

# NUMERICAL SOLUTION OF THE BARENBLATT – ZHELTOV – KOCHINA EQUATION WITH ADDITIVE "WHITE NOISE" IN SPACES OF DIFFERENTIAL FORMS ON A TORUS

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The paper is devoted to the search for numerical solutions to the Cauchy problem for the linear stochastic Barenblatt – ZheltoV – Kochina equation in space of smooth differential forms on a torus. Based on the previously obtained results on the type of analytical solution to the stochastic version of the Barenblatt – ZheltoV – Kochina equation in spaces of smooth differential forms on smooth compact Riemannian manifolds without boundary, we choose several terms from the analytical solution in order to construct graphs of the numerical solution for various values of the coefficients and the inhomogeneous term. Since these equations are Sobolev type equations with a degenerate operator at the derivative, we can solve various initial-boundary value problems using the theory of degenerate analytic groups and semigroups of resolving operators. In the deterministic case, the solution is based on the phase subspace of the original space. In spaces of differential forms, we use the invariant form of the Laplacian, i.e. the Laplace – Beltrami operator. The phase space method is also used in non-deterministic case, but we use the Nelson – Gliklikh derivative due to the non-differentiability of "white noise" in the usual sense. In this paper, a two-dimensional torus plays the role of a smooth compact oriented Riemannian manifold without boundary. Numerical solutions are found using the Galerkin method and are presented for several fixed time points as graphs of the coefficients of differential forms obtained in Maple.

*Keywords:* Sobolev type equation; Nelson–Gliklikh derivative; Laplace–Beltrami operator.

## Introduction

In its initial statement [1], the Barenblatt – ZheltoV – Kochina equation

$$(\lambda - \Delta)u_t = \alpha\Delta u + f$$

simulates the pressure of a viscoelastic fluid filtered in a fractured porous medium. There exist other physical interpretations of this equation [2], [3]. Different types (including cases of different functional spaces) of initial boundary value problems for this equation were solved by reducing to abstract Sobolev type equations (with noninvertible operator at derivative) [4]

$$Lu = Mu + f. \quad (1)$$

In turn, the abstract Sobolev type equation with operators  $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{U})$  is reduced to the equivalent system on splitting spaces

$$\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1; \mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1. \quad (2)$$

by the phase space method proposed by G. A. Sviridyuk. Recently, in the Chelyabinsk scientific school of Sobolev type equations, various results were obtained for the

Barenblatt – Zheltov – Kochina equation, in particular, in spaces of differential forms defined on a Riemannian manifold without boundary [5], [6], and, for the stochastic version [15], [16] of this equation, in the space of differential  $K$ -"noises" [7], [8]. Note that in this case we use the invariant generalization of the Laplace operator in spaces of differential forms, i.e. the Laplace – Beltrami operator [9], and differentiability of, generally speaking, just continuous stochastic processes in the sense of Nelson – Gliklikh [10], which is extended to the Sobolev type equations [11].

The actual problem is to find numerical solutions to Sobolev type equations [17]. The aim of this paper is to obtain and analyze a numerical solution to the Cauchy problem

$$u(0) = u_0, \tag{3}$$

for equation (1) on a two-dimensional torus, which we use as an example of a Riemannian manifold. To this end, we use a numerical algorithm based on the Galerkin approximation method in order to carry out computational experiments for some set of initial parameters of the Barenblatt – Zheltov – Kochina equation in Maple. The results are presented in the form of sets of time-fixed values of the coefficients of differential forms.

The paper consists of Introduction, three sections, Conclusion, and References. Section 1 contains preliminary information and describes spaces of differential  $K$  - "noises" [7], [8]. Section 2 describes an example for the Barenblatt – Zheltov – Kochina equation on a torus. In Section 3, we use the Galerkin approximation method in order to construct a numerical solution to the Cauchy problem. References do not pretend to be complete, but only meet the preferences of the author.

## 1. Sobolev Type Equations in the Deterministic Case and Differential $K$ -"Noises" on Differential Form Spaces

Let  $\mathfrak{U}$  and  $\mathfrak{F}$  be Banach spaces, and the operators  $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$  (i.e., the operators are linear and continuous). Consider the  $L$ -resolvent set  $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$  and  $L$ -spectrum  $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$  of the operator  $M$ . If  $L$ -spectrum  $\sigma^L(M)$  of the operator  $M$  is bounded, then the operator  $M$  is called  $(L, \sigma)$ -bounded. If the operator  $M$  is  $(L, \sigma)$ -bounded, then there exist the projectors

$$P = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^L(M) d\mu \in \mathcal{L}(\mathfrak{F}).$$

Here  $R_{\mu}^L(M) = (\mu L - M)^{-1}L$  and  $L_{\mu}^L(M) = L(\mu L - M)^{-1}$  are the *right* and the *left  $L$ -resolvents* of the operator  $M$ , respectively, and the closed contour  $\gamma \subset \mathbb{C}$  bounds the domain containing  $\sigma^L(M)$ . Set  $\mathfrak{U}^0(\mathfrak{U}^1) = \ker P(\text{im}P)$ ,  $\mathfrak{F}^0(\mathfrak{F}^1) = \ker Q(\text{im}Q)$  and denote by  $L_k(M_k)$  the restriction of the operator  $L(M)$  on  $\mathfrak{U}^k$ ,  $k = 0, 1$ .

