MSC 57M99 DOI: 10.14529/jcem200103

# CLASSIFICATION OF PRIME KNOTS IN THE THICKENED SURFACE OF GENUS 2 HAVING DIAGRAMS WITH AT MOST 4 CROSSINGS

***A. A. Akimova,*** South Ural State University, Chelyabinsk, Russian Federation, akimovaaa@susu.ru

The goal of this paper is to tabulate all prime knots in the thickened surface of genus 2 having diagrams with at most 4 crossings. First, we introduce definition of prime knot in the thickened surface of genus 2. Second, we construct a table of prime knots. To this end, we use the table of prime knot projections in the surface of genus 2 to construct a preliminary set of diagrams. In order to remove duplicates and prove that all the rest knots are inequivalent, as well as to prove that all tabulated knots admit no destabilisations, we propose an invariant called the Kauffman bracket frame, which is a simplification of the generalized Kauffman bracket polynomial. The idea is to take into account only the order and values of coefficients and disregard the degrees of one of the variables. However, the proposed simplification is more compact, and at the same time is not weaker than the original generalized Kauffman bracket polynomial in the sense of, for example, tabulation of prime knots up to complexity 4 inclusively. Finally, we prove that each tabulated knot can not be represented as a connected sum under the hypothesis that the complexity of a connected sum is not less than the sum of complexities of the terms that form the sum.

*Keywords: prime knot; thickened surface of genus 2; classification; generalised Kauffman bracket polynomial; Kauffman bracket frame.*

## Introduction

During last 150 years, many researchers study the problem to construct complete tables of knots and links arranged with respect to some their numerical characteristics. Most of the constructed tables consider knots and links in the 3-dimensional sphere, see [5, 11, 16]. Recently, increasing interest in the theory of global knots (i.e., knots in arbitrary 3-manifolds) leads to tabulation of knots in manifolds different from the 3-dimensional sphere. Note tables of links in the projective space [6], knots in the solid torus [8], knots in the thickened Klein bottle [15], as well as prime knots in the lens spaces [9]. Note that, in the knot theory, recent tables include only the so-called prime objects, which can not be obtained by some known operations from already tabulated objects. Virtual knots and knots in the thickened surfaces have been of particular interest during the last 20 years. Therefore, some tables of such knots were also constructed. In particular, the works [10] and [17] present perfect tables of virtual knots arranged with respect to number of classical crossing and construct a list of some properties of each knot. However, these tables are constructed without taking into account primeness and such important property of a knot as the genus determined by the minimal genus of the thickened surface which can contain the given knot. The natural idea is to classify virtual knots taking into account both parameters, i.e. not only number of classical crossings, but also the genus of a knot, see the papers [1] and [2] for tables of prime knots and links in the thickened torus. In a sense, such tables can be considered as tables of prime virtual knots and links of genus 1.

In this paper, we classify prime knots in the thickened surface of genus 2. To this end, we use the result of the first step [4], i.e. a table of prime knot projections in the surface of genus 2 having at most 4 crossings, in order to construct a table of prime diagrams, i.e. table of prime knots. In order to remove duplicates and prove that all the rest knots are inequivalent, as well as to prove that all tabulated knots admit no destabilisations, we propose an invariant called the Kauffman bracket frame, which is a simplification of the generalized Kauffman bracket polynomial. The idea is to take into account only the order and values of coefficients and disregard the degrees of one of the variables. However, the proposed simplification is more compact, and at the same time is not weaker than the original generalized Kauffman bracket polynomial in the sense of, for example, tabulation of prime knots up to complexity 4 inclusively. Finally, we prove that each tabulated knot can not be represented as a connected sum under the hypothesis that the complexity of a connected sum is not less than the sum of complexities of the terms that form the sum. In addition, we construct one-to-one correspondence of the tabulated knots and some virtual knots of genus 2 given in [10].

The paper is organized as follows. Section 1 gives some required definitions. In Section 2, we describe some modifications of the Kauffman bracket and, in particular, introduce the so-called Kauffman bracket frame. Finally, Section 3 presents main ideas of the tabulation of prime knots in the thickened surface of genus 2 and proves the main theorem that there exist no more than 75 pairwise inequivalent prime such knots having diagrams with at most 4 crossings.

The author is grateful to V.V. Tarkaev and Ph.G. Korablev for useful discussions.

## 1. Definitions

A direct product of two copies of an 1-dimensional sphere $S^1$ is called a 2-dimensional torus $T = S^1 \times S^1$. Further, for shortness, we refer to a 2-dimensional torus $T$ as a torus $T$.

A surface $F^o$ with a hole is obtained from the original surface $F$ by removing the interior of a 2-dimensional disk $D^2$. Further, for shortness, we refer to a 2-dimensional disk $D^2$ as a disk $D^2$. Fig. 1(a) shows an example: a torus $T^o$ with a hole is obtained from a torus $T$ by removing the interior of a disk $D^2$. Hereinafter, we write $^o$ to show that a surface has one hole, $^{oo}$ to show that a surface has two holes.

By a 2-dimensional surface $T_2$ of genus 2 we mean a surface formed by gluing of two copies of a torus $T^o$ with a hole constructed by identifying their holes. Here each torus $T^o$ is called a handle of a 2-dimensional surface $T_2$ of genus 2. In other words, a 2-dimensional surface $T_2$ of genus 2 is a connected sum of two copies of a torus $T$. Further, for shortness, we refer to a 2-dimensional surface $T_2$ of genus 2 as a surface $T_2$.

A simple closed curve $C \subset T_2$ is said to be cut, if the complement $T_2 \backslash C$ consists of two components, and noncut, if the complement $T_2 \backslash C$ consists of the unique component.

Recall that the intersection number of two oriented curves is defined as a sum of signs of their intersection points. Here we say that an intersection point has the sign «+1» if the rotation from the direction of the first curve to the direction of the second curve is counterclockwise. Otherwise the intersection point has the sign «−1».

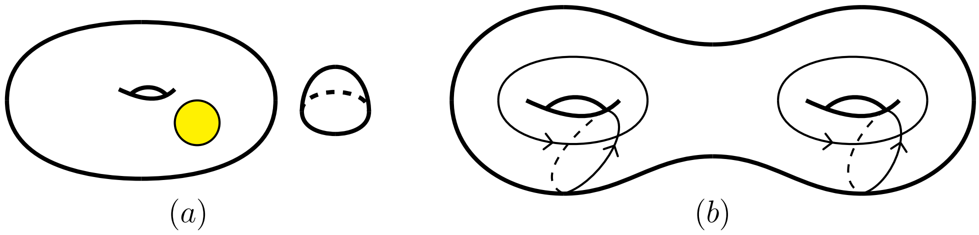


**Fig. 1**. (a) A torus $T^o$ with a hole and a disk $D^2$, (b) a surface $T_2$ endowed with oriented pairs «meridian-longitude» of its handles

For two fixed oriented pairs «meridian-longitude» of handles of the surface $T_2$ (throughout the paper, see Fig. 1(b)) and any oriented noncut curve $C \subset T_2$, we calculate the numbers $a$ and $c$ (respectively, $b$ and $d$) as intersection numbers of the curve $C$ and the corresponding meridian (respectively, longitude) of the surface $T_2$ and say that the curve $C$ has the homology class $(a, b, c, d)$. Since direction of orientation of $C$ is arbitrary, we consider homology classes $(a, b, c, d)$ and $(-a, -b, -c, -d)$ to be equal. Geometrically, $C$ goes $a$ times along the longitude and $b$ times along the meridian of the left handle of the surface $T_2$, and $c$ times along the longitude and $d$ times along the meridian of the right handle of the surface $T_2$. The signs of the numbers $a, b, c, d$ are positive, if the the direction of orientation of $C$ coincides with the direction of the corresponding longitude or meridian. Note that, in contrast to the case of the torus $T$, where the greatest common divisor $gcd(a, b) = 1$, there exist noncut curves such that $gcd(a, b) \neq 1$ or $gcd(c, d) \neq 1$. For example, we can consider the curve having homology class $(2, 1, 0, -2)$.

Consider a surface $T_2$ and an interval $I = [0, 1]$. By a thickened surface of genus 2 we mean a 3-dimensional manifold homeomorphic to the direct product $T_2 \times I$.

A smooth embedding of a simple closed curve in the interior of $T_2 \times I$ is called a knot in $T_2 \times I$ and denoted by $K \subset T_2 \times I$. Two knots $K_1 \subset T \times I$ and $K_2 \subset T \times I$ are said to be equivalent, if there exists a homeomorphism of $T \times I$ onto itself that takes $K_1$ to $K_2$.

As in the classical case, knots in $T_2 \times I$ can be given by their diagrams. A diagram $D \subset T_2$ of a knot $K \subset T_2 \times I$ is defined by analogy with the diagram of the classical knot except that the knot is projected into the surface $T_2$ instead of the plane.

