MSC 60H30, 60H10

# STOCHASTIC LEONTIEFF TYPE EQUATIONS IN TERMS OF CURRENT VELOCITIES OF THE SOLUTION

***Y. E. Gliklikh***[1]***,*** Voronezh State University, Voronezh, Russian Federation, yeg@math.vsu.ru
***E. Yu. Mashkov,*** Kursk State University, Kursk, Russian Federation, mashkovevgen@yandex.ru

In papers by A.L. Shestakov and G.A. Sviridyuk [1, 2] a new model of the description of dynamically distorted signals in some radio devises is suggested. In [3, 4] the influence of noise is taken into account in terms of the so-called current velocities of the Wiener process instead of using white noise. This allows the authors to avoid using the generalized function. It should be pointed out that by physical meaning the current velocity is a direct analog of physical velocity for the determinitic processes. Note that the use of current velocity of the Wiener process means that in the construction of mean derivatives the $\sigma$-algebra "present" for the Wiener process is under consideration while there is another possibility to deal with the "present" $\sigma$-algebra of the solution as it is in the usual case in the theory of stochastic differential equation with mean derivatives. In this paper we consider stochastic Leontiev type equation of some special sort given in terms of current velocities of the solution and obtain existence of solution theorem as well as some formulae for the density of the solution.

*Keywords: Mean derivatives, current velocities, stochastic Leontieff type equations.*

## Introduction

The idea of mean derivatives of stochastic processes was suggested by E. Nelson in 60-th years of XX century (see [5, 6, 7]). Unlike ordinary derivatives, the mean derivatives are well-posed for a very broad class of stochastic processes and equations with mean derivatives naturally arise in many mathematical models of physics (in particular, E. Nelson introduced the mean derivatives for the needs of Stochastic Mechanics, a version of quantum mechanics). Nelson introduced forward and backward mean derivatives while only their half-sum, which is symmetric mean derivative called current velocity, is a direct analog of an ordinary velocity for deterministic processes. Another mean derivative, which is called quadratic, is introduced in [8]. It gives information about the diffusion coefficient of the process. Using both Nelson's and quadratic mean derivatives, one can in principle recover the process from its mean derivatives.

In papers by A.L. Shestakov and G.A. Sviridyuk [1, 2] a new model of the description of dynamically distorted signals in some radio devises was suggested. In [3, 4] the influence of noise is taken into account in terms of the so-called current velocities of the Wiener process instead of the use of white noise that allowed the authors to avoid using the generalized function. Note that the use of current velocity of the Wiener process means that in the construction of mean derivatives the "present"$\sigma$-algebra for the Wiener process is under consideration. At the some time there is another possibility to deal with the "present" $\sigma$-algebra of the solution as it is in the usual case in the theory of stochastic differential equation with mean derivatives.

[1] ***Corresponding author***

In this paper we consider the stochastic Leontieff type equations given in terms of the current velocities of the solution, i.e., we use the solution "present" $\sigma$-algebra in the construction of mean derivatives. Note that in this case the Wiener process is not included in the equation explicitly but it is involved into the final formula for the solution. We consider the equation whose matrix pencil is regular and satisfies the so-called rank-degree condition (Chistyakov's condition). As a result, we obtain the existence of solution theorem and explicit formulae for the solution in terms of its probabilistic density.

The structure of the paper is as follows. In Section 1 we introduce the necessary material from the Theory of Mean Derivatives. In the second Section we present some facts from the Theory of Matrices . Section 3 is devoted to the main result and in Section 4 we consider some generalization of our construction that allows us to investigate some equations that are more general than those in Section 3.

## 1. Preliminaries on mean derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is $L_1$-random variable for all $t$.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of $\sigma$-subalgebras of $\sigma$-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \leq s \leq t$;
(ii) the "future" $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \leq s \leq l$;
(iii) the "present" ("now") $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.
All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$.

