18}$$
$$x_3(u,t) = \frac{u_3(t)}{a(t)}.$$

3. Optimal Control

Let L, M, and C be square matrices of order n, and, perhaps, $\det L = 0$, the matrix M be (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$ be the order of pole of the L-resolvent of the matrix M at the point ∞, $u : [0,\tau] \to \mathbb{R}^n$, $a : [0,\tau] \to \mathbb{R}_+$.

Consider the Leontief type system

$$L\dot{x} = aMx + u, \tag{19}$$

where $x = (x_1, \ldots, x_n)$ and $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$ are the vector functions of production volumes and their rate of change, respectively; L and M are matrices representing the mutual influence of the rates of change in production volumes and production volumes, respectively; the matrix C characterizes the relationship between the observed and planned values of production volumes; $u = (u_1, \ldots, u_n)$ is the vector function of controls; n is the number of state system parameters; N_q are symmetric positive defined matrices.

Fix $\tau \in \mathbb{R}_+$ and consider the *space of solutions*

$$\chi = \left\{x \in L_2\left((0,\tau), \mathbb{R}^n\right) : \dot{x} \in L_2\left((0,\tau), \mathbb{R}^n\right)\right\},$$

space of controls $\mathfrak{U} = \left\{u \in L_2\left((0,\tau), \mathbb{R}^n\right) : u^{(p+1)} \in L_2\left((0,\tau), \mathbb{R}^n\right)\right\}$,
and *space of observations* $\mathfrak{Y} = C[\chi]$. Note that not always $\mathfrak{Y} = \chi$, but always $\mathfrak{Y}$ is isomorphic to some subspace in χ. In $\mathfrak{U}$, consider the compact and convex subset $\mathfrak{U}_\partial$, which is the *set of admissable controls*. By admissible controls we mean the controls such

that

$$\sum_{q=0}^{p+1} \int_0^\tau \left\| u^{(q)}(t) \right\|^2 dt \leq d,$$

where $d = const$ is the maximum admissable value of the vector function of controls. Consider the *problem on optimal control* to solutions of the non-stationary Granberg model.

It is necessary to find the vector function $v \in \mathfrak{U}_\partial$ minimizing the value of the functional

$$J(u) = \sum_{q=0}^{1} \int_0^\tau \left\| Cx^{(q)}(u,t) - y_0^{(q)}(t) \right\|^2 dt + \sum_{q=0}^{\theta} \int_0^\tau \langle N_q u^{(q)}(t), u^{(q)}(t) \rangle dt, \tag{20}$$

i.e.

$$J(v) = \min_{u \in \mathfrak{U}_\partial} J(u), \tag{21}$$

such that the vector function $x(v) \in \chi$ satisfies system (2.14) almost everywhere on $(0, \tau)$ and, for some $x_0 \in \mathbb{R}^n$ and $\alpha \in \rho^L(M)$, the vector function $x(v) \in \chi$ satisfies the Showalter - Sidorov condition

$$\left((\alpha L - M)^{-1} L\right)^{p+1} (x(0) - x_0) = 0, \tag{22}$$

where $\| \cdot \|$ is the Euclidean norm of the space $\mathbb{R}^n$, and $y_0(t) = col\,(y_{01}(t), \ldots, y_{0n}(t))$ are the planned values of production volumes at the certain point in time t.

Definition 1. A vector function $u \in \mathfrak{U}_\partial$ satisfying (20) is called an *admissible control* of problem (19)–(22) under the condition that $x(u) \in \chi$ satisfies (19) and (22).

By construction, the functional (20) is continuous and strongly convex on $\mathfrak{U}_\partial$, then there exists the unique minimum point of the functional on $\mathfrak{U}_\partial$.

Definition 2. A vector function $v \in \mathfrak{U}_\partial$ satisfying (21) is called an *optimal control* of problem (19)–(22), if $x(v) \in \chi$ satisfies (19) and (22).