**Theorem 1.** [4] (Sviridyuk's splitting theorem)

*Let the operator  $M$  be  $(L, \sigma)$ -bounded. Then*

*(i) the operators  $L_k(M_k) \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$ ,  $k = 0, 1$ ;*

*(ii) there exist the operators  $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$  and  $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ .*

Construct the operators  $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ ,  $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$ . Under the conditions of the theorem,

$$(\mu L - M)^{-1} = - \sum_{k=0}^{\infty} \mu^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q$$

for all  $\mu \in \rho^L(M)$ . The operator  $M$  is called  $(L, p)$ -bounded,  $p \in \{0\} \cup \mathbb{N}$ , if  $\infty$  is a removable singular point (i.e.  $H \equiv \mathbb{O}, p = 0$ ) or a pole of the order  $p \in \mathbb{N}$  (i.e.  $H^p \neq \mathbb{O}, H^{p+1} \equiv \mathbb{O}$ ) of  $L$ -resolvents  $(\mu L - M)^{-1}$  of the operator  $M$ .

A vector function  $u \in C^\infty(\mathbb{R}; \mathfrak{U})$  is a *solution to equation* (1), if  $u$  satisfies the equation on  $\mathbb{R}$ . A solution  $u$  to equation (1) is a *solution to problem* (1), (3), if  $u$  satisfies condition (3).

**Theorem 2.** [4] *For the  $(L, p)$ -bounded operator  $M$ ,  $p \in \{0\} \cup \mathbb{N}$  and for any initial  $u_0 \in \mathfrak{P} \subset \mathfrak{U}^1$  and  $f \in C^\infty(\mathbb{R}; \mathfrak{U})$ , for problem (1),(3) there exist analytic resolving groups of operators of the form*

$$U^t = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) e^{\mu t} d\mu \in \mathcal{L}(\mathfrak{U}), \quad F^t = \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^L(M) e^{\mu t} d\mu \in \mathcal{L}(\mathfrak{F}), \quad (4)$$

and the solution has the form

$$u(t) = U^t u_0 + - \sum_{q=0}^p H^q M_0^{-1} \frac{d^q(I - Q)f}{dt^q}(t) + \int_0^t U^{t-s} Q f ds.$$

Let  $\Omega \equiv (\Omega, \mathcal{S}, \mathbf{P})$  be a complete probability space with the probability measure  $\mathbf{P}$  associated with the  $\sigma$ -algebra  $\mathcal{S}$  of subsets of the set  $\Omega$ . Denote by  $\mathbb{R}$  the set of real numbers endowed with the structure of  $\sigma$ -algebra. Then the mapping  $\chi : \mathcal{S} \rightarrow \mathbb{R}$  is called a *random variable*. The set of random variables  $\{\chi\}$  with zero mathematical expectation ( $\mathbf{E}\chi = 0$ ) and finite dispersion ( $\mathbf{D}\chi < +\infty$ ) forms the Hilbert space  $\mathbf{L}_2$  with the scalar product  $(\chi_1, \chi_2) = \mathbf{E}\chi_1\chi_2$  and the norm  $\|\chi\|_{\mathbf{L}_2}$ . Let  $\mathcal{S}_0$  be a  $\sigma$ -subalgebra of the  $\sigma$ -algebra  $\mathcal{S}$ . Construct the subspace  $\mathbf{L}_2^0 \subset \mathbf{L}_2$  of random variables measurable with respect to  $\mathcal{S}_0$ . Denote by  $\Pi : \mathbf{L}_2 \rightarrow \mathbf{L}_2^0$  the orthoprojector. Consider the random variable  $\chi \in \mathbf{L}_2$ , then  $\Pi\chi$  is called a *conditional expectation* and is denoted by  $\mathbf{E}(\chi|\mathcal{S}_0)$ .