Let $D \subset T_2$ be a knot diagram. We say that a noncut curve $C \subset T_2$ is a cancellation curve for the pair $(D, T_2)$, if an intersection of $C$ and $D$ is empty. In order to perform destabilization of the surface $T_2$, it is enough to cut $T_2$ along a cancellation curve $C$ and glue each obtained component of the boundary by a disk $D^2$. Fig. 2 shows a torus $T$ as a result of destabilization of the surface $T_2$ of genus 2.

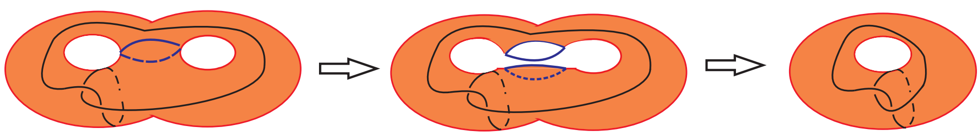

**Fig. 2**. Destabilization of the surface $T_2$ of genus 2

We consider the following types of knots in $T_2 \times I$ (compare with the types of knot projections in the surface $T_2$ given in [4]).

A knot $K \subset T_2 \times I$ is said to be essential, if any diagram of $K$ admits no destabilization. In other words, any annulus $\mathcal{A}$, which is isotopic to $C \times I \subset T_2 \times I$, where $C \subset T_2$ is a noncut curve, has nonempty intersection with $K$.

A knot $K \subset T_2 \times I$ is said to be trivial, if $K$ has a diagram without crossings.

A knot $K \subset T_2 \times I$ is said to be composite, if at least one of the following three conditions $(a)$, $(b)$, or $(c)$ holds.

$(a)$ $K$ is a connected sum of nontrivial knots $K_1 \subset T_2 \times I$ and $K_2 \subset S^3$, which is defined by analogy with the classical connected sum of two classical knots in the 3-dimensional sphere $S^3$. Namely, in $T_2 \times I$ (respectively, $S^3$), remove an open 3-dimensional ball $B^3$ that intersects $K_1$ (respectively, $K_2$) by an unknotted arc. As a result, both knots $K_1$ and $K_2$ are transformed to knotted arcs. Then, glue the resulting thickened surfaces (a thickened surface $T_2^o \times I$ with a hole and a thickened disk $D^2 \times I$) into one new $T_2 \times I$ by a homeomorphism that identifies the obtained spherical boundaries such that endpoints of different knotted arcs are glued pairwise.

$(b)$ $K$ is a circular connected sum of nontrivial knots $K_1 \subset T_2 \times I$ and $K_2 \subset T \times I$, which is defined by analogy with the circular connected sum introduced by S.V. Matveev in [14]. Namely, consider $K_1$ and $K_2$ to be such that there exist annuli $\mathcal{A}_1 \subset T_2 \times I$ and $\mathcal{A}_2 \subset T \times I$, where $\mathcal{A}_i$ is isotopic to $C_i \times I$ (here $C_i$ is a noncut curve in $T_2$ or $T$, respectively), and $\mathcal{A}_i$ intersects $K_i$ transversally at exactly one point, $i = 1, 2$. Cut $T_2 \times I$ along $\mathcal{A}_1$ and $T \times I$ along $\mathcal{A}_2$. As a result, both knots $K_1$ and $K_2$ are transformed to knotted arcs. Then, glue the resulting thickened surfaces (a thickened torus $T^{oo} \times I$ with two holes and a thickened annulus $\mathcal{A} \times I$) into one new $T_2 \times I$ by a homeomorphism that identifies the obtained annular boundaries such that endpoints of different knotted arcs are glued pairwise.

$(c)$ $K$ is a connected sum of two nontrivial knots $K_1 \subset T \times I$ and $K_2 \subset T \times I$ defined as follows. In each $T \times I$, remove a thickened disk $D^2 \times I$, where $D^2 \subset T$, that intersects a knot by an unknotted arc. As a result, both knots $K_1$ and $K_2$ are transformed to knotted arcs. Then, glue the resulting thickened surfaces (two copies of a thickened torus $T^o \times I$ with a hole) into one new $T_2 \times I$ by a homeomorphism that identifies the obtained annular boundaries such that endpoints of different knotted arcs are glued pairwise.

A knot $K \subset T_2 \times I$ is said to be prime, if $K$ is essential and noncomposite.

The natural idea is to tabulate only prime knots. Indeed, nonessential knots correspond to knots that can be found in already existing tables of knots in the 3-dimensional sphere $S^3$ [11], [16], [5], thickened annulus $\mathcal{A} \times I$ (solid torus) [8], or thickened torus $T \times I$ [1]. In their turn, composite knots correspond to knots, which can be constructed using already known knots by connected sums described in types $(a) - (c)$.

## 2. Generalizations of the Kauffman Bracket

Recall a definition of the surface bracket polynomial $\langle \cdot \rangle$ proposed in [7], which is a generalisation of the Kauffman bracket [13] (see also [12] for the original version called the Jones polynomial). Such an invariant is enough to prove that the main part of tabulated knots are inequivalent, see Subsection 3.2. Moreover, in Subsection 3.3, we consider this invariant as a tool to prove impossibility to realize any of tabulated knots as a knot in the thickened surface having smaller genus.

Let $F = T_1 \# T_2 \# \ldots \# T_m$ be a surface constructed as a connected sum of $m$ copies of a torus $T$, and $D$ be a diagram of a knot $K$ in the thickened surface $F \times I$. For a fixed representation of a knot diagram $D$, we refer to the surface-knot pair, $(F, K)$, to indicate a specific choice of the surface $F$ and embedding of the knot $K$.

Endow each angle of each crossing of $D$ with a marker $A$ or $B$ according to the rule given in the center of Fig. 3(a). Each state $s$ of the diagram $D$ is defined by a combination of ways to smooth each crossing of $D$ such as to join together either two angles endowed with a marker $A$, or two angles endowed with a marker $B$, see Fig. 3(a) on the left and right, respectively. By a surface-state pair, $(F, s)$, we mean a collection of disjoint simple closed curves in the surface $F$ obtained as a result of smoothing according to the state $s$.

Obviously, if the diagram $D$ has $n$ crossings, then there exist exactly $2^n$ states of $D$. Therefore, we obtain $2^n$ surface-state pairs, denoted $\{(F, s_1), ..., (F, s_{2^n})\}$, by assignment of smoothing type. We denote the collection of all surface-state pairs as $(F, S)$.

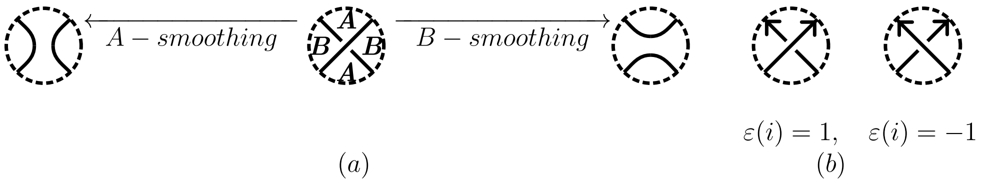


**Fig. 3**. (a) $A$- and $B$-smoothings of a crossing, (b) rules to define the sign $\varepsilon(i)$ of the $i$-th crossing

The formula of the surface bracket polynomial $\langle \cdot \rangle$ [7] is as follows:

$$\langle (F, K) \rangle = \sum_{i=1}^{2^n} a^{\alpha(s_i) - \beta(s_i)} (-a^2 - a^{-2})^{\gamma(s_i)} [s_i]. \tag{1}$$

Here $\alpha(s_i)$ and $\beta(s_i)$ are the numbers of markers $A$ and $B$ in the given state $s_i$, while $\gamma(s_i)$ is the number of cut curves in the surface obtained by smoothing according to the state $s_i$, and $[s_i]$ represents a formal sum of the disjoint noncut curves in the surface-state pair $(F, s_i)$. Note that $[s_i]$ may be regarded either as a formal sum of homology classes in the surface $F$ or as a sum of isotopy classes in the surface $F$ mod orientation preserving homeomorphisms of $F$. The sum is taken over all $2^n$ surface-state pairs $(F, s_i) \in (F, S)$.

The surface bracket polynomial $\langle \cdot \rangle$ [7] is invariant under the Reidemeister moves $\Omega_2$ and $\Omega_3$. As usual, in order to obtain a complete invariant (i.e., under the Reidemeister move $\Omega_1$ as well), it is enough to use the writhe $w(D)$ as follows:

$$\widetilde{\mathscr{X}}(D) = (-a)^{-3w(D)} \sum_{i=1}^{2^n} a^{\alpha(s_i) - \beta(s_i)} (-a^2 - a^{-2})^{\gamma(s_i)} [s_i], \tag{2}$$

where $w(D)$ is the writhe of an oriented knot diagram $D$ with $n$ crossings, i.e. the sum $w(D) = \sum_{i=1}^{n} \varepsilon(i)$ over all crossings of $D$, where $\varepsilon(i)$ is the sign of the $i$-th crossing of $D$ defined by the rules given in Fig. 3(b).

