

NUMERICAL SOLUTION OF OPTIMAL CONTROL PROBLEM FOR THE MODEL OF LINEAR WAVES IN PLASMA

*A. A. Zamyshlyeva*¹, zamyshliaevaaa@susu.ru,

*O. N. Tsyplenkova*¹, tcyplenkovaon@susu.ru

¹ South Ural State University, Chelyabinsk, Russian Federation

The article considers the optimal control in the mathematical model of ion-acoustic waves in plazma in an external magnetic field. For this model, on the basis of theoretical results, an algorithm for numerical solution was developed based on the modified Galerkin method and the Ritz method. The algorithm was implemented in Maple. Using the developed program, the result of a computational experiment is presented. The mathematical model considered in the article was first obtained by Yu.D. Pletner.

Keywords: Sobolev type equations of higher order; model of linear waves in plasma; optimal control; Galerkin method; numerical solution.

Introduction

In the cylinder $\Omega \times \mathbb{R}$, consider equation

$$(\lambda - \Delta)x_{ttt}(s, t) = (\Delta - \lambda')x_{tt}(s, t) + \alpha \frac{\partial^2 x(s, t)}{\partial s_3^2} + u(s, t), \quad s = (s_1, s_2, s_3) \in \Omega, t \in [0, \tau], \quad (1)$$

with the boundary condition

$$x(s, t) = 0, \quad (s, t) \in \partial\Omega \times [0, \tau]. \quad (2)$$

Here $\Omega = (0, a) \times (0, b) \times (0, c) \subset \mathbb{R}^3$. Equation (1) describes ion-acoustic waves in plasma in an external magnetic field [1], [2]. The function $x(s, t)$ represents the generalized potential of an electric field, $u(s, t)$ is an external influence, the constants in the equation relate the ionic frequency, the Debye radius, and the Langmoor frequency.

In suitable Hilbert spaces \mathfrak{X} , \mathfrak{Y} , \mathfrak{U} , problem (1), (2) can be reduced to the Sobolev type operator-differential equation

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \dots + B_0x + y + Cu, \quad (3)$$

where the operators $A, B_{n-1}, \dots, B_0 \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{U}; \mathfrak{Y})$, the functions $u : [0, \tau] \subset \mathbb{R}_+ \rightarrow \mathfrak{U}$, $y : [0, \tau] \subset \mathbb{R}_+ \rightarrow \mathfrak{Y}$ ($\tau < \infty$).

Consider the Showalter – Sidorov conditions

$$P(x^{(m)}(0) - x_m) = 0, \quad m = \overline{0, n-1}, \quad (4)$$

where P is a spectral projector in space \mathfrak{X} . Therefore the optimal control problem is to find a pair (\hat{x}, \hat{u}) , where \hat{x} is a solution to (3), (4), and the control $\hat{u} \in \mathfrak{U}_{ad}$ satisfies the relation

$$J(\hat{x}, \hat{u}) = \min_{(x, u) \in \mathfrak{X} \times \mathfrak{U}_{ad}} J(x, u). \quad (5)$$

Here $J(x, u)$ is the penalty functional, \mathfrak{U}_{ad} is a closed and convex set in the control space \mathfrak{U} .

The paper [3] proves the existence and uniqueness of a strong solution and optimal control of solutions to problem (3), (4). Based on these results, this article proposes an algorithm for finding optimal control of solutions to the Showalter – Sidorov problem for the model under consideration. The Showalter – Sidorov conditions are a generalization of the Cauchy conditions [4]. The paper [5] proposes a numerical algorithm for finding a solution to the optimal control problem for linear Sobolev type equation of the first order. Further it was developed in [6 – 11].

1. Numerical Algorithm

Based on the obtained theoretical results, we develop and implement in Maple 15.0 an algorithm for finding optimal control of solutions to the Showalter – Sidorov problem for the model of ion-acoustic waves in plasma in an external magnetic field. The program uses the phase space method [12, 13], the modified Galerkin method, and the Ritz method. Describe an algorithm of the numerical method. The block diagram of the algorithm of the numerical method is shown in Fig. 1.

Step 1. Enter the coefficients of the equation of ion-acoustic waves in plasma $\lambda, \lambda_1, \alpha$, the number of terms of the Galerkin sum N , the number of terms in the form of optimal control components M , initial values x_0, x_1, x_2, x_3 , the lengths of the segments a, b, c on which the solution is sought and the time interval $t \in [0, \tau]$.