Theorem 1. [12] *Let the matrix M be (L,p)-regular, $p \in \{0\} \cup \mathbb{N}$, $\tau \in \mathbb{R}_+$, $\det M \neq 0$. Then for any $x_0 \in \mathbb{R}^n$, $y_0 \in \mathfrak{Y}$ there exists the unique solution $v \in \mathfrak{U}_\partial$ to problem* (19)–(22), *which is the optimal control, moreover, $x(v)$ satisfies system* (19) *under initial condition* (22) *and has the form*

$$\begin{gathered} x(u,t) = \lim_{k\to\infty} x_k(u,t) = \lim_{k\to\infty} \left[-\sum_{q=0}^{p} \left(M^{-1}(\mathbb{I} - Q_k)L\right)^q M^{-1}(\mathbb{I} - Q_k) \left(\frac{1}{a(t)}\frac{d}{dt}\right)^q \frac{u(t)}{a(t)} + \right. \\ \left. + \left(\left(L - \frac{1}{k} M \int_0^t a(\zeta)d\zeta\right)^{-1} L\right)^k x_0 + \int_0^t \left(\left(L - \frac{1}{k} M \int_s^t a(\zeta)d\zeta\right)^{-1} L\right)^k \left(L - \frac{1}{k}M\right)^{-1} Q_k u(s) ds \right], \end{gathered} \tag{23}$$

where $\left(\frac{1}{a(t)}\frac{d}{dt}\right)^q$ *means that we apply the given operator q times successively,* $Q_k = \left(kL_k^L(M)\right)^{p+1}$.

Let us find the optimal control to the solutions of the problem considered above. As the scalar function $a(t)$, we take the linear function $a(t) = t$ on the interval $[0;1]$, and find the vector function of control $u(t) = (u_1(t), u_2(t), u_3(t))$ in the form of a vector function of polynomials of the third degree

$$u(t) = \begin{pmatrix} a_1t^3 + b_1t \\ a_2t^3 + b_2t \\ a_3t^3 + b_3t \end{pmatrix}. \tag{24}$$

As the planned values of production volumes, we consider

$$y_0(t) = \begin{pmatrix} t - t^2 \\ 0 \\ 0 \end{pmatrix}; \tag{25}$$

$N_q = \mathbb{I}$, $q = 0, 1$; the matrix characterizing the relationship between the observed and planned values of production volumes has the form

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Substitute (24), (25), (26) and (18) into formula (20). We obtain the functional

$$J(u) = \int_0^1 \left(1,083(a_2t^2 + b_2) + 0,736(a_3t^2 + b_3) - t + t^2\right)^2 dt+$$

$$+\int_0^1 (2,166a_2t + 1,472a_3t - 1 + 2t)^2 dt + \int_0^1 \left((a_1t^3 + b_1t)^2 + (a_2t^3 + b_2t)^2 + (a_3t^3 + b_3t)^2\right) dt + \tag{27}$$

$$+\int_0^1 \left((3a_1t^2 + b_1)^2 + (3a_2t^2 + b_2)^2 + (3a_3t^2 + b_3)^2\right) dt.$$

Therefore, it is necessary to find the coefficients a_1, a_2, a_3, b_1, b_2, b_3 in (24) such that (27) takes the minimum value. Note that (27) is a function of six variables, the minimum of which can be found by an environment designed to perform various mathematical and technical calculations on a computer. We obtain

$$J(v) = 0,281$$

for

$$a_1 = 0, a_2 = -0,116, a_3 = -0,079, b_1 = 0, b_2 = 0,127, b_3 = 0,086,$$

i.e.

$$v(t) = \begin{pmatrix} 0 \\ -0,116t^3 + 0,127t \\ -0,079t^3 + 0,086t \end{pmatrix}.$$

Taking into account the form (25), (26), we present a graphic illustration of the first component of the observed (see Fig. 1) and planned (see Fig. 2) production volumes.