Associate each formal sum $[s_i]$ of the disjoint noncut curves in the surface-state pair $(F, s_i)$ with a product of the corresponding variables $y_j$, which take values in homology classes $(a_j, b_j, c_j, d_j)$ of the noncut curves that form $[s_i]$, see Table 1. Then, (2) takes the form

$$\widetilde{\mathscr{X}}(D) = (-a)^{-3w(D)} \sum_{i=1}^{2^n} a^{\alpha(s_i)-\beta(s_i)} (-a^2 - a^{-2})^{\gamma(s_i)} \prod_j y_j^{\delta_j(s_i)}, \quad (3)$$

where $\delta_j(s_i)$ is the number of noncut curves having the homology class $(a_j, b_j, c_j, d_j)$ associated with the variable $y_j$, see Table 1.

**Table 1**

Values of the variables $y_j$ in terms of homology classes $(a_j, b_j, c_j, d_j)$ of noncut curves in $T_2$

| | | | |
|---|---|---|---|
| $y_1 = (0,0,0,1)$ | $y_{18} = (0,1,-2,0)$ | $y_{35} = (-1,1,0,1)$ | $y_{52} = (1,2,0,-2)$ |
| $y_2 = (0,0,1,0)$ | $y_{19} = (0,-2,0,1)$ | $y_{36} = (1,1,1,0)$ | $y_{53} = (1,-2,2,0)$ |
| $y_3 = (0,0,1,1)$ | $y_{20} = (0,2,1,-1)$ | $y_{37} = (1,1,-1,0)$ | $y_{54} = (2,1,0,0)$ |
| $y_4 = (0,0,1,-1)$ | $y_{21} = (1,0,0,0)$ | $y_{38} = (1,-1,1,0)$ | $y_{55} = (2,-1,0,0)$ |
| $y_5 = (0,0,1,2)$ | $y_{22} = (1,0,0,1)$ | $y_{39} = (-1,1,1,0)$ | $y_{56} = (2,1,0,-1)$ |
| $y_6 = (0,0,2,-1)$ | $y_{23} = (1,0,0,-1)$ | $y_{40} = (1,1,1,1)$ | $y_{57} = (2,1,0,-2)$ |
| $y_7 = (0,1,0,0)$ | $y_{24} = (1,0,0,-2)$ | $y_{41} = (1,1,1,-1)$ | $y_{58} = (-2,1,1,0)$ |
| $y_8 = (0,1,0,-1)$ | $y_{25} = (1,0,1,0)$ | $y_{42} = (1,1,-1,1)$ | $y_{59} = (2,-1,1,0)$ |
| $y_9 = (0,1,1,0)$ | $y_{26} = (1,0,-1,0)$ | $y_{43} = (1,-1,1,1)$ | $y_{60} = (2,1,1,1)$ |
| $y_{10} = (0,1,-1,0)$ | $y_{27} = (1,0,1,1)$ | $y_{44} = (1,1,-1,-1)$ | $y_{61} = (2,1,1,-1)$ |
| $y_{11} = (0,1,1,1)$ | $y_{28} = (1,0,1,-1)$ | $y_{45} = (1,-1,1,-1)$ | $y_{62} = (2,-1,1,1)$ |
| $y_{12} = (0,1,1,-1)$ | $y_{29} = (1,0,2,0)$ | $y_{46} = (1,1,1,-2)$ | $y_{63} = (-2,1,1,1)$ |
| $y_{13} = (0,1,-1,1)$ | $y_{30} = (1,1,0,0)$ | $y_{47} = (1,-1,1,2)$ | $y_{64} = (2,-1,1,-1)$ |
| $y_{14} = (0,-1,1,1)$ | $y_{31} = (1,-1,0,0)$ | $y_{48} = (1,-1,1,-2)$ | $y_{65} = (2,-1,1,2)$ |
| $y_{15} = (0,1,1,2)$ | $y_{32} = (1,1,0,1)$ | $y_{49} = (1,1,2,-1)$ | $y_{66} = (2,-1,1,-2)$ |
| $y_{16} = (0,1,1,-2)$ | $y_{33} = (1,1,0,-1)$ | $y_{50} = (1,2,0,0)$ | $y_{67} = (2,-2,1,1)$ |
| $y_{17} = (0,-1,1,2)$ | $y_{34} = (1,-1,0,1)$ | $y_{51} = (1,2,0,-1)$ | $y_{68} = (2,2,2,-1)$ |

For shortness, we propose to use the following simplification of (3).

Let us order terms of (3) in nondecreasing order of the powers of the variable $a$ and collect terms having the same power of the variable $a$, i.e. represent (3) as $\sum_m P_m a^m$, where $P_m$ is a polynomial in the variables $y_j$. Then, we associate the polynomial (3) with an ordered set of nonzero polynomials $P_m$ in the variables $y_j$, which is called the Kauffman bracket frame $\mathfrak{F}(\cdot)$. For example, $\widetilde{\mathscr{X}}(D) = -2a^{-8}y_{12} - a^{-12}y_{62} - a^{-8}y_3y_7 - a^{-6}y_4y_7$ is associated with $\mathfrak{F}(D) = (-y_{62}, -2y_{12} - y_3y_7, -y_4y_7)$.

We say that the Kauffman bracket frames $\mathfrak{F}(D_1)$ and $\mathfrak{F}(D_2)$ are inverted to each other, if the elements of $\mathfrak{F}(D_1)$ are the corresponding elements of $\mathfrak{F}(D_2)$, where the polynomials $P_m$ are taken in reverse order. This transformation of the Kauffman bracket frames is called an inversion. For example, $\mathfrak{F}(D_1) = (-y_{62}, -2y_{12} - y_3y_7, -y_4y_7)$ is inverted to $\mathfrak{F}(D_2) = (-y_4y_7, -2y_{12} - y_3y_7, -y_{62})$.

By virtue of the arguments similar to those given in [3] for another simplification of the generalised Kauffman bracket polynomial called the Kauffman bracket skeleton, the following lemma is true.

**Lemma 1.** *The Kauffman bracket frame $\mathfrak{F}(\cdot)$ considered up to inversion, multiplication by $-1$, and changes of variables $y_j$ associated with changes of the corresponding homology classes of noncut curves in the surface $T_2$ generated by orientation preserving homeomorphisms of $T_2$ is an invariant of knots in the thickened surface $T_2 \times I$ of genus 2.*

## 3. Table of Prime Knots

**Theorem 1.** *In the thickened surface of genus 2, there exist no more than 75 pairwise inequivalent prime knots having diagrams with at most 4 crossings, see Figs. 4 – 5.*