Step 2. In the cycle, compile the required approximate solution $x(s_1, s_2, s_3, t)$ in the form of the Galerkin sum $\sum_{i,j,k=1}^N a_{ijk}(t) \sin \frac{\pi i s_1}{a} \sin \frac{\pi j s_2}{b} \sin \frac{\pi k s_3}{c}$ and compile the approximate control $u(s_1, s_2, s_3, t)$ in the form of the Galerkin sum $\sum_{i,j,k=1}^M u_{ijk}(t) \sin \frac{\pi i s_1}{a} \sin \frac{\pi j s_2}{b} \sin \frac{\pi k s_3}{c}$.

Step 3. Based on the entered data, generate a differential equation with respect to unknowns $a_{ijk}(t)$ and $u_{ijk}(t)$.

Step 4. In order to find the components $u_{ijk}(t)$ of optimal control, represent them in the form

$$u_{ijk}(t) = u(0) + \frac{u(\tau) - u(0)}{\tau} t + \sum_{l=1}^M c_{ijk}^l \sin \left(\frac{\pi l}{\tau} t \right) \quad (6)$$

based on the Ritz method.

Step 5. Multiply the generated differential equation by functions $\varphi_{ijk}, i = \overline{1, N}, j = \overline{1, N}, k = \overline{1, N}$. Obtain a system of differential equations for the coefficients of the Galerkin approximation of the solution $x(s_1, s_2, s_3, t)$.

Step 6. Begin the cycle by i, j, k from 1 to N .

Step 7. Check if λ belongs to the spectrum of the Laplace operator.

If step 7 is true, then go to step 8, otherwise go to step 12.

Step 8. Check if $\lambda = \lambda_{ijk}$.

If step 8 is true, then go to step 9, otherwise go to step 10.

Step 9. Solve an algebraic equation for $a_{ijk}(t)$.

Step 10. Using the Showalter – Sidorov conditions obtain two initial conditions for the components of the solution.