For some interval  $\mathfrak{J} \subset \mathbb{R}$ , the measurable mapping  $\eta : \mathfrak{J} \times \mathcal{S} \rightarrow \mathbb{R}$  is called a *stochastic process*, and the random variable  $\eta(\cdot, \omega)$  is called a *section* of the stochastic process, and the function  $\eta(t, \cdot)$ ,  $t \in \mathfrak{J}$  is called a *trajectory* of the stochastic process. The stochastic process  $\eta = \eta(t, \cdot)$  is called *continuous*, if a.s. (almost sure), i.e. for a.a. (almost all)  $\omega \in \mathcal{S}$ , the trajectories  $\eta(t, \omega)$  are continuous functions. The set  $\{\eta = \eta(t, \omega)\}$  of all continuous stochastic processes with values in  $\mathbf{L}_2$  forms the Banach space  $\mathbf{CL}_2$  with the norm

$$\|\eta\|_{\mathbf{CL}_2} = \sup_{t \in \mathfrak{J}} (\mathbf{D}\eta(t, \omega))^{1/2}.$$

Fix an arbitrary stochastic process  $\eta \in \mathbf{CL}_2$ .

**Definition 1.** A random variable

$$\overset{\circ}{\eta}(\cdot, \omega) = \frac{1}{2} \left( \lim_{\Delta t \rightarrow 0+} \mathbf{E}_t^\eta \left( \frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t} \right) + \lim_{\Delta t \rightarrow 0+} \mathbf{E}_t^\eta \left( \frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\Delta t} \right) \right)$$

is called the *Nelson – Gliklikh derivative* of the stochastic process  $\eta$  at the point  $t \in \mathfrak{J}$ , if the limit exists in the sense of a uniform metric on  $\mathbb{R}$ .

Here  $\mathbf{E}_t^\eta = \mathbf{E}(\cdot | \mathcal{N}_t^\eta)$ , and  $\mathcal{N}_t^\eta \subset \mathcal{S}$  is a  $\sigma$ -algebra generated by the random variable  $\eta(t, \omega)$ .

If the Nelson – Gliklikh derivatives  $\overset{\circ}{\eta}(\cdot, \omega)$  of the stochastic process  $\eta(\cdot, \omega)$  there exist for a.a. points of the interval  $\mathfrak{J}$ , then there exists the Nelson – Gliklikh derivative  $\overset{\circ}{\eta}(\cdot, \omega)$  on the interval  $\mathfrak{J}$  (a.s. on  $\mathfrak{J}$ ). Denote the set of continuous stochastic processes with continuous Nelson – Gliklikh derivatives  $\overset{\circ}{\eta} \in \mathbf{C}\mathbf{L}_2(\mathfrak{J})$  by  $\mathbf{C}^1\mathbf{L}_2(\mathfrak{J})$ . By induction, we can define Banach spaces  $\mathbf{C}^l\mathbf{L}_2(\mathfrak{J})$ ,  $l \in \mathbb{N}$ , of the stochastic processes having continuous Nelson – Gliklikh derivatives on  $\mathfrak{J}$  up to the order  $l \in \mathbb{N}$  inclusively.

The norms of these spaces have the form

$$\|\eta\|_{\mathbf{C}^l\mathbf{L}_2} = \sup_{t \in \mathfrak{J}} \left( \sum_{k=0}^l \mathbf{D} \overset{\circ}{\eta}^k(t, \omega) \right)^{1/2},$$

where  $\overset{\circ}{\eta}^0 \equiv \eta$ . Since "white noise" belongs to all the spaces  $\mathbf{C}^l\mathbf{L}_2(\mathbb{R}_+)$ ,  $l \in \{0\} \cup \mathbb{N}$ , then these spaces are called the *spaces of "noises"*.

We also use the spaces of *random  $\mathbf{K}$ -variables*. Let space  $\mathfrak{H}$  be a separable Hilbert space with an orthonormal basis  $\{\varphi_k\}$ , a monotone sequence  $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$  ( $\sum_{k=1}^{\infty} \lambda_k^2 < +\infty$ ), and a sequence  $\{\xi_k\} = \xi_k(\omega) \subset \mathbf{L}_2$  of random variables with norm  $\|\xi_k\|_{\mathbf{L}_2} \leq C$  for one  $C \in \mathbb{R}_+$  and for all  $k \in \mathbb{N}$ . We define a  $\mathfrak{H}$ -valued random  $\mathbf{K}$ -variable  $\xi(\omega) = \sum_{k=1}^{\infty} \lambda_k \xi_k(\omega) \varphi_k$ . Complete the linear span of the set  $\{\lambda_k \xi_k \varphi_k\}$  with the norm

$$\|\eta\|_{\mathbf{H}\mathbf{K}\mathbf{L}_2}^2 = \left( \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D} \xi_k \right)^{1/2}$$

and call the result by a *space of ( $\mathfrak{H}$ -valued) random  $\mathbf{K}$ -variables*. Denote the space by  $\mathbf{H}\mathbf{K}\mathbf{L}_2$ . The obtained space  $\mathbf{H}\mathbf{K}\mathbf{L}_2$  is a Hilbert space and contains a random  $\mathbf{K}$ -variable  $\xi = \xi(\omega) \in \mathbf{H}\mathbf{K}\mathbf{L}_2$ . Similar, a Banach space of ( $\mathfrak{H}$ -valued)  $\mathbf{K}$ -*"noises"* is defined as a completion of the linear span of the set  $\{\lambda_k \xi_k \varphi_k\}$  with the norm