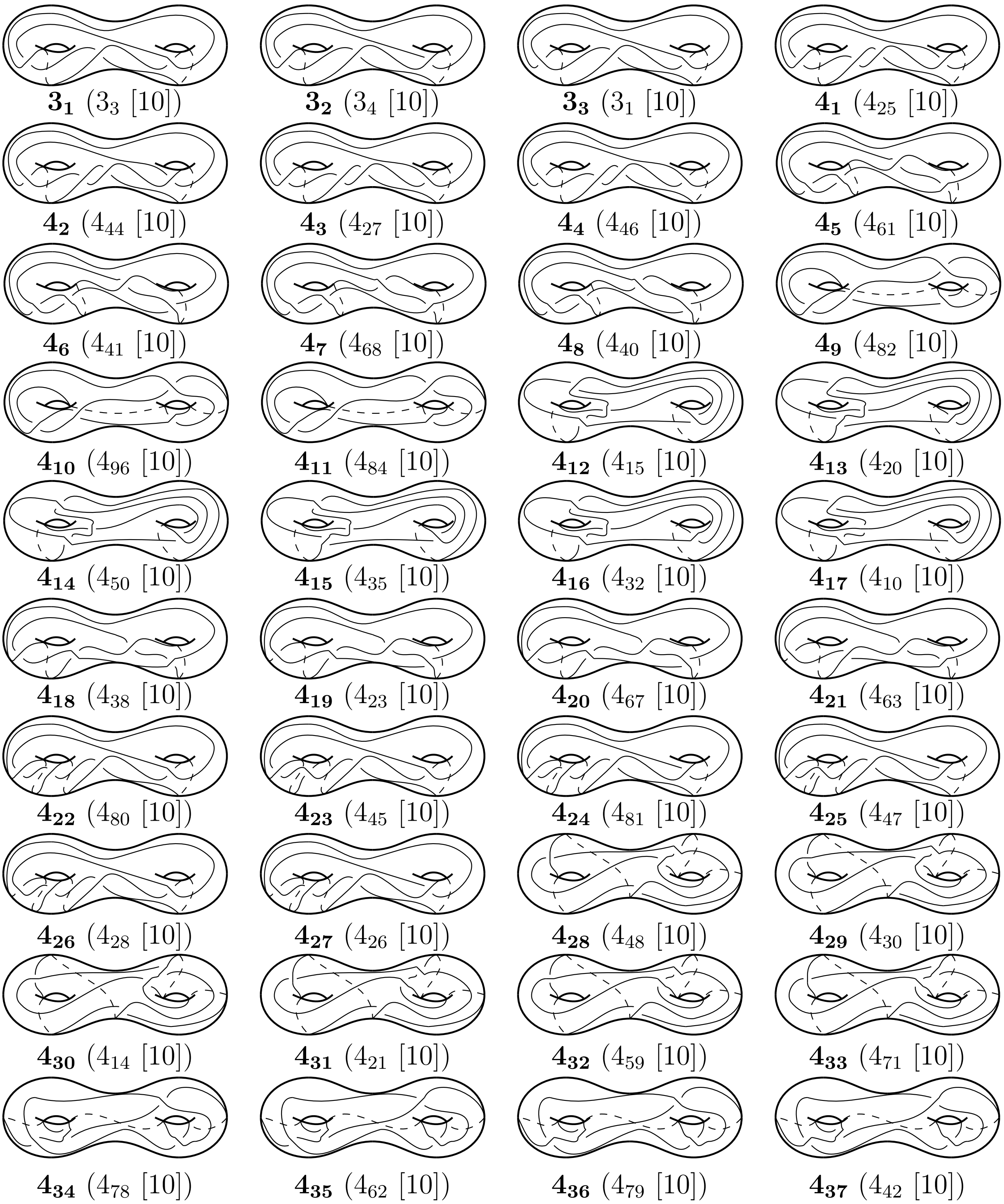


**Fig. 4**. Diagrams $3_1$ – $3_3$, $4_1$ – $4_{37}$ of prime knots in the thickened surface $T_2 \times I$

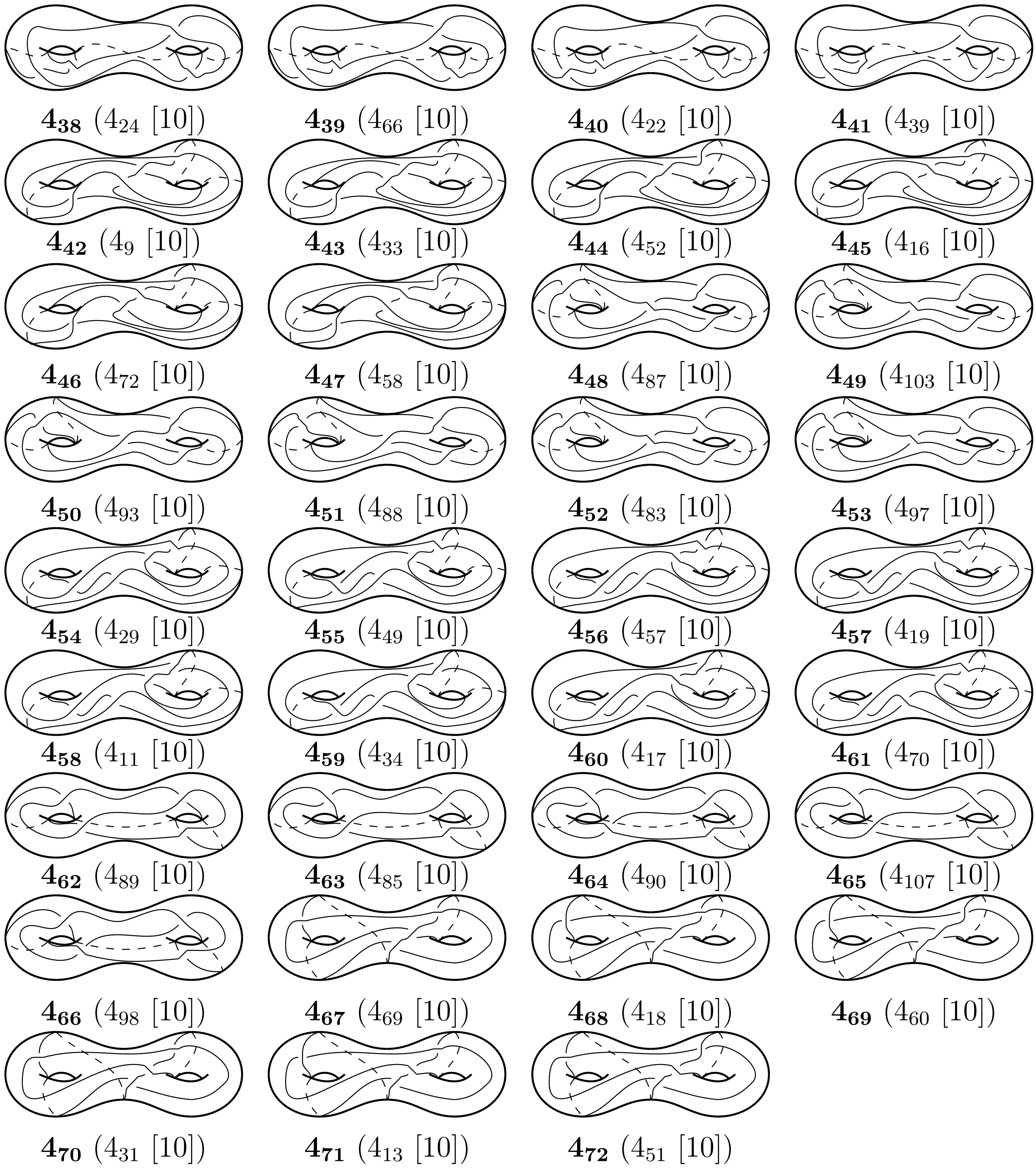


**Fig. 5**. Diagrams $4_{38}$ – $4_{72}$ of prime knots in the thickened surface $T_2 \times I$

Theorem 1 is proved by three steps described in Subsections 3.1 – 3.3. In addition, we construct one-to-one correspondence of the tabulated knots and some virtual knots of genus 2 given in [10]. Namely, in Figs. 4 – 5, each tabulated knot is provided with both a number in our table and a number in the table presented in [10].

### 3.1. Construction of a Preliminary List of Diagrams on Prime Projections

Let us convert each projection constructed in [4] to the set of corresponding diagrams. To this end, enumerate all possible ways to consider each crossing of a projection to be either an over- or undercrossing of a diagram. Obviously, there are $2^n$ diagrams on each projection with $n$ crossings. Therefore, direct construction by tabulated 14 projections [4] leads to $2^3 + 13 \cdot 2^4 = 216$ diagrams. However, we can significantly reduce this procedure by the following ideas [1].

First, the simultaneous switching of all crossings convert any diagram to the equivalent one. Therefore, we can fix the type of a crossing of each projection and, consequently, to halve the set of diagrams on the projection.

Second, if a diagram is based on a projection having biangle face, then both crossings of the face have the same type. Otherwise, we can reduce the number of crossings by the second Reidemeister move $\Omega_2$.

### 3.2. Formation of Equivalence Classes of the Constructed Diagrams

In order to compare the obtained diagrams, we use the software «Wolfram Mathematica» to calculate the Kauffman bracket frames for all diagrams constructed in Subsection 3.1. Therefore, we find few tens of groups formed by diagrams having the same Kauffman bracket frames. Each group includes from 2 to 6 diagrams, and there exist groups having diagrams on inequivalent projections $4_4$ and $4_5$ [4]. Then, by hand, we construct sequences of Reidemeister moves, homeomorphisms of the surface of genus 2 onto itself and simultaneous switching of all crossings in order to show that diagrams having the same Kauffman bracket frame are equivalent. Unexpectedly and fortunately, with the exception of two pairs ($4_{49}$, $4_{50}$) and ($4_{65}$, $4_{66}$) discussed below, the list of the Kauffman bracket frames presented below is enough to prove that all tabulated knots are pairwise inequivalent.