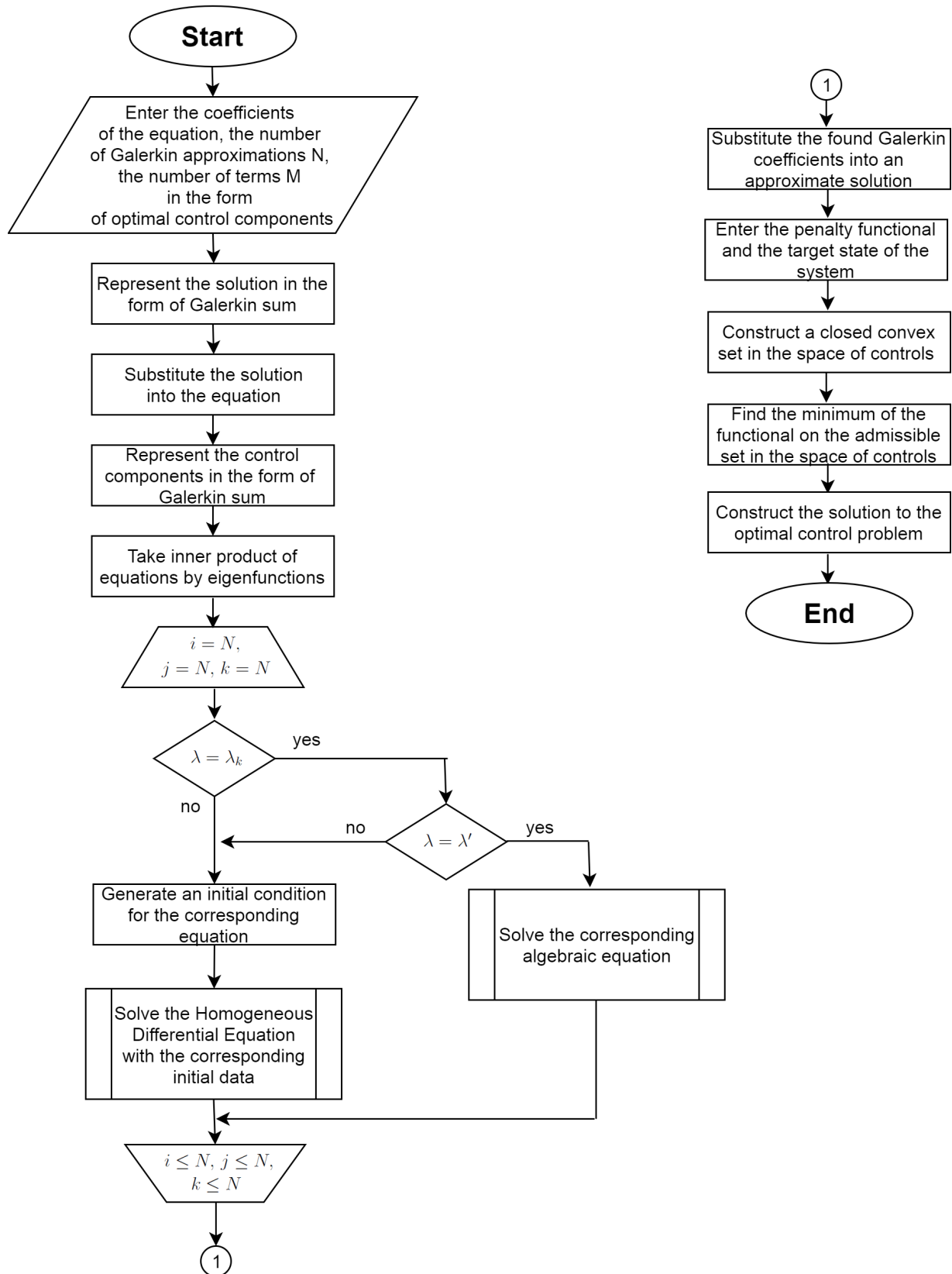


Fig. 1. The block diagram of the algorithm for the numerical solution of the problem

Step 11. Solve of the differential equation of the second order for the current number i, j, k .

Step 12. Using the Showalter – Sidorov conditions obtain four initial conditions for the components of the solution.

Step 13. Solve of the differential equation of the fourth order for the current number i, j, k .

Step 14. End of cycle by i, j, k .

Step 15. Assign the planned observation, construct the penalty functional J . Substitute the solution of the system of equations into J .

Step 16. Construct a closed and convex subset of admissible controls as a restriction on $u_{ijk}(t)$.

Step 17. Determine the minimum point of the functional J in terms of the coefficients in the control components.

Step 18. Substitute the coefficients into optimal control $\hat{u}(s, t)$ and solution $\hat{x}(s, t)$.

Step 19. Display the graphs of projections functions of the solution $\hat{x}(s_1, s_2, s_3, t)$ and the optimal control $\hat{u}(s_1, s_2, s_3, t)$.

The calculations were made in a computer math system Maple 15.0. The program allows

- to find Galerkin approximations of the solution to the equation of ion-acoustic waves in plasma;
- to take into account the Showalter – Sidorov conditions;
- to find a solution to the optimal control problem (a couple of functions (\hat{x}, \hat{u}));
- to find the minimum value of a given functional;
- to plot the obtained numerical solution.

At the first stage of the implementation of the algorithm, the generation of the equation of ion-sound waves in plasma, the form of the solution $x(s_1, s_2, s_3, t)$ and the optimal control $u(s_1, s_2, s_3, t)$ are carried out using the cycle **for()**. At the second stage of the algorithm, the inner product formula is used. At the third stage, the initial values are found from the Showalter – Sidorov conditions as the coefficients of the functions $x_0^0(s_1, s_2, s_3)$, $x_1^0(s_1, s_2, s_3)$, $x_2^0(s_1, s_2, s_3)$, $x_3^0(s_1, s_2, s_3)$ in their expansions in the eigenfunctions $\{\varphi_{ijk}\}$, the functional and components of optimal control are compiled. The resulting system of equations is analytically solved with the initial conditions by connecting the *PDEtools* package using the **Solve()** procedure. At the fourth stage of the program, on a given closed and convex subset of admissible controls, minimum point of the functional as a function of several variables is found by using the package *Optimization* namely the procedure **Minimize()**. Then the components of functions $x(s_1, s_2, s_3, t)$, $u(s_1, s_2, s_3, t)$ are displayed.