$$\|\eta\|_{\mathbf{C}^l\mathbf{H}\mathbf{K}\mathbf{L}_2}^2 = \sup_{t \in \mathfrak{J}} \left( \sum_{k=1}^{\infty} \lambda_k^2 \sum_{m=1}^l \mathbf{D} \overset{\circ}{\eta}_k^m \right)^{1/2},$$

where a sequence  $\{\eta_k\} \subset \mathbf{C}^l\mathbf{L}_2$ ,  $l \in \{0\} \cup \mathbb{N}$ . The vector  $\eta(t, \omega) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t, \omega) \varphi_k$  belongs to the space  $\mathbf{C}^l(\mathfrak{J}; \mathbf{H}\mathbf{K}\mathbf{L}_2)$ , if the elements of the sequence of vectors  $\{\eta_k\} \subset \mathbf{C}^l\mathbf{L}_2$  and all the Nelson – Gliklikh derivatives of these elements of the sequence up to the order  $l \in \{0\} \cup \mathbb{N}$  inclusively are uniformly bounded with respect to the existing norm  $\|\cdot\|_{\mathbf{C}^l\mathbf{L}_2}$ .

Further, in the spaces of  $\mathbf{K}$ -*"noises"*, we determine the coefficients of differential forms given on a manifold without boundary. Let  $M_d$  be a smooth connected oriented compact

Riemannian manifold without boundary of the class  $C^\infty$  and of dimension  $d$ . On the manifold  $M_d$ , consider the vector space  $E^q(M_d)$  of the  $q$ -forms

$$a(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_q} a_{i_1, i_2, \dots, i_q}(x_1, \dots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_q},$$

where  $a_{i_1, i_2, \dots, i_q}(x_1, \dots, x_n) \in C^\infty(M_d)$ ,  $q = \{0, 1, \dots, d\}$ . On the space  $E^q$ , define the scalar product

$$(a, b)_0 = \int_{M_d} a \wedge *b,$$

where  $*$  is the Hodge operator that establishes a linear isomorphism of  $q$ -forms and  $(p-q)$ -forms on  $M_d$ , but (possible) up to a sign by virtue of  $** = (-1)^{q(p-q)}$ . In the spaces  $E^q$ , the manifold  $(M_d)$  is defined by the formula  $\Delta = d\delta + \delta d$ , where  $d$  is the external differentiation operator,  $\delta = (-1)^{n(k+1)+1} * d*$  is the Laplace – Beltrami operator. Introduce the following two scalar products:

$$(a, b)_1 = (a, b)_0 + (\Delta a, b)_0, (a, b)_2 = (a, b)_1 + (\Delta a, \Delta b)_0.$$

Denote by  $\mathbf{H}_k^q$ ,  $k = 0, 1, 2$  a Hilbert space obtained by completion of  $E^q$  in the norm  $\|\cdot\|_k$ , induced by the scalar product  $(\cdot, \cdot)_k$ ,  $k = 0, 1, 2$ ,  $q \in \{0, 1, \dots, d\}$ . The obtained separable Hilbert space  $H_k^q$  has a basis of the eigenfunctions of the Laplace – Beltrami operator, be orthonormal with respect to the scalar product  $(\cdot, \cdot)_k$ ,  $k = 0, 1, 2$ .

*Spaces of random  $\mathbf{K}$ -variables* defined on the manifold  $M_d$ :  $\mathbf{U}_{\mathbf{K}}\mathbf{L}_2 = \mathbf{H}_{0\mathbf{K}}^q\mathbf{L}_2$  and  $\mathbf{F}_{\mathbf{K}}\mathbf{L}_2 = \mathbf{H}_{2\mathbf{K}}^q\mathbf{L}_2$ , where  $\mathbf{K} = \{\lambda_k\}$  is a monotone sequence of eigenvalues of the Green operator (the eigenvalues are inverted to the eigenvalues of the Laplace – Beltrami operator). The elements of these spaces are the vectors  $\alpha = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k$  and  $\beta = \sum_{k=1}^{\infty} \lambda_k \xi_k \psi_k$ , respectively, where  $\{\varphi_k\}$  and  $\{\psi_k\}$  are eigenvectors of the operator orthonormal with respect to  $(\cdot, \cdot)_0$  and  $(\cdot, \cdot)_2$ .

Consider the spaces of  $\mathbf{K}$ -"noises"  $\mathbf{C}^l(\mathcal{J}; \mathbf{H}_{0\mathbf{K}}^q\mathbf{L}_2)$  and  $\mathbf{C}^l(\mathcal{J}; \mathbf{H}_{2\mathbf{K}}^q\mathbf{L}_2)$ ,  $l \in \{0\} \cup \mathbb{N}$ ,  $q \in \{0, 1, \dots, d\}$ ,  $\mathcal{J} \subset \mathbb{R}$  is an interval.