Ordinary ("unconditional") expectation is denoted by $E$.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all $t$. Thus its "classical"derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [5, 6, 7]) we give

**Definition 1.** *(i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t$ is an $L_1$-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t}\right), \tag{1}$$

*where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that $\Delta t$ tends to 0 and $\Delta t > 0$.*

*(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at $t$ is an $L_1$-random variable*

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\right), \tag{2}$$

*where the conditions and the notation are the same as in (i).*

Note that mainly $D\xi(t) \neq D_*\xi(t)$, but if, say, $\xi(t)$ a.s. has smooth sample paths, these derivatives evidently coinside.

From the properties of conditional expectation it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t,x) = \lim_{\Delta t \to +0} E(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x),$$

$$Y^0_*(t,x) = \lim_{\Delta t \to +0} E(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x) \tag{3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y^0_*(t,\xi(t))$.

**Definition 2.** *The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called anti-symmetric mean derivative.*

Consider the vector fields

$$v^\xi(t,x) = \frac{1}{2}(Y^0(t,x) + Y^0_*(t,x))$$

and

$$u^\xi(t,x) = \frac{1}{2}(Y^0(t,x) - Y^0_*(t,x)).$$

**Definition 3.** $v^\xi(t) = v^\xi(t,\xi(t)) = D_S\xi(t)$ *is called current velocity of* $\xi(t)$*;*
$u^\xi(t) = u^\xi(t,\xi(t)) = D_A\xi(t)$ *is called osmotic velocity of* $\xi(t)$*.*

For stochastic processes the current velocity is a direct analogue of ordinary physical velocity of deterministic processes (see, e.g., [5, 6, 7, 9]). The osmotic velocity measures how fast the "randomness" grows up.

Below we often deal with the processes of the form

$$\xi(t) = \xi_0 + \int_0^t \beta(s)ds + w(t), \tag{4}$$

where $w(t)$ is a Wiener process. For such processes the above-mantioned "physical" properties of current and osmotic velocities become clear from the following propositions.

Denote by $\rho^\xi(t,x)$ the density of process ((4)) with respect to Lebesgue measure $\lambda$ on $[0,l]\times\mathbb{R}^n$. This means that for every continuous inntegrable function $f(t,x)$ on $[0,l]\times\mathbb{R}^n$ the following equality takes place:

$$\int_{[0,l]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)d\lambda = \int_{\Omega\times[0,l]} f(t,\xi(t))d\mathsf{P}dt.$$

**Lemma 1.** *For porcess ((4)) in $\mathbb{R}^n$ the vector field $u^\xi(t,x)$ is represented in the form*

$$u^\xi(t,x) = \frac{1}{2}\mathrm{grad}\, log\, \rho^\xi(t,x). \tag{5}$$

**Lemma 2.** *For process ((4)) in $\mathbb{R}^n$ the vector field $v^\xi(t,x)$ and the density $\rho^\xi(t,x)$ satisfy the equation of continuity*

$$\frac{\partial \rho^\xi(t,x)}{\partial t} = -\mathrm{div}(\rho^\xi v^\xi). \tag{6}$$

Introduce the increment $\Delta\xi(t)$ of process $\xi(t)$: $\Delta\xi(t) = \xi(t + \Delta t) - \xi(t)$. We consider $\Delta\xi(t)$ and a column-vector in $\mathbb{R}^n$. So, its conjugate $\Delta\xi(t)^*$ and is a row-vector. We define the so called quadratic mean derivative $D_2$ by the formula

$$D_2\xi(t) = \lim_{\Delta t \downarrow 0} E_t^\xi \frac{\Delta\xi(t) \cdot \Delta\xi(t)^*}{\Delta t}, \tag{7}$$

where $\cdot$ denotes the matrix multiplication. If $D_2\xi(t)$ exists, it takes values in symmetric positive semi-definite $(2, 0)$-tensors (see details, e.g., in [9]).