2. Computational Experiment

The following examples illustrate the operation of the program. For convenience and clarity, the below computational experiments were considered on interval $(0, \pi)$, since in this case the eigenfunctions of the Dirichlet problem for the Laplace operator have the simplest form. In the first example, we consider the solution to the problem in the case when $\lambda \notin \sigma(\Delta)$.

Example 1. It is required to find a solution to (3), (4) in $[0, \pi] \times [0, \pi] \times [0, \pi]$ for the given coefficients $\lambda = \frac{1}{25}, \lambda' = \frac{1}{9}, \alpha = 2, N = 2, M = 2, a = b = c = \pi$.

Consider the Showalter – Sidorov conditions

$$\langle \varphi_k, x(s_1, s_2, s_3, 0) - x_k(s_1, s_2, s_3) \rangle = 0, k = \overline{0, 3}, \quad (7)$$

where

$$\begin{aligned} x_0(s_1, s_2, s_3) &= \sin(s_1) \sin(s_2) \sin(s_3), \\ x_1(s_1, s_2, s_3) &= \frac{1}{2} \sin(s_1) \sin(s_2) \sin(s_3), \\ x_2(s_1, s_2, s_3) &= -\frac{1}{3} \sin(s_1) \sin(s_2) \sin(s_3), \\ x_3(s_1, s_2, s_3) &= \sin(s_1) \sin(s_2) \cos(s_3). \end{aligned}$$

The eigenfunctions are $\varphi_{ijk} = \sin(is_1) \sin(js_2) \sin(ks_3), s_1 \in [0, \pi], s_2 \in [0, \pi], s_3 \in [0, \pi]$, so

$$x(s_1, s_2, s_3, t) = \sum_{i,j,k=1}^2 a_{ijk}(t) \sin(is_1) \sin(js_2) \sin(ks_3), \quad (8)$$

$$u(s_1, s_2, s_3, t) = \sum_{i,j,k=1}^2 u_{ijk}(t) \sin(is_1) \sin(js_2) \sin(ks_3). \quad (9)$$

Substitute (8), (9) into equation (1) with condition (7) and obtain a system of differential equations consisting of 8 equations with 32 conditions derived from (7).

Construct the penalty functional and set the desired state of the system

$$\tilde{x}(t) = \frac{2}{\pi} \sin(s_1) \sin(s_2) \sin(s_3)$$

and the subset of admissible controls $\mathfrak{U}_{ad} = \{u \in \mathfrak{U} : \sum_{i,j,k=1}^2 u_{ijk}^2(t) \leq 1\}$ and get the solution to the problem.

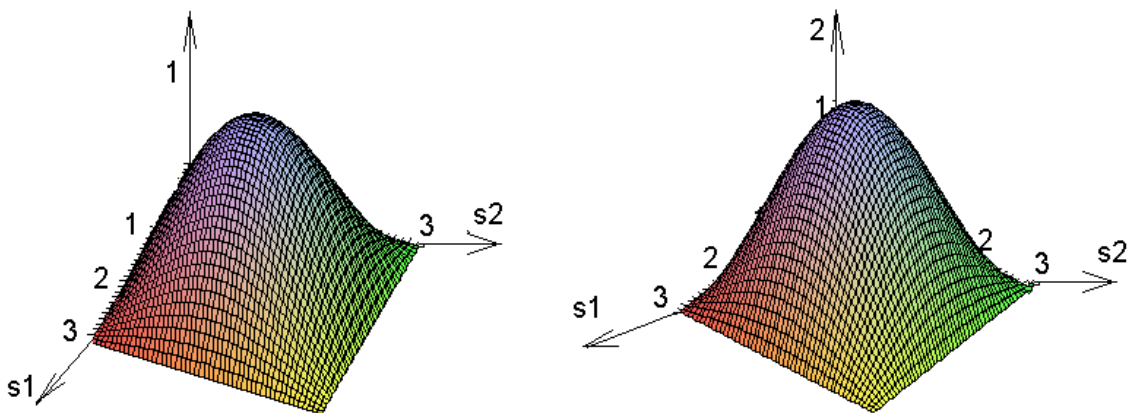


Fig. 2. The graphs for $\hat{x}(s_1, s_2, \frac{\pi}{3}, 0)$ and $\hat{x}(s_1, s_2, \frac{\pi}{3}, 1)$

The solution graph for fixed variables $s_3 = \frac{\pi}{3}$ and $t = 0$ or $t = 1$ is shown in Fig. 2, and the graph of the control function is given in Fig. 3.