## 2. Solution of the Stochastic Variant of the Barenblatt – Zheltov – Kochina Equation on the Torus

Consider the homogeneous Barenblatt – Zheltov – Kochina equation

$$(\lambda - \Delta)u_t = \alpha \Delta u \tag{5}$$

and the Cauchy problem

$$u(0) = u_0 \tag{6}$$

We can [5], [7] to reduce (5), (6) to problem (1), (3). To this end, we define the operators

$$L = (\lambda - \Delta) = (\lambda + d\delta + \delta d), M = \alpha \Delta = \alpha(d\delta + \delta d) \tag{7}$$

and consider the stochastic equation

$$L \overset{\circ}{\eta} = M\eta \tag{8}$$

with the condition

$$\eta(0) - \eta_0 = 0. \tag{9}$$

**Theorem 3.** [8] *Let the operator  $M$  be  $(L, p)$ -bounded,  $p \in \{0\} \cup \mathbb{N}$ . Then for any  $\eta_0 \in \mathbf{U}_{\mathbf{K}}\mathbf{L}_2$  a.s. there exists the unique solution  $\eta \in C^1(\mathcal{J}; \mathbf{U}_{\mathbf{K}}\mathbf{L}_2)$  to Cauchy problem (9) for equation (8) of the form  $u(t) = U^t u_0$ .*

Let us consider the application of the obtained results to the Barenblatt – Zhelтов – Kochina equation in the spaces of  $\mathbf{K}$ -"noises defined on a smooth compact Riemannian oriented manifold without boundary. As such a manifold, we consider the two-dimensional torus  $\mathbb{T}^2 = [0, 2\pi] \times [0, \pi]$  and consider the stochastic variant of the Barenblatt – Zhelтов – Kochina equation

$$(\lambda + \Delta)u_t = \alpha\Delta u. \tag{10}$$

Define the operators  $L$  and  $M$  by the formulas

$$L = (\lambda + d\delta + \delta d), M = \alpha(d\delta + \delta d). \tag{11}$$

For the two-dimensional torus with coordinates  $x_1, x_2$ , taking into account the general representation of the Laplace – Beltrami operator on the manifold  $M_d$  with the Riemannian metric  $g$

$$\Delta_{M_d} = \frac{1}{\sqrt{|g|}}(\partial_i g^{ij} \partial_j),$$

we obtain the Laplace – Beltrami operator in the form

$$\Delta_{\mathbb{T}^2} = (8\pi^2 \partial_{x_1}^2 - 4\pi \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2).$$

The eigenvalues  $\lambda_k$  are

$$\lambda_k = \min_{E \subset \mathbb{Z}^2, |E|=k+1} \max_{(c_1, c_2) \subset E} 4\pi^2 \left[ c_1^2 \left(1 + \frac{2\pi^2}{\pi^2}\right) - 2c_1 c_2 \frac{2\pi}{\pi^2} + \frac{c_2^2}{\pi^2} \right].$$

Therefore, we have a non-negative, non-decreasing, finite multiple, converging only to  $\infty$  sequence of eigenvalues  $\{\lambda_k\}$ , and the sequence of corresponding eigenfunctions  $\{\varphi_k\}$  forms the necessary orthonormal basis in  $\mathfrak{U}^1 = \mathbf{H}_{0\mathbf{K}\Delta}^{q\perp} \mathbf{L}_2$ , where  $\mathbf{H}_{0\mathbf{K}\Delta}^{q\perp} \mathbf{L}_2$  is a subspace  $\mathbf{H}_{0\mathbf{K}}^q \mathbf{L}_2$  obtained by the Hodge – Kodair splitting [9], be orthogonal to harmonic forms for  $\lambda \neq \lambda_k$ .

Since the dimension of the manifold is  $d = 2$ , we have solutions of two types. The first type takes place for 0-forms (and 2-forms isomorphic to them), and the second type takes place for 1-forms.

The relative spectrum of the Barenblatt – Zhelтов – Kochina equation has the form

$$\sigma^L(M) = \left\{ \mu_k = \frac{\alpha}{\lambda - \lambda_k} \right\}, \tag{12}$$

therefore, the operator  $M$  is  $(L, p)$ -bounded. For the inhomogeneous Barenblatt – Zhelтов – Kochina equation

$$L \overset{\circ}{\eta} = M\eta + f, \tag{13}$$

it is necessary to find the projections of the inhomogeneity  $f$  onto the corresponding subspaces by the existing projectors  $Q, (I - Q)$ , see Theorem 2.