$\mathfrak{F}(3_1) = (-y_{62}, -y_{11} - y_{14} - y_{25}y_{34}, -2y_{12} - y_3y_7, -y_4y_7)$,
$\mathfrak{F}(3_2) = (-y_{25}y_{34}, -2y_{12} - y_{62}, -y_{11} - y_{14} - y_4y_7, -y_3y_7)$,
$\mathfrak{F}(3_3) = (-y_{14}, -y_{12} - y_{62} - y_3y_7, -y_{11} - y_{25}y_{34} - y_4y_7, -y_{12})$,
$\mathfrak{F}(4_1) = (y_{25}y_{26}, 2y_{12} + y_{62}, y_{11} + y_{14} + y_4y_7, -y_{12} + y_3y_7, -y_{11} - y_4y_7, -y_3y_7)$,
$\mathfrak{F}(4_2) = (y_{12}, y_{14} + y_{25}y_{26} + y_4y_7, y_{12} + y_{62} + y_3y_7, y_{11} - y_4y_7, -y_{12} - y_3y_7, -y_{11})$,
$\mathfrak{F}(4_3) = (y_{62}, y_{11} + y_{14} + y_{25}y_{26}, 2y_{12} + y_3y_7, -y_{11} + y_4y_7, -y_{12} - y_3y_7, -y_4y_7)$,
$\mathfrak{F}(4_4) = (y_{14}, y_{12} + y_{62} + y_3y_7, y_{11} + y_{25}y_{26} + y_4y_7, y_{12} - y_3y_7, -y_{11} - y_4y_7, -y_{12})$,
$\mathfrak{F}(4_5) = (-y_{12}, -y_{14} - y_4y_7, y_{12} - y_{13}, y_{14} + y_{22}y_{36} + y_4y_7, y_{12} + y_{13} + y_{60}, y_{61})$,
$\mathfrak{F}(4_6) = (-y_4y_7, -y_{12} - y_{13}, -y_{14} + y_4y_7, 2y_{12} + y_{13}, y_{14} + y_{22}y_{36} + y_{61}, y_{60})$,
$\mathfrak{F}(4_7) = (-y_{14}, -y_{12} - y_{13}, y_{14} - y_4y_7, y_{12} + y_{13} + y_{60}, y_{22}y_{36} + y_{61} + y_4y_7, y_{12})$,
$\mathfrak{F}(4_8) = (-y_{13}, -y_{14} - y_4y_7, -y_{12} + y_{13}, y_{14} + y_{61} + y_4y_7, 2y_{12} + y_{60}, y_{22}y_{36})$,
$\mathfrak{F}(4_9) = (y_1y_{56}, y_{54} + y_{57} + y_1y_8, y_{18} + 2y_7, y_{10}y_2 - y_1y_8, -2y_7, -y_{10}y_2)$,
$\mathfrak{F}(4_{10}) = (y_{54}, y_{18} + y_1y_{56} + y_7, y_{10}y_2 + y_{57} + y_1y_8, -y_{10}y_2 - y_1y_8, -y_7)$,
$\mathfrak{F}(4_{11}) = (y_{18}, y_{10}y_2 + y_{54} + y_{57}, y_1y_{56} + 2y_7, -y_{10}y_2 + y_1y_8, -2y_7, -y_1y_8)$,
$\mathfrak{F}(4_{12}) = (y_{30}y_4 + y_{44}, y_{10}y_{23} + y_3y_{30} + y_{41} + y_{42}, y_{44} + y_{45} + y_{40}, y_{43})$,
$\mathfrak{F}(4_{13}) = (y_{10}y_{23}, 2y_{44} + y_{45}, y_3y_{30} + y_{41} + y_{42} + y_{43}, y_{30}y_4 + y_{40})$,
$\mathfrak{F}(4_{14}) = (y_{30}y_4, y_3y_{30} + y_{41} + y_{42}, 2y_{44} + y_{45} + y_{40}, y_{10}y_{23} + y_{43})$,
$\mathfrak{F}(4_{15}) = (y_{42}, y_{30}y_4 + y_{44} + y_{45}, y_{10}y_{23} + y_3y_{30} + y_{41} + y_{43}, y_{44} + y_{40})$,
$\mathfrak{F}(4_{16}) = (y_3y_{30}, y_{30}y_4 + 2y_{44} + y_{40}, y_{10}y_{23} + 2y_{41} + y_{42} + y_{43}, 2y_{45}, -y_{41}, -y_{45})$,
$\mathfrak{F}(4_{17}) = (-y_{41}, -y_{45}, 2y_{41}, y_{30}y_4 + y_{44} + 2y_{45} + y_{40}, y_{10}y_{23} + y_3y_{30} + y_{42} + y_{43}, y_{44})$,
$\mathfrak{F}(4_{18}) = (y_{11} + y_{25}y_{34}, 2y_{12} + y_{62} + y_3y_7, y_{14} + y_{64} + y_4y_7, y_{13})$,
$\mathfrak{F}(4_{19}) = (-y_4y_7, -y_{12}, 2y_4y_7, 3y_{12} + y_{13} + y_3y_7, y_{11} + y_{14} + y_{25}y_{34} + y_{64}, y_{62})$,