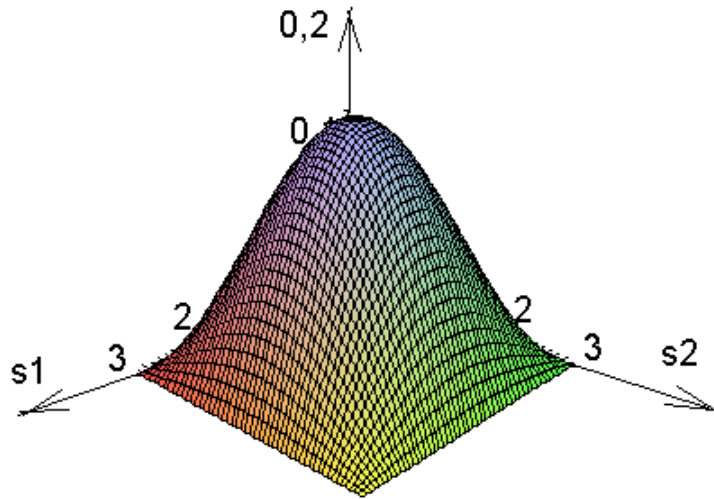


Fig. 3. The control function graph $\hat{u}(s_1, s_2, \frac{\pi}{3}, 0)$

The minimum value of the functional $J_{min} = 31,07$.

Example 2. It is required to find a solution to (3), (4) in $[0, \pi] \times [0, \pi] \times [0, \pi]$ for the given coefficients $\lambda = \lambda' = -12, \alpha = 2, N = 2, M = 2, a = b = c = \pi$.

Consider the Showalter – Sidorov conditions

$$\langle \varphi_k, x(s_1, s_2, s_3, 0) - x(s_1, s_2, s_3) \rangle = 0, k = \overline{0, 3},$$

where

$$\begin{aligned} x_0(s_1, s_2, s_3) &= -\sin(s_1) \sin(s_2) \sin(s_3), \\ x_1(s_1, s_2, s_3) &= \sin(s_1) \sin(s_2) \sin(s_3), \\ x_2(s_1, s_2, s_3) &= \frac{1}{10} \sin(s_1) \sin(s_2) \sin(s_3), \\ x_3(s_1, s_2, s_3) &= \frac{1}{2} \sin(s_1) \sin(s_2) \sin(s_3). \end{aligned}$$

The eigenfunctions are $\varphi_{ijk} = \sin(is_1) \sin(js_2) \sin(ks_3), s_1 \in [0, \pi], s_2 \in [0, \pi], s_3 \in [0, \pi]$, so

$$\begin{aligned} x(s_1, s_2, s_3, t) &= \sum_{i,j,k=1}^2 a_{ijk}(t) \sin(is_1) \sin(js_2) \sin(ks_3), \\ u(s_1, s_2, s_3, t) &= \sum_{i,j,k=1}^2 u_{ijk}(t) \sin(is_1) \sin(js_2) \sin(ks_3). \end{aligned}$$

Since $\lambda = \lambda' = \lambda_{222} = -12$, after substitution we get a system consisting of 7 differential equations and 1 algebraic equation

$$a_{222} = \frac{1}{8}c_{21} + \frac{1}{8}c_{22} + \frac{1}{8}c_{23}.$$

Construct the penalty functional and set the desired state of the system

$$\tilde{x}(t) = \sin(s_1) \sin(s_2) \cos(s_3)$$

and the subset of admissible controls

$$\mathcal{U}_{ad} = \{u \in \mathcal{U} : \sum_{i,j,k=1}^2 u_{ijk}^2(t) \leq 1\}$$

get the solution to the problem.

The solution graph for the fixed variables $s_3 = \frac{\pi}{2}$ and $t = 0$ or $t = 1$ is shown in Fig. 4, and the graph of the control function is given in Fig. 5.