## 2. Glossary of some facts from matrix theory

We need also some facts from the theory of matrices. Detailed explanation of this material can be found, e.g., in [10, 11].

**Definition 4.** *Let two $n \times n$ constant matrices $A$ and $B$ be given. The expression $\lambda A + B$ where $\lambda$ is a real or complex valued parameter, is called the matrix pencil. The polynomial* $\det(\lambda A + B)$ *(with respect to $\lambda$) is called the characteristic polynomial of the pencil. If $det(\lambda A + B)$ is not identical zero, the pencil is called regular.*

**Theorem 1.** *Let the matrix pencil $\lambda A + B$ be regular. Then there exist non-degenerate matrices $P$ and $Q$ such that*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{8}$$

*where $I_d$ and $I_{n-d}$ are unit matrices of the corresponding dimensions, $N$ is an upper triangle matrix consisting of Jordan boxes with zeros on diagonal and $J$ is a certain $d \times d$ block.*

**Definition 5.** *If the characteristic polynomial satisfies the equality*

$$rank(A) = \deg(\det(\lambda A + B)), \tag{9}$$

*we say that the polynomial satisfies the* rank-degree *condition or equivalently* Chistyakov's condition.

**Theorem 2.** *If the characteristic polynomial satisfies the rank-degree condition, assertion of Theorem 1 holds true and formula (8) takes the form*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{10}$$

## 3. Main result

Let $\tilde{L}$ be a degenerate matrix and $\tilde{M}$ be a non-degenerate matrix such that the pencil $\lambda\tilde{L} + \tilde{M}$ is regular. Evereywhere below we suppose that this pencil satisfies the rank-degree (Chistyakov's) condition. Take matrices $P$ and $Q$ from Theorem 2 and construct matrices $L = P\tilde{L}Q$ and $M = P\tilde{M}Q$. From Theorem 2 it follows that $L = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}$ Note that since $\tilde{M}$ is non-degenerate, $J$ is also non-degenerate.

Introduce the matrix $\overline{L} = QLQ^*$. For $C^\infty$-smooth vector-function $f(t)$ we consider the system

$$\begin{cases} \tilde{L}D_S\xi(t) = \tilde{M}\xi(t) + \tilde{f}(t), \\ D_2\xi(t) = \overline{L} \end{cases} \tag{11}$$

that we call *the stochastic Leontieff type equation with current velocities.* The adequate initial conditions for equation (11) will be described below.

Consider $\eta(t) = Q^{-1}\xi(t)$ and $f(t) = P\tilde{f}(t)$. Then according to the transformation of equation (11), explained in Theorem 2, we obtain the first line in (11) in the form $LD_S\eta(t) = M\eta(t) + f(t)$. Since $\eta(t) = Q^{-1}\xi(t)$, from the definition of $L$ and definition of $D_2$ by formula (7) we get that the second line of (11) for $\eta(t)$ takes the form $D_2\eta(t) = L$. Thus, equation (11) is transformed into the equation for $\eta(t)$ in the form

$$\begin{cases} LD_S\eta(t) = M\eta(t) + f(t), \\ D_2\eta(t) = L \end{cases} \tag{12}$$

Thus $\mathbb{R}^n$ is expanded into the direct sum of two subspaces $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$ so that equation (12) is expanded into two independent equations in those spaces:

$$\begin{cases} D_S\eta^{(1)}(t) = J\eta^{(1)}(t) + f^{(1)}(t), \\ D_2\eta^{(1)}(t) = I_d \end{cases} \tag{13}$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) + f^{(2)}(t) = 0, \\ D_2\eta^{(2)}(t) = 0 \end{cases} \tag{14}$$

in $\mathbb{R}^{n-d}$.

From the second line of ((14)) it follows that the solution of (14) is not random. Then from the first line it follows that the solution takes the form $\eta^{(2)}(t) = -f^{(2)}(t)$ with obvious initial condition $\eta^{(2)}(0) = -f^{(2)}(0)$.