### 3. The Numerical Solution of the Stochastic Barenblatt – Zheltov – Kochina Equation on the Torus

Numerical solution of the Cauchy problem for the Barenblatt – Zheltov – Kochina equation was carried out in Maple using the operators and variables implemented in Maple. The solution algorithm is presented by the block diagram given in Fig. 1. The numerical

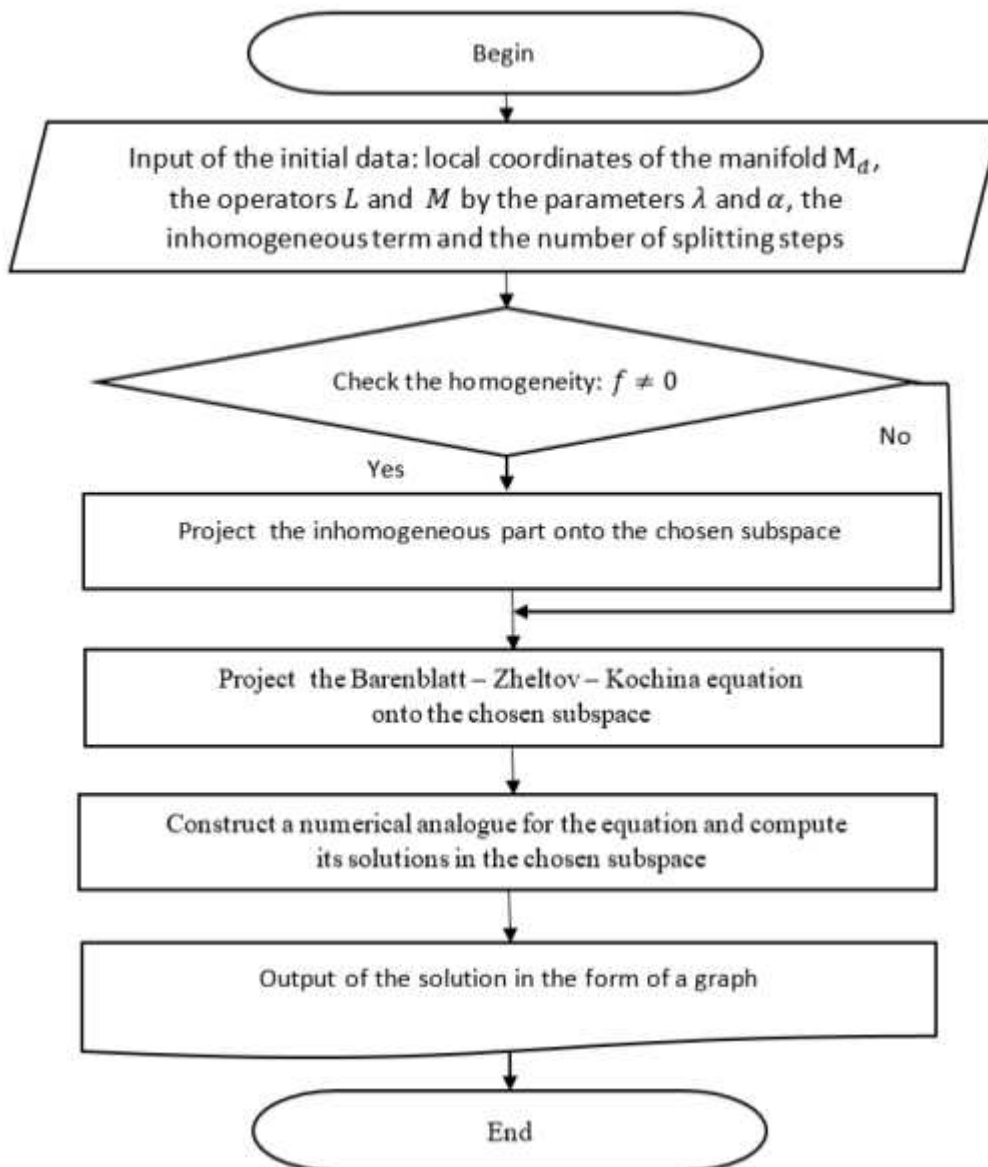
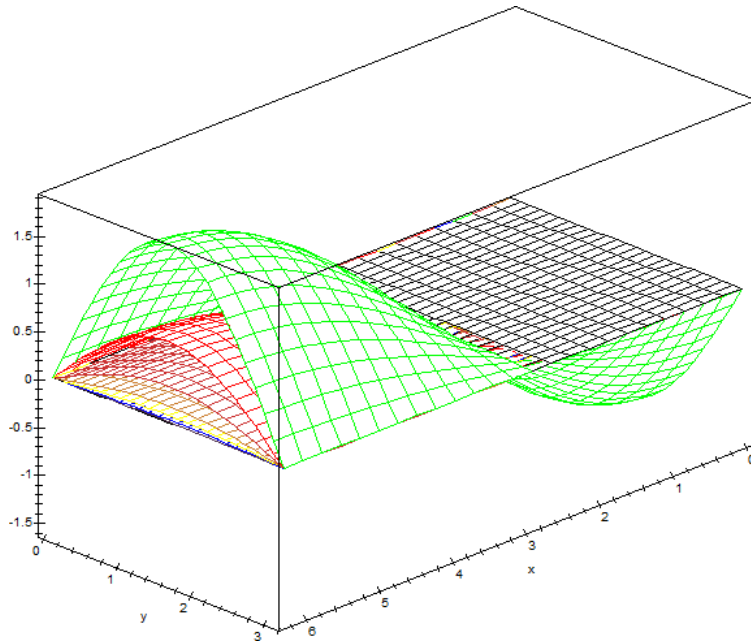