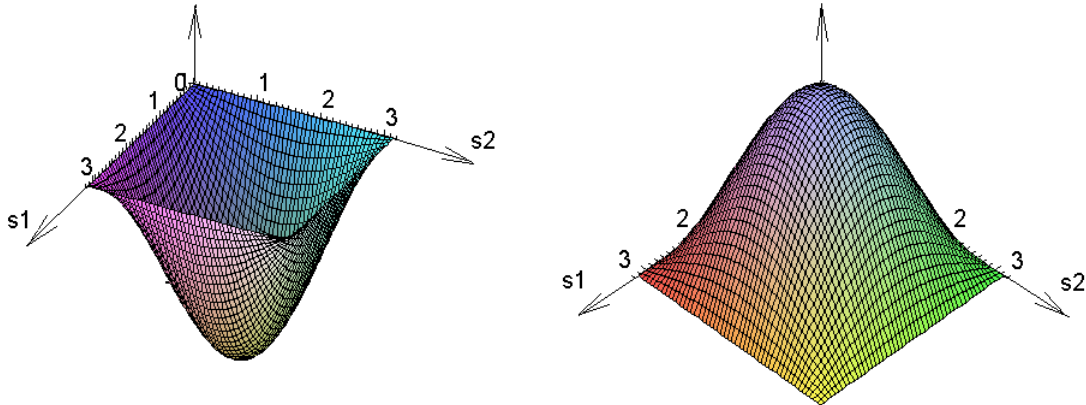


Fig. 4. The graphs for $\hat{x}(s_1, s_2, \frac{\pi}{2}, 0)$ and $\hat{x}(s_1, s_2, \frac{\pi}{2}, 2)$

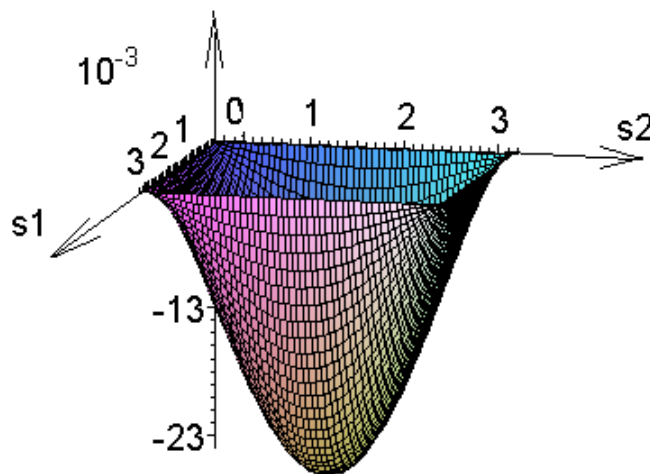


Fig. 5. The control function graph $\hat{u}(s_1, s_2, \frac{\pi}{2}, 0)$

The minimum value of the functional $J_{min} = 32,89$.

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Alyona A. Zamyshlyayeva, DSc (Math), Professor, Department of Applied Mathematics and Computer Science, South Ural State University (Chelyabinsk, Russian Federation), zamyshlyayeva@susu.ru

Olga N. Tsyplenkova, PhD (Math), Department of Equations of Mathematical Physics, South Ural State University, (Chelyabinsk, Russian Federation), tcyplenkovaon@susu.ru

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ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ МОДЕЛИ ЛИНЕЙНЫХ ВОЛН В ПЛАЗМЕ

А. А. Замышляева, О. Н. Цыпленкова

В статье рассмотрено оптимальное управление в математической модели ионно-звуковых волн во внешнем магнитном поле. Для данной модели на основе теоретических результатов был разработан алгоритм для численного решения задачи, основанный на модифицированном методе Галеркина и методе Рунге. Минимум функционала определяется как минимум по коэффициентам в компонентах управления. Алгоритм реализован в среде Maple. При помощи разработанной программы приведен результат вычислительного эксперимента. Математическая модель, рассмотренная в статье, впервые была получена Ю. Д. Плетнером.

Ключевые слова: уравнения соболевского типа высокого порядка; модель линейных волн в плазме; оптимальное управление; метод Галеркина; численное решение.

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Замышляева Алена Александровна, доктор физико-математических наук, профессор, кафедра прикладной математики и программирования, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), zamyshlyayeva@susu.ru.

Цыпенкова Ольга Николаевна, кандидат физико-математических наук, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), tsyplenkovaon@susu.ru.

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