For investigation of ((13)) we apply Theorem 8.50 of [9].

For simplicity denote the $C^\infty$-smooth vector field $Jx + f^{(1)}(t)$ on $\mathbb{R}^d$ by the symbol $v(t,x)$ and denote the flow of this vector field by $g_t$.

From the second line of ((13)) it follows that the solution, if it exists, must have the form of (4). Specify a probabilistic density $\rho_0$ on $\mathbb{R}^d$ such that it is nowhere equals zero. In this case it follows from Theorem 8.50 of [9] that the density $\rho(t)$ of the solution with initial density $\rho_0$ takes the form $\rho(t) = e^{p(t)}$ where $p(t,x) = p_0(g_{-t}(x)) - \int_0^t (div\ v)(s, g_s(g_{-t}(x))ds$, $p_0 = log\rho_0$.

Since $\rho(t,x)$ is well-defined for all $t \in [0,T]$, it determines a process $\xi(t)$ with this probability density and so with initial density $\rho_0$. By $\eta_0^{(1)}$ we denote the random variable in $\mathbb{R}^d$ with density $\rho_0$.

By construction $D_S\eta^{(1)}(t) = v(t, \eta^{(1)}(t))$. Let $u = \frac{1}{2}grad\ p = grad \log\sqrt{\rho}$ and $a(t,x) = v(t,x) + u(t,x)$. Then $\eta^{(1)}(t)$ satisfies the stochastic differential equation

$$\eta^{(1)}(t) = \eta_0^{(1)} + \int_0^t a(s, \eta^{(1)}(s))ds + w(s) \tag{15}$$

and so it is a solution of (13) in form (4) we are looking for.

Thus we have proved the following

**Theorem 3.** *Under the conditions mentioned above, equation (11) transformed into form (12), with initial conditions* $\eta^{(2)}(0) = -f^{(2)}(0)$ *in* $\mathbb{R}^{n-d}$ *and a random variable with density* $\rho_0$ *nowhere equals to zero in* $\mathbb{R}^d$*, has a solution.*

## 4. Certain generalization

Consider a certain symmetric positive-definite matrix $\Xi$ in $\mathbb{R}^d$. Since it is positive definite, it is non-degenerate and there exists a non-degenrate matrix $A$ in $\mathbb{R}^d$ such that $\Xi = AA^*$ where $A^*$ is a transpose matrix $A$. Introduce the matrix $\Theta = \begin{pmatrix} \Xi & 0 \\ 0 & 0 \end{pmatrix}$ and the matrix $\overline{\Theta} = Q\Theta Q^*$ in $\mathbb{R}^n$. We deal with the equation

$$\begin{cases} \tilde{L}D_S\xi(t) = \tilde{M}\xi(t) + \tilde{f}(t), \\ D_2\xi(t) = \overline{\Theta}. \end{cases} \tag{16}$$

In the same way as (11) is transformed to (12), by applying $P$ and $Q$ we transform (16) into the equation

$$\begin{cases} LD_S\eta(t) = M\eta(t) + f(t), \\ D_2\eta(t) = \Theta. \end{cases} \tag{17}$$

As well as above, (17) is expanded into two independent equations

$$\begin{cases} D_S\eta^{(1)}(t) = J\eta^{(1)}(t) + f^{(1)}(t), \\ D_2\eta^{(1)}(t) = \Xi \end{cases} \tag{18}$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) + f^{(2)}(t) = 0, \\ D_2\eta^{(2)}(t) = 0 \end{cases} \tag{19}$$

in $\mathbb{R}^{n-d}$. Note that (19) coincides with (14) and the arguments for its investigation are the same as for (14).