Fig. 1. Block diagram of the algorithm

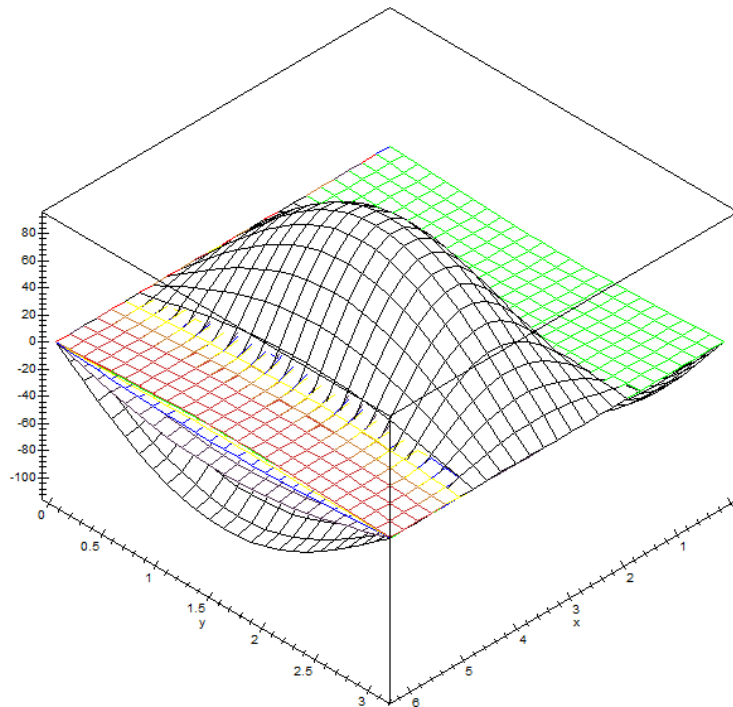
solution was obtained with the given step up to the time  $T = 4$ . An approximate solution is obtained by approximating the first three basic functions of the Galerkin method in the form of product of expansions in cosines of the trigonometric system in the variable  $x$  and in sines of the trigonometric system in the variable  $y$ . Random values are introduced using the randomize procedure. The initial condition is expanded in a series in the eigenvalues of the Green operator for the Laplace equation, which are inverse with respect to the

eigenvalues of the Laplace operator. The graphs show the solutions at the time instants  $t_k, k = 1, \dots, 8$ , by the corresponding colors: green, red, orange, gold, yellow, blue, violet, black (see Fig. 2).



**Fig. 2.** The graph of the solution with  $\lambda = 2, \alpha = 1, f = 0$

Fig. 2 shows the graphs of the solution in the first eight time cutoffs of the homogeneous Cauchy problem with  $\lambda = 2, \alpha = 1$ . Fig. 3 shows the graphs of the solution in the first eight time cutoffs of the homogeneous Cauchy problem with  $\lambda = -2.5, \alpha = 0.5$ .



**Fig. 3.** The graph of the solution with  $\lambda = -2.5, \alpha = 0.5, f = 0$



Further, we present a graph of the solution with  $\lambda = 7, \alpha = 0.5$  for a homogeneous equation and inhomogeneous equation with  $f = 5 \sin(t)$  in Fig. 4 and in Fig. 5, respectively.