$\mathfrak{F}(4_{20}) = (-y_{12}, -y_4y_7, 2y_{12}, y_{11} + y_{25}y_{34} + y_{64} + 2y_4y_7, y_{12} + y_{13} + y_{62} + y_3y_7, y_{14}),$
$\mathfrak{F}(4_{21}) = (y_{12}, y_{14} + y_{25}y_{34} + y_4y_7, y_{12} + y_{13} + y_{62} + y_3y_7, y_{11} + y_{64}),$
$\mathfrak{F}(4_{22}) = (y_{31}y_{43}, 2y_3 + y_{67}, y_{20} + y_4 + y_{11}y_7 + y_{14}y_7, -y_3 + 2y_{12}y_7 + y_3y_7^2, -y_4 + y_4y_7^2),$
$\mathfrak{F}(4_{23}) = (y_3, y_{20} + y_{31}y_{43} + y_{11}y_7 + y_{14}y_7, y_3 + y_{67} + 2y_{12}y_7 + y_3y_7^2, y_4 + y_4y_7^2, -y_3, -y_4),$
$\mathfrak{F}(4_{24}) = (y_{67}, y_{20} + y_4 + y_{31}y_{43}, 2y_3 + 2y_{12}y_7, -y_4 + y_{11}y_7 + y_{14}y_7 + y_4y_7^2, -y_3 + y_3y_7^2),$
$\mathfrak{F}(4_{25}) = (y_{20}, y_3 + y_{67} + 2y_{12}y_7, y_4 + y_{31}y_{43} + y_{11}y_7 + y_{14}y_7 + y_4y_7^2, y_3 + y_3y_7^2, -y_4, -y_3),$
$\mathfrak{F}(4_{26}) = (y_{31}y_{43} + y_{14}y_7, y_3 + y_{67} + y_{12}y_7 + y_3y_7^2, y_{20} + y_{11}y_7 + y_4y_7^2, y_{12}y_7),$
$\mathfrak{F}(4_{27}) = (y_{67} + y_{12}y_7, y_{20} + y_{31}y_{43} + y_{14}y_7 + y_4y_7^2, y_3 + y_{12}y_7 + y_3y_7^2, y_{11}y_7),$
$\mathfrak{F}(4_{28}) = (y_{59}, y_{10} + y_{23}y_{43} + y_2y_{55}, y_{17} + y_{23}y_{45} + y_{58} + y_{59} + y_2y_7, y_{10} + y_{66} + y_9, y_{16}),$
$\mathfrak{F}(4_{29}) = (y_{17} + y_{59} + y_2y_7, 2y_{10} + y_{23}y_{43} + y_2y_{55} + y_9, y_{16} + y_{23}y_{45} + y_{58} + y_{59}, y_{66}),$
$\mathfrak{F}(4_{30}) = (-y_{10}, 3y_{10}, y_{16} + y_{17} + y_{23}y_{45} + y_{58} + y_{59} + y_2y_7, y_{23}y_{43} + y_2y_{55} + y_{66} + y_9, y_{59}),$
$\mathfrak{F}(4_{31}) = (y_{23}y_{43}, y_{17} + y_{23}y_{45} + 2y_{59}, 2y_{10} + y_2y_{55} + y_{66} + y_9, y_{16} + y_{58} + y_2y_7),$
$\mathfrak{F}(4_{32}) = (y_{17}, y_{10} + y_{23}y_{43} + y_9, y_{16} + y_{23}y_{45} + 2y_{59} + y_2y_7, y_{10} + y_2y_{55} + y_{66}, y_{58}),$
$\mathfrak{F}(4_{33}) = (y_{23}y_{45}, y_{10} + y_{23}y_{43} + y_{66}, y_{16} + y_{17} + y_{58} + 2y_{59}, y_{10} + y_2y_{55} + y_9, y_2y_7),$
$\mathfrak{F}(4_{34}) = (y_1y_{22}, y_{21} + y_1^2y_{21} + y_1y_{51}, 2y_1y_{23} + y_2y_{25} + y_{50} + y_{52}, y_{21} + y_{24} + y_{29}, y_{53}),$
$\mathfrak{F}(4_{35}) = (y_{21}, y_1y_{22} + y_1y_{23} + y_2y_{25} + y_{52}, y_{21} + y_1^2y_{21} + y_{24} + y_{29} + y_1y_{51}, y_1y_{23} + y_{50} + y_{53}),$
$\mathfrak{F}(4_{36}) = (-y_{21} + y_1^2y_{21}, y_1y_{22} + 2y_1y_{23}, 3y_{21} + y_{24} + y_1y_{51}, y_2y_{25} + y_{50} + y_{52} + y_{53}, y_{29}),$
$\mathfrak{F}(4_{37}) = (y_1y_{23}, y_{21} + y_1^2y_{21} + y_{24}, y_1y_{22} + y_1y_{23} + y_2y_{25} + y_{52} + y_{53}, y_{21} + y_{29} + y_1y_{51}, y_{50}),$
$\mathfrak{F}(4_{38}) = (y_1y_{51}, y_1y_{22} + y_1y_{23} + y_{50} + y_{52}, 2y_{21} + y_1^2y_{21} + y_{24} + y_{29}, y_1y_{23} + y_2y_{25} + y_{53}),$
$\mathfrak{F}(4_{39}) = (y_{52}, y_{21} + y_{24} + y_{29} + y_1y_{51}, y_1y_{22} + 2y_1y_{23} + y_2y_{25} + y_{50} + y_{53}, 2y_{21} + y_1^2y_{21}, -y_{21}),$
$\mathfrak{F}(4_{40}) = (y_1y_{23}, y_1^2y_{21} + y_{24} + y_1y_{51}, y_1y_{22} + y_1y_{23} + y_{50} + y_{52} + y_{53}, 2y_{21} + y_{29}, y_2y_{25}),$
$\mathfrak{F}(4_{41}) = (y_1y_{22} + y_2y_{25} + y_{50}, 2y_{21} + y_1^2y_{21} + y_{29} + y_1y_{51}, 2y_1y_{23} + y_{52} + y_{53}, y_{24}),$
$\mathfrak{F}(4_{42}) = (y_9, y_{17} + y_{27}y_{34} + y_{59} + y_2y_7, y_{10} + y_{28}y_{34} + y_{23}y_{43} + y_{65} + y_5y_7 + y_9, y_{15} + y_{16} + y_{17} + y_{59}, y_9),$
$\mathfrak{F}(4_{43}) = (y_{17}, y_{23}y_{43} + y_{65} + y_5y_7 + y_9, y_{15} + y_{17} + y_{27}y_{34} + 2y_{59} + y_2y_7, y_{10} + y_{28}y_{34} + 2y_9, y_{16}),$
$\mathfrak{F}(4_{44}) = (y_{23}y_{43}, 2y_{17} + 2y_{59}, y_{10} + y_{28}y_{34} + y_{65} + y_5y_7 + 2y_9, y_{15} + y_{16} + y_{27}y_{34} + y_2y_7, y_9),$
$\mathfrak{F}(4_{45}) = (y_{27}y_{34}, y_{28}y_{34} + y_{65} + 2y_9, y_{15} + y_{16} + y_{17} + 2y_{59} + y_2y_7, y_{10} + y_{23}y_{43} + y_5y_7 + y_9, y_{17}),$
$\mathfrak{F}(4_{46}) = (y_{65}, y_{15} + y_{17} + y_{27}y_{34} + y_{59}, y_{28}y_{34} + y_{23}y_{43} + y_5y_7 + 3y_9, y_{16} + y_{17} + y_{59} + y_2y_7, y_{10}),$
$\mathfrak{F}(4_{47}) = (y_{28}y_{34}, y_{16} + y_{27}y_{34} + 2y_{59}, y_{10} + y_{23}y_{43} + y_{65} + 3y_9, y_{15} + 2y_{17} + y_2y_7, y_5y_7),$
$\mathfrak{F}(4_{48}) = (y_{30}y_{49}, y_{30}y_{32} + y_{30}y_{33} + y_6 + y_{68}, 2y_1 + y_{19} + y_1y_{30}^2, 2y_7y_8, -2y_1 + y_1y_7^2),$
$\mathfrak{F}(4_{49}) = (y_6, y_1 + y_{19} + y_{30}y_{49}, y_{30}y_{32} + y_{30}y_{33} + y_{68} + 2y_7y_8, y_1y_{30}^2 + y_1y_7^2, -y_1),$
$\mathfrak{F}(4_{50}) = (y_{68}, y_1 + y_{19} + y_{30}y_{49}, y_{30}y_{32} + y_{30}y_{33} + y_6 + 2y_7y_8, y_1y_{30}^2 + y_1y_7^2, -y_1),$
$\mathfrak{F}(4_{51}) = (y_{19}, y_6 + y_{68} + 2y_7y_8, 2y_1 + y_{30}y_{49} + y_1y_7^2, y_{30}y_{32} + y_{30}y_{33}, -2y_1 + y_1y_{30}^2),$
$\mathfrak{F}(4_{52}) = (y_{30}y_{32}, y_1y_{30}^2 + y_{30}y_{49}, y_{30}y_{33} + y_6 + y_{68} + y_7y_8, y_{19} + y_1y_7^2, y_7y_8),$
$\mathfrak{F}(4_{53}) = (y_{30}y_{32} + y_6 + y_7y_8, y_{19} + y_1y_{30}^2 + y_{30}y_{49} + y_1y_7^2, y_{30}y_{33} + y_{68} + y_7y_8),$
$\mathfrak{F}(4_{54}) = (y_{36}, y_{21}y_{30} + y_{38} + y_{39} + y_{27}y_8, 2y_2y_{31} + y_{36} + y_{37} + y_{47} + y_{28}y_8, 2y_{38} + y_{39} + y_{46}, y_{48}),$
$\mathfrak{F}(4_{55}) = (y_{39}, 2y_2y_{31} + y_{36} + y_{47}, y_{21}y_{30} + 3y_{38} + y_{39} + y_{27}y_8, y_{36} + y_{37} + y_{48} + y_{28}y_8, y_{46}),$
$\mathfrak{F}(4_{56}) = (y_{27}y_8, 2y_{36} + y_{47} + y_{28}y_8, y_{21}y_{30} + 3y_{38} + y_{39} + y_{46}, 2y_2y_{31} + y_{37} + y_{48}, y_{39}),$
$\mathfrak{F}(4_{57}) = (y_{47}, 2y_{38} + y_{39} + y_{27}y_8, 2y_2y_{31} + 2y_{36} + y_{48} + y_{28}y_8, y_{21}y_{30} + y_{38} + y_{39} + y_{46}, y_{37}),$
$\mathfrak{F}(4_{58}) = (y_{38}, y_2y_{31} + y_{36} + y_{37} + y_{28}y_8, y_{21}y_{30} + y_{38} + 2y_{39} + y_{46} + y_{27}y_8, y_2y_{31} + y_{36} + y_{47} + y_{48}, y_{38}),$
$\mathfrak{F}(4_{59}) = (y_2y_{31}, 2y_{38} + 2y_{39}, y_2y_{31} + y_{36} + y_{37} + y_{47} + y_{48} + y_{28}y_8, y_{21}y_{30} + y_{38} + y_{46} + y_{27}y_8, y_{36}),$
$\mathfrak{F}(4_{60}) = (y_{28}y_8, 2y_{38} + y_{46} + y_{27}y_8, y_2y_{31} + 2y_{36} + y_{37} + y_{47} + y_{48}, y_{21}y_{30} + y_{38} + 2y_{39}, y_2y_{31}),$
$\mathfrak{F}(4_{61}) = (y_{21}y_{30}, y_2y_{31} + 2y_{36} + y_{37}, 2y_{38} + 2y_{39} + y_{46} + y_{27}y_8, y_2y_{31} + y_{47} + y_{48} + y_{28}y_8, y_{38}),$
$\mathfrak{F}(4_{62}) = (-1, -5 + y_{14}^2 + y_3^2 + y_{30}^2 + y_{33}^2, y_1y_{30}y_{33} + y_{14}y_3y_7, -4 + 2y_1^2 + 2y_7^2, y_1y_7y_8),$
$\mathfrak{F}(4_{63}) = (-3 + y_{14}^2, y_{14}y_3y_7, -5 + y_1^2 + y_3^2 + y_{30}^2 + y_{33}^2 + 2y_7^2, y_1y_{30}y_{33} + y_1y_7y_8, -2 + y_1^2),$