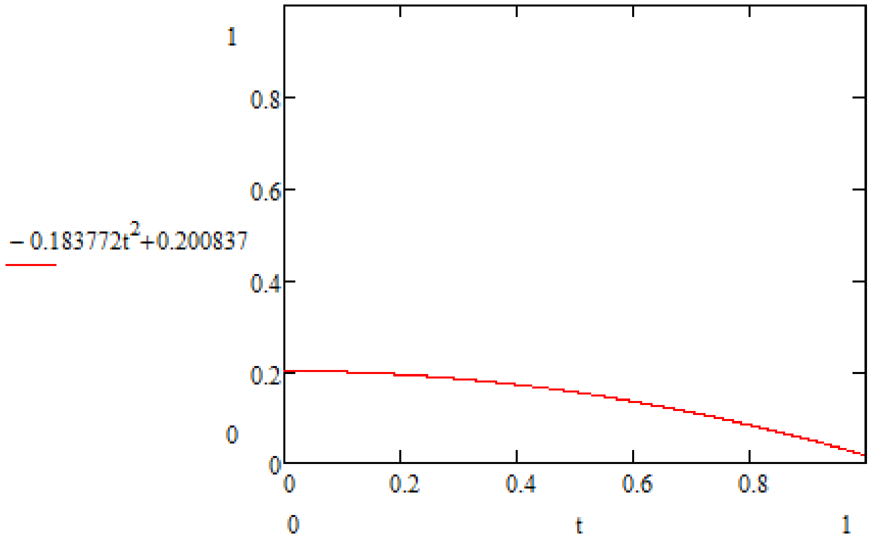

Fig. 1. Graph of the function $x_1(v, t)$

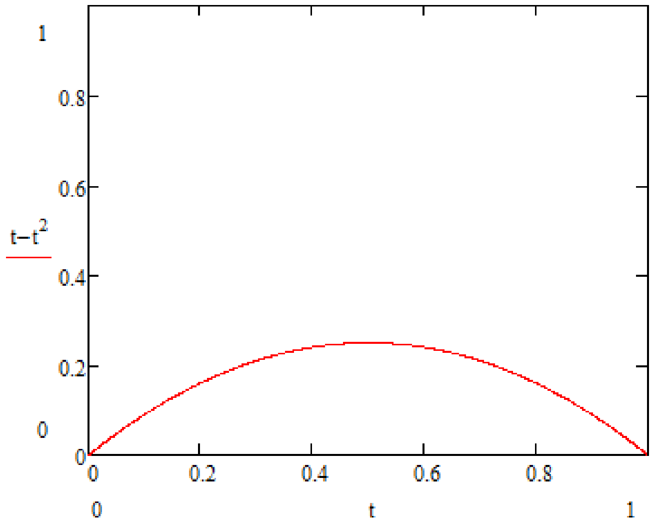

Fig. 2. Graph of the function $y_{01}(v, t)$

4. Algorithm for the Numerical Solution of the Problem

In order to find a numerical solution to problem (19)–(22), which exists by virtue of Theorem 1, we represent the admissible control $u \in \mathfrak{U}_\partial$ as a vector function of polynomials of degree $l > p$:

$$u^l = col\left(\sum_{j=0}^{l} a_{1j}t^j, \sum_{j=0}^{l} a_{2j}t^j, \ldots, \sum_{j=0}^{l} a_{nj}t^j\right). \tag{28}$$

Denote by v_k^l the approximate solution to the optimal control problem, where v_k^l is the minimum point of the quality functional $J_k(u^l)$ on $\mathfrak{U}_\partial \bigcap \mathfrak{U}_\partial^l$

$$J_k(u^l) = \sum_{q=0}^{1} \int_0^\tau \left\| Cx_k^{(q)}\left(u^l, t\right) - y_0^{(q)}(t) \right\|^2 + \sum_{q=0}^{\theta} \int_0^\tau \langle N_q(u^l)^{(q)}(t), (u^l)^{(q)}(t)\rangle dt, \qquad (29)$$

and $x_k^l = x_k\left(v_k^l, t\right)$ satisfies conditions (19), (22) for $k \geq K$, $K = \max\{k_1, k_2\}$. The values k_1, k_2 are determined by the formulas given below, in which $h = n$.

Denote by $\tilde{z}_k(u,t)$ the numerical solution to problem (19), (22), and

$$\tilde{z}_k(u,t) = z_k(u,t),$$

if $k \geq K$ and $K = \max\{k_1, k_2\}$:

$$\begin{gathered} k_1 > \frac{1}{|\alpha_{h-p}|} \sum_{i=0}^{h-p} |\alpha_i| + 1, \\ k_2 > \frac{1}{|\alpha_{h-p}|\, p^p} \sum_{i=0}^{h-p} |\alpha_i| \, (p+1)^{h-i} + 1, \end{gathered} \qquad (30)$$

where α_i are the coefficients of the polynomial $\det(\mu L - M)$ of degree $(h-p)$,

$$\alpha_i = (-1)^{h-i} \sum_{r=1}^{C_h^{h-i}} \Delta_{h-i}^r, \quad i = \overline{0, h},$$

Δ_{h-i}^r are determinants obtained from the determinant of the matrix L by replacing $(h-i)$ columns with the corresponding columns of the matrix M, and r is the order number of the determinant, $(h-p) \leq \operatorname{rank} L$.