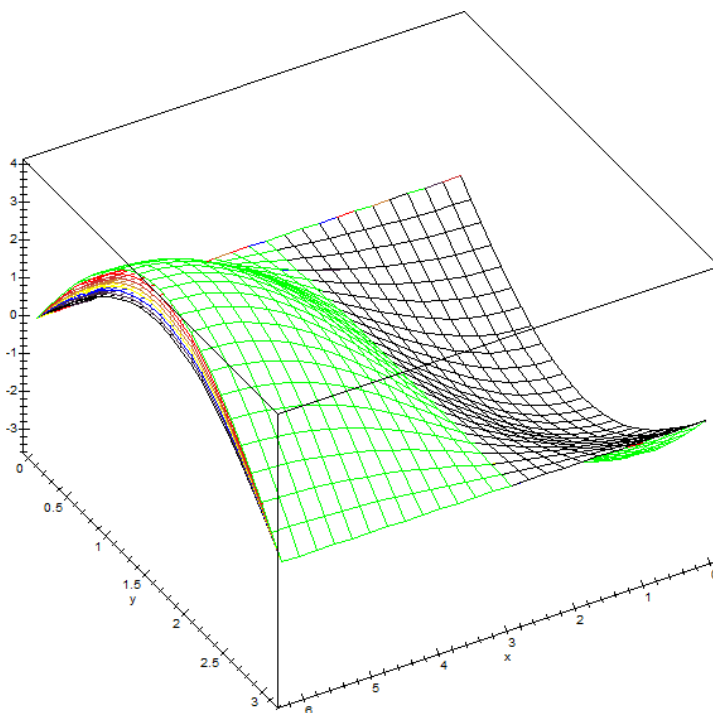


Fig. 4. The graph of the solution with  $\lambda = 7, \alpha = 0.5, f = 0$

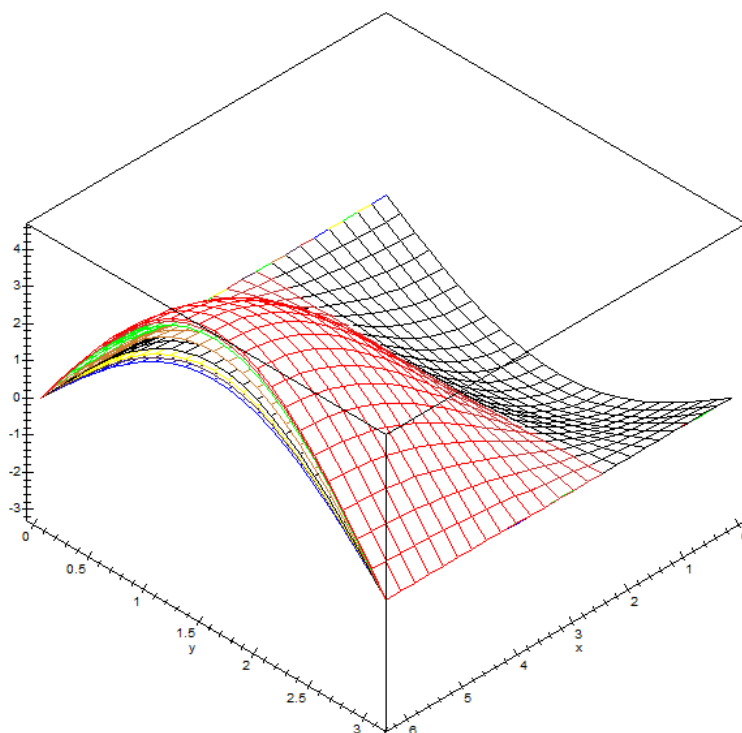


Fig. 5. The graph of the solution with  $\lambda = 7, \alpha = 0.5, f = 5 \sin(t)$

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## **ЧИСЛЕННОЕ РЕШЕНИЕ УРАВНЕНИЯ БАРЕНБЛАТТА – ЖЕЛТОВА – КОЧИНОЙ С АДДИТИВНЫМ "БЕЛЫМ ШУМОМ" В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ НА ТОРЕ**

*Д. Е. Шафранов*

Работа посвящена поиску численных решений задачи Коши для линейного стохастического уравнения Баренблатта – Желтова – Кочкиной в пространстве гладких дифференциальных форм на торе. Исходя из ранее полученных результатов по виду аналитического решения стохастического варианта уравнения Баренблатта – Желтова – Кочкиной в пространствах гладких дифференциальных форм на гладких компактных римановых многообразиях без края и выбирая из аналитического решения несколько слагаемых, строятся графики численного решения для различных значений коэффициентов и неоднородного члена. Это уравнения относятся к уравнениям соболевского типа с вырожденным оператором при производной, что и позволило решить различные начально-краевые задачи с помощью теории вырожденных аналитических групп и полугрупп разрешающих операторов. В детерминированном случае решение строится на фазовом подпространстве исходного пространства. В пространствах дифференциальных форм используется инвариантная форма лапласиана – оператор Лапласа – Бельтрами. Метод фазового пространства используется и в недетерминированном

случае, но, в силу недифференцируемости "белого шума" в обычном понимании, мы используем производную Нельсона – Гликлиха. Двумерный тор в нашей статье играет роль гладкого компактного ориентированного риманового многообразия без края. Численные решения находятся при помощи метода Галеркина и представлены для нескольких фиксированных моментов времени, как графики коэффициентов дифференциальных форм, полученных в системе Maple.

*Ключевые слова: уравнения соболевского типа; производная Нельсона – Гликлиха; оператор Лапласа – Бельтрами.*

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