$\mathfrak{F}(4_{64}) = (y_{14}y_3y_7, -5 + y_{14}^2 + y_3^2 + 2y_7^2, y_1y_7y_8, -5 + 2y_1^2 + y_{30}^2 + y_{33}^2, y_1y_{30}y_{33}),$
$\mathfrak{F}(4_{65}) = (-1, -4 + y_1^2 + y_{14}^2 + y_{30}^2 + y_7^2, y_1y_{30}y_{33} + y_{14}y_3y_7 + y_1y_7y_8, -4 + y_1^2 + y_3^2 + y_{33}^2 + y_7^2, -1),$
$\mathfrak{F}(4_{66}) = (-1, -4 + y_1^2 + y_3^2 + y_{30}^2 + y_7^2, y_1y_{30}y_{33} + y_{14}y_3y_7 + y_1y_7y_8, -4 + y_1^2 + y_{14}^2 + y_{33}^2 + y_7^2, -1),$
$\mathfrak{F}(4_{67}) = (y_{11}, y_{13} + y_{63} + 2y_3y_7, y_{11} + 2y_{14} + y_{26}y_{35} + y_3y_{55} + y_4y_7, y_{12} + y_{13} + y_{62} + y_{63}, y_{64}),$
$\mathfrak{F}(4_{68}) = (y_{63}, y_{11} + y_{14} + y_{26}y_{35} + y_3y_{55}, 2y_{13} + y_{62} + y_{63} + 2y_3y_7, y_{11} + y_{14} + y_{64} + y_4y_7, y_{12}),$
$\mathfrak{F}(4_{69}) = (y_{26}y_{35}, 2y_{13} + 2y_{63}, y_{11} + 2y_{14} + y_3y_{55} + y_{64} + y_4y_7, y_{12} + y_{62} + 2y_3y_7, y_{11}),$
$\mathfrak{F}(4_{70}) = (y_3y_7, 2y_{11} + y_{14} + y_4y_7, y_{12} + 2y_{13} + y_{62} + y_{63} + y_3y_7, y_{14} + y_{26}y_{35} + y_3y_{55} + y_{64}, y_{63}),$
$\mathfrak{F}(4_{71}) = (y_{14}, y_{13} + y_{62} + y_{63} + y_3y_7, 2y_{11} + y_{26}y_{35} + y_3y_{55} + y_{64} + y_4y_7, y_{12} + y_{13} + y_{63} + y_3y_7, y_{14}),$
$\mathfrak{F}(4_{72}) = (y_4y_7, y_{12} + 2y_{13} + y_3y_7, 2y_{11} + 2y_{14} + y_{26}y_{35} + y_{64}, y_{62} + 2y_{63} + y_3y_7, y_3y_{55}).$

Indeed, for shortness, consider all $y_j$ to be equal to the same value $y$ and consider only the first and the last elements of each Kauffman bracket frame taking into account the inversion. As a result, all the Kauffman bracket frames presented in the list of the Kauffman bracket frames are divided into 22 groups having from 1 to 13 Kauffman bracket frames. Within each of the obtained group, it is easy to see that all the Kauffman bracket frames are pairwise inequivalent in the sense of coefficients and powers of $y$ in the inner elements.

Moreover, the list of the Kauffman bracket frames presented in the list of the Kauffman bracket frames is enough to show that all tabulated knots admit no destabilisations, see Subsection 3.3.

As mentioned above, there exist two pairs of the tabulated diagrams having the same Kauffman bracket frames: ($4_{49}$, $4_{50}$) and ($4_{65}$, $4_{66}$). In order to show that diagrams that form each pair are inequivalent, it is enough to calculate and compare their cabled Jones polynomials, see the invariants of the virtual knots ($4_{103}$, $4_{93}$) and ($4_{107}$, $4_{98}$) given in [10], respectively.

### 3.3. On Primarily of the Tabulated Knots

In order to prove that a knot is prime, it is enough to show that the knot is essential and noncomposite.

We use the following two obvious statements in order to show that each of the knots given in Figs. 4 – 5 is essential, i.e. admits no destabilisation.

**Lemma 2.** *In the surface $T_2$, the intersection number of two noncut curves having homology classes $(a, b, c, d)$ and $(a', b', c', d')$ equals to $(a \cdot b' - b \cdot a') + (c \cdot d' - d \cdot c')$.*

*Proof.* For the torus $T$, it is well known that the intersection number of two noncut curves having homology classes $(a, b)$ and $(a', b')$ equals to $(a \cdot b' - b \cdot a')$. For the surface $T_2$, take into account that pairs «meridian-longitude» do not intersect each other.

□

**Lemma 3.** *Suppose that the Kauffman bracket frame $\mathfrak{F}(D)$ of a knot diagram $D \subset T_2$ contains terms corresponding to 4 noncut curves having inequivalent homology classes $(a_k, b_k, c_k, d_k)$, $k \in \{1, 2, 3, 4\}$, such that the system of 4 linear equations of the form*

$$b_k \cdot a - a_k \cdot b + d_k \cdot c - c_k \cdot d = 0, \quad k \in \{1, 2, 3, 4\},$$

*where $a$, $b$, $c$, $d$ are the variables and $a_k$, $b_k$, $c_k$, $d_k$ are known coefficients, has only zero solution. Then the knot diagram $D$ admits no destabilisation.*

*Proof.* The system is based on the following two ideas. First, intersection number of two noncut curves can be calculated according to Lemma 2. Second, intersection number of the cancellation curve and any of noncut curves involved in the terms of $\mathfrak{F}(D)$ should be equal to zero.

□

Following Lemma 3, for each tabulated diagram $D$, we construct a set of 4 noncut curves involved in the Kauffman bracket frame $\mathfrak{F}(D)$, which is enough to show that there exists no cancellation curve for the corresponding knot $K \subset T_2 \times I$. To this end, we use the following obvious interesting property of the Kauffman bracket frames. Namely, the Kauffman bracket frames of the tabulated diagrams based on the same projection involve the same set of the variables $y_j$. Moreover, it turned out that the Kauffman bracket frames of the tabulated diagrams based on the projections $3_1$, $4_1$, $4_5$, and $4_{13}$ [4] involve the variables $y_{11}, y_{12}, y_{14}, y_{62}$, while the Kauffman bracket frames of diagrams based on the projections $4_7$ and $4_9$ [4] involve the variables $y_2, y_{10}, y_{23}, y_{43}$, which are enough to show that the corresponding knots are essential. More precisely, Table 2 give the set of the variables $y_j$ involved in the Kauffman bracket frames that are enough to show that the corresponding tabulated knots are essential.

**Table 2**

Sets of tabulated knots associated with sets of 4 variables $y_j$

| | |
|---|---|
| $3_1 - 3_3$, $4_1 - 4_4$, $4_{18} - 4_{21}$, $4_{67} - 4_{72}$: $y_{11}, y_{12}, y_{14}, y_{62}$ | $4_{28} - 4_{33}$, $4_{42} - 4_{47}$: $y_2, y_{10}, y_{23}, y_{43}$ |
| $4_5 - 4_8$: $y_7, y_{12}, y_{14}, y_{22}$ | $4_{34} - 4_{41}$: $y_2, y_{21}, y_{50}, y_{52}$ |
| $4_9 - 4_{11}$: $y_2, y_7, y_8, y_{54}$ | $4_{48} - 4_{53}$: $y_6, y_7, y_8, y_{49}$ |
| $4_{12} - 4_{17}$: $y_3, y_{23}, y_{41}, y_{45}$ | $4_{54} - 4_{61}$: $y_2, y_8, y_{21}, y_{27}$ |
| $4_{22} - 4_{27}$: $y_3, y_{11}, y_{20}, y_{43}$ | $4_{62} - 4_{66}$: $y_3, y_7, y_8, y_{30}$ |

In order to prove that all 75 tabulated knots are noncomposite, it is enough to show that each knot can not be represented as a connected sum of the type $(a)$, $(b)$, or $(c)$ under the hypothesis that the complexity of a connected sum is not less than the sum of complexities of the terms that form the sum. More precisely, we assume that there exists no a pair of nontrivial knots such that the connected sum of these knots admits a diagram having number of crossings, which is smaller than a minimal sum of numbers of crossings of the diagrams corresponding to both knots formed the pair. Within the considered problem on tabulation of knots having diagrams with either 3 or 4 crossings, the impossibility of representation as a connected sum of the type $(a)$, $(b)$, or $(c)$ is obvious. Indeed, for the connected sums of the types $(a)$ and $(b)$, we note that all nontrivial knots in $T \times I$ have diagrams with at least 2 crossings, while all nontrivial knots in $S^3$ and $T_2 \times I$ have diagrams with at least 3 crossings. For the connected sum of the type $(c)$, we note that such a sum can be constructed only in the case then both terms are given by the unique nontrivial knot $2_1$ [1] in $T \times I$ having a diagram with 2 crossings. Due to specific form of $2_1$, the surface bracket polynomial $\langle \cdot \rangle$ of the obtained connected sum admits no surface-state pair that contains noncut curves having homology classes of the form $(0, 0, c, d)$ or $(a, b, 0, 0)$ only. As a result, all 75 tabulated knots can not be represented as the connected sum of the type $(c)$, since the Kauffman bracket frame $\mathfrak{F}(\cdot)$ of each knot contains at least one polynomial $P_m$ having at least one term formed only by the variables that belong to the set $\{y_1,\ y_2,\ y_4,\ y_7,\ y_{21},\ y_{30}\}$.