Step 1.

Consider initial data: L, M, C are the matrices, $a(t)$ is the function, which is a polynomial of degree l_1, n is the number of industries under consideration; l is the degree of polynomials in representation (28) of the admissible control; Y_0 is the vector function of planned production volumes, the elements of which are presented in the form of polynomials of degree $l - l_1$; d is a ball in the space $\mathbb{R}^{l+1}$, which is taken as the set of admissible controls $\mathfrak{U}_\partial^l$; $[0;\tau]$ is the considered period of time; ε is the accuracy with which the approximate value of the quality functional is calculated; h_{min}, h_{max} are the minimum and maximum optimization steps, respectively; δ is the coefficient of change in the optimization step.

Step 2.

Verify the condition $\det M = 0$. If the condition is satisfied, then go to Step 3, otherwise replace $x = e^{\lambda t} z$.

Step 3.

Divide the interval $[0;\tau]$ into the points and calculate the values of the necessary functions at each of these points.

Step 4.

Calculate the values of p, that is the order of pole of the L-resolvent of the matrix M at the point ∞, and determine the matrices N_q, $q = \overline{0, p+1}$.

Step 5.

Calculate the value of K, starting from which we can find a numerical solution to the problem such that not to be near the points of the L-spectrum of the operator M.

Step 6.

Determine Z_k^t, Q_k, and R_k^t:

$$Z_k^t = \left[\left(L - \frac{1}{k} M \int\limits_0^t a(\zeta) d\zeta \right)^{-1} L \right]^k,$$

$$Q_k = \left[k L_k^L (M) \right]^{p+1}, \tag{31}$$

$$R_k^t = \left[\left(L - \frac{1}{k} M \int\limits_s^t a(\zeta) d\zeta \right)^{-1} L \right]^k \left(L - \frac{1}{k} M \right)^{-1}.$$

Note that the value $\int a(\zeta)d\zeta$ at the points of division is calculated at Step 3.

Step 7.

As initial value, take $u = col(0, \ldots, 0)$, i.e. $a_{ij} = 0$ in (28), and determine the value of the vector function of control at each of the points found at Step 3. Then, calculate the corresponding value of the functional $J_k(u^l)$ for the initial u.

Step 8.

Implement the optimization procedure, i.e. find values of the coefficients of the polynomials in (28) such that the value of functional (29) is minimal.

Step 9.

As a result of the optimization procedure, determine the coefficients of polynomials in the representation of the admissible control at which the quality functional takes a minimum value, i.e. calculate the approximate value v_k^l and the value of the quality functional J_k. Write the answer.

Note that, in view of condition (14), when numerically solving the problem on optimal control to the solutions of the Granberg model in the considered algorithm, it is necessary to include a procedure that verifies that this condition is fulfilled, when the components of the vector function of the production volume are calculated.

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ЗАДАЧА ОПТИМАЛЬНОГО УПРАВЛЕНИЯ РЕШЕНИЯМИ ОДНОЙ МОДЕЛИ ГРАНБЕРГА

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Активное развитие методов решения неоднородных систем дифференциальных уравнений с вырожденной матрицей при производной связано, в первую очередь, с широким кругом прикладных задач. Оптимальное управление решениями этих задач также является актуальным направлением исследований. В статье рассматривается задача оптимального управления решениями нестационарной модели Гранберга. Основными методами проведенного исследования являются методы теории вырожденных (полу)групп и оптимального управления для уравнений соболевского типа. Приведенный пример решения задачи из монографии А.Г. Гранберга иллюстрирует преимущества применяемых методов решения, не требующих введения допущений, применяемых ранее и не соответствующих реальным ситуациям при решении подобного рода задач. Также в качестве примера приведено точное решение задачи оптимального управления, в которой плановые значения экономических показателей взяты в виде полинома второго порядка, при управляющем воздействии в виде полинома третьего порядка. Кроме того, в работе предложен алгоритм численного решения поставленной задачи оптимального управления.

Ключевые слова: уравнения леонтьевского типа; алгоритм численного решения; модель Гранберга; оптимальное управление; нестационарная модель.

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