For investigation of (18), introduce in $\mathbb{R}^d$ a new inner product $\langle\cdot,\cdot\rangle$ such that for arbitrary vertors $X$ and $Y$ in $\mathbb{R}^d$ its value takes the form $\langle X, Y\rangle = (\Xi^{-1}X, Y)$. Specify an initial distribution $\rho_0$ in $\mathbb{R}^d$ that nowhere equals zero. Consider the vector field $v(t,x) = Mx + f(t)$ and denote by $g_t$ its flow. Then from Theorem 8.50 of [9] it follows that the density $\rho(t)$ of the solution with initial density $\rho_0$ takes the form $\rho(t) = e^{p(t)}$ with $p(t,x) = p_0(g_{-t}(x)) - \int_0^t (Div\ v)(s, g_s(g_{-t}(x))ds$ where $p_0 = log\rho_0$ and $Div$ denotes the divergence in $\mathbb{R}^d$ with inner product $\langle\cdot,\cdot\rangle$. An additional modification of the construction here is that $\eta^{(1)}(t) = \eta_0^{(1)} + \int_0^t a(s, \eta^{(1)}(s))ds + Aw(s)$(an analog of (15)). Thus we obtain

**Theorem 4.** *Under the conditions mentioned above, equation (16) transformed into form (17), with initial conditions* $\eta^{(2)}(0) = -f^{(2)}(0)$ *in* $\mathbb{R}^{n-d}$ *and a random variable with density* $\rho_0$ *nowhere equals to zero in* $\mathbb{R}^d$*, has a soltion.*

*The research, whose results are described in Section 3, is supported in part by RFBR Grant 12-01-00183, the ones whose results are described in Section 4 – by RSF Grant 14-21-00066.*

## References

1. Shestakov A.L., Sviridyuk G.A. A New Approach to Measurement of Dynamically Perturbed Signals. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2010, no. 16(192), issue 5, pp. 116–120. (in Russian)

2. Shestakov A.L., Sviridyuk G.A. Optimal Measurement of Dynamically Distorted Signals. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2011, no. 17(234), issue 8, pp. 70–75.

3. Shestakov A.L., Sviridyuk G.A. On the Measurement of the "White Noise". *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2012, no. 27(286), issue 13, pp. 99–108.

4. Gliklikh Yu.E., Mashkov E.Yu. Stochastic Leontieff Type Equations and Mean Derivatives of Stochastic Processes. *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming & Computer Software"*, 2013, vol. 6, no. 2, pp. 25–39.

5. Nelson E. Derivation of the Schrödinger Equation from Newtonian Mechanics. *Phys. Reviews*, 1966, vol. 150, no. 4, pp. 1079–1085

6. Nelson E. *Dynamical Theory of Brownian Motion.* Princeton, Princeton University Press, 1967.

7. Nelson E. *Quantum Fluctuations.* Princeton, Princeton University Press, 1985.

8. Azarina S.V., Gliklikh Yu.E. Differential Inclusions with Mean Derivatives. *Dynamic Systems and Applications*, 2007, vol. 16, no. 1, pp. 49–71.

9. Gliklikh Yu. E. *Global and Stochastic Analysis with Applications to Mathematical Physics.* London, Springer, 2011. DOI: 10.1007/978-0-85729-163-9

10. Chistyakov C. F., Shcheglova A. A. [*Selected Chapters of the Theory of Differential-Algebraic Systems*]. Novosibirsk, Nauka Publ., 2003. (in Russian)

11. Gantmakher F. R. [*The Theory of Matrices*]. Moscow, Fizmatlit Publ., 1967. (in Russian)

*Yuri E. Gliklikh, Doctor of Physics and Mathematics Science, Professor, Department of Algebra and Topological Analysis Methods, Voronezh State University, Voronezh, Russian Federation, yeg@math.vsu.ru*

*Evgeny Yu. Mashkov, Postgraduate student, Department of Mathematical Analysis and Applied Mathematics, Kursk State University, Kursk, Russian Federation, mashkovevgen@yandex.ru*

*Received* October 18, 2014