*The work is supported by the RFBR grant no. 20-01-00127.*

## References

1. Akimova A. A., Matveev S. V. Classification of Genus 1 Virtual Knots Having at Most Five Classical Crossings. *Journal of Knot Theory and Its Ramifications*, 2014, vol. 23, no. 6, art. no. 1450031. DOI: 10.1142/S021821651450031X.
2. Akimova A. A., Matveev S. V., Tarkaev V. V. Classification of Links of Small Complexity in the Thickened Torus. *Proceedings of the Steklov Institute of Mathematics*, 2018, vol. 303 (suppl. 1), no. 12, pp. 12–24. DOI: 10.1134/S008154381809002X.
3. Akimova A. A. Computation of the Kauffman bracket skeleton. *Journal of Computational and Engineering Mathematics*, 2019, vol. 6, no. 3, pp. 3–13. DOI: 10.14529/jcem190301.
4. Akimova A. A. Classification of Prime Projections of Knots in the Thickened Torus of Genus 2 with at most 4 Crossings. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2020, vol. 12, no. 1, pp. 5–13. DOI: 10.14529/mmph200101.
5. Bar-Natan D. *The Hoste-Thistlethwaite Link Table*, available at: http://katlas.org/wiki/Main_Page (accessed on July 25, 2018).
6. Drobotukhina Yu. V. Classification of Links in $\mathbb{R}P^3$ with at most Six Crossings. *Zap. Nauchn. Sem. LOMI*, 1991, vol. 193, pp. 39–63. (in Russian)
7. Dye H. A., Kauffman L. H. Minimal Surface Representations of Virtual Knots and Links. *Algebraic and Geometric Topology*, 2005, vol. 5, pp. 509–535.
8. Gabrovšek B., Mroczkowski M. Knots in the Solid Torus up to 6 Crossings. *Journal of Knot Theory and Its Ramifications*, 2012, vol. 21, no. 11, 43 pp. DOI: 10.1142/S0218216512501064.
9. B. Gabrovšek, Tabulation of Prime Knots in Lens Spaces, *Mediterr. J. Math*, 2017, vol. 14, no. 2, art.no. 88, 24 pp. DOI: 10.1007/s00009-016-0814-5.
10. Green J. *A Table of Virtual Knots*, available at: http://katlas.math.toronto.edu/wiki/ (accessed on July 25, 2018).
11. Hoste J., Thistlethwaite M., Weeks J., The First 1,701,936 Knots. *The Mathematical Intelligencer*, 1998, vol. 20, no. 4, pp. 33–47.
12. Jones V. F. R. A Polynomial Invariant for Knots via von Neumann Algebra. *Bulletin of the American Mathematical Society*, 1985, vol. 12, no. 1, pp. 103–111.
13. Kauffman L. H. State Models and the Jones Polynomial. *Topology*, 1987, vol. 26, no. 3, pp. 395–407. DOI: 10.1016/0040-9383(87)90009-7.
14. Matveev S. V. Prime Decompositions of Knots in $T \times I$. *Topology and its Applications*, 2011, vol. 159, no. 7, pp. 1820–1824.
15. Matveev S. V., Nabeeva L. R. Tabulating Knots in the Thickened Klein Bottle. *Siberian Mathematical Journal*, 2016, vol. 57, issue 3, pp. 542–548. DOI: 10.1134/S0037446616030174.
16. Rolfsen D. Knots and Links. *Berkeley, CA: Publish or Perish*, 1976.
17. Stenlund E. *Classification of Virtual Knots*, available at: http://evertstenlund.se/knots/Virtual%20Knots.pdf (accessed on July 25, 2018).

*Alyona A. Akimova, PhD (Math), Department of Mathematical and Computer Modelling, South Ural State University (Chelyabinsk, Russian Federation), akimovaaa@susu.ru.*

*Received* February 19, 2020.

**УДК 515.162** **DOI: 10.14529/jcem200103**

# КЛАССИФИКАЦИЯ ПРИМАРНЫХ УЗЛОВ В УТОЛЩЕННОМ КРЕНДЕЛЕ РОДА 2, ИМЕЮЩИХ ДИАГРАММЫ С НЕ БОЛЕЕ ЧЕМ 4 ПЕРЕКРЕСТКАМИ

***А. А. Акимова***

Цель настоящей работы – составить таблицу всех примарных узлов в утолщенном кренделе рода 2, имеющих диаграммы с не более чем 4 перекрестками. Прежде всего, мы вводим определение примарного узла в утолщенном кренделе рода 2. После этого, мы строим таблицу примарных узлов в утолщенном кренделе рода 2, имеющих диаграммы с не более чем 4 перекрестками. Для этого мы используем таблицу примарных проекций узлов на кренделе рода 2, чтобы построить предварительный набор диаграмм. Для того, чтобы удалить дубликаты и доказать, что все оставшиеся узлы неэквивалентны, а также доказать, что все табличные узлы не допускают дестабилизации, мы предлагаем инвариант, называемый каркас скобки Кауфмана, который является упрощением обобщенного полинома скобки Кауфмана. Идея состоит в том, чтобы принимать во внимание только порядок и значения коэффициентов и игнорировать степени одной из переменных. Предлагаемое упрощение является более компактным и в то же время не слабее, чем исходный обобщенный полином скобки Кауфмана в смысле, например, табулирования примарных узлов до сложности 4 включительно. На заключительном шаге мы доказываем, что ни один из табулированных узлов не может быть представлен в виде связной суммы в рамках гипотезы, что наименьшее число перекрестков связной суммы узлов не меньше суммы наименьших чисел перекрестков слагаемых.

*Ключевые слова: примарный узел; утолщенный крендель рода 2; классификация; обобщенный полином скобки Кауфмана; каркас скобки Кауфмана.*

## Литература

1. Akimova, A. A. Classification of Genus 1 Virtual Knots Having at Most Five Classical Crossings / A. A. Akimova, S. V. Matveev // Journal of Knot Theory and Its Ramifications. – 2014. – V. 23, № 6. – Article: 1450031, 19 p.
2. Акимова, А. А. Классификация зацеплений малой сложности в утолщенном торе / А. А. Акимова, С. В. Матвеев, В. В. Таркаев // Труды института математики и механики УрО РАН. – 2017. – Т. 23, № 4. – С. 18–31.
3. Akimova, A. A. Computation of the Kauffman bracket skeleton / A. A. Akimova // Journal of Computational and Engineering Mathematics. – 2019. – V. 6, № 3. – P. 3–13.
4. Akimova, A. A. Classification of Prime Projections of Knots in the Thickened Torus of Genus 2 with at most 4 Crossings / A. A. Akimova // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2020. – V. 12, №. 1. – P. 5–13.

5. Bar-Natan, D. The Hoste-Thistlethwaite Link Table [Электронный ресурс]. – URL: http://katlas.org/wiki/Main_Page (дата обращения: 25.07.2018).
6. Дроботухина, Ю. В. Классификация зацеплений в $\mathbb{R}P^3$ с небольшим числом точек скрещивания / Ю. В. Дроботухина // Зап. научн. сем. ЛОМИ. – 1991. – Т. 193. – С. 39–63.
7. Dye, H. A. Minimal Surface Representations of Virtual Knots and Links / H. A. Dye, L. H. Kauffman // Algebraic and Geometric Topology. – 2005. – V. 5. – P. 509–535.
8. Gabrovšek, B. Knots in the Solid Torus up to 6 Crossings / B. Gabrovšek, M. Mroczkowski // Journal of Knot Theory and Its Ramifications. – 2012. – V. 21, № 11. – 43 p.
9. Gabrovšek, B. Tabulation of Prime Knots in Lens Spaces / B. Gabrovšek // Mediterranean Journal of Mathematics. – 2017. – V. 14, iss. 2. – Article: 88.
10. Green, J. A. Table of Virtual Knots [Электронный ресурс]. – URL: http://katlas.math.toronto.edu/wiki/ (дата обращения: 25.07.2018).
11. Hoste, J. The First 1,701,936 Knots / J. Hoste, M. Thistlethwaite, J. Weeks // The Mathematical Intelligencer. – 1998. – V. 20, №. 4. – P. 33–47.
12. Jones, V. F. R. A Polynomial Invariant for Knots via von Neumann Algebra / V. F. R. Jones // Bulletin of the American Mathematical Society. – 1985. V. 12, № 1. – P. 103–111.
13. Kauffman, L. H. State Models and the Jones Polynomial / L. H. Kauffman // Topology. – 1987. V. 26, № 3. – P. 395–407.
14. Matveev, S. V. Prime Decompositions of Knots in $T \times I$ / S. V. Matveev // Topology and its Applications. – 2011. – V. 159, № 7. – P. 1820–1824.
15. Матвеев, С. В. Табулирование узлов в утолщенной бутылке Клейна / С. В. Матвеев, Л. Р. Набеева // Сибирский математический журнал. – 2016. – Т. 57, вып. 3. – С. 688–696.
16. Rolfsen, D. Knots and Links / D. Rolfsen. – Berkeley, CA: Publish or Perish, 1976.
17. Stenlund, E. Classification of Virtual Knots [Электронный ресурс]. – URL: http://evertstenlund.se/knots/Virtual%20Knots.pdf (дата обращения: 25.07.2018).


*Акимова Алена Андреевна, кандидат физико-математических наук, доцент кафедры математического и компьютерного моделирования, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), akimovaaa@susu.ru.*

*Поступила в редакцию* 19 февраля 2020 г.