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# STABLE AND UNSTABLE INVARIANT SPACES OF ONE STOCHASTIC NON-CLASSICAL EQUATION WITH A RELATIVELY RADIAL OPERATOR ON A 3-TORUS

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In this paper, we consider a stochastic analogue of the Dzektser equation, which is a model of the evolution of the free surface of a filtered fluid in the spaces of differential forms defined on a smooth compact oriented manifold without boundary. We consider a three-dimensional torus (3-torus) as such a manifold. Also, we consider the question of the stability of solutions to the Dzektser equation in the spaces of "noises"on this manifold in terms of invariant spaces. To this end, the stochastic Dzektser equation is reduced to a linear stochastic Sobolev type equation. We show the existence of stable and unstable invariant spaces and dichotomies of solutions to the stochastic Dzektser equation on a three-dimensional torus. A computational experiment is carried out. An algorithm is developed in the form of a program in the Maple environment. As a result of the implementation of this algorithm, we obtain the following. First, we construct a graph of solutions when the coefficients of the Dzektser equation satisfy sufficient conditions for the existence of only a stable invariant space of this equation. Second, we construct graphs of solutions in the case of the existence of exponential dichotomies of solutions. In this case, we show that the space of solutions splits into stable and unstable invariant spaces such that solutions increase in one of the spaces and decrease in another space.

*Keywords: Sobolev type equations; stochastic equations; three-dimensional torus; invariant spaces; exponential dichotomies.*

## Introduction

The equation

$$(\lambda - \Delta)u_t = \alpha\Delta u - \beta\Delta^2 u, \tag{1}$$

where $\alpha,\ \beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, simulates the evolution of the free surface of a filtered fluid [1]. Here the parameters $\alpha,\ \beta,\ \lambda$ characterize the environment.

The solvability of the initial-boundary value problem for equation (1) is considered, for example, in [2]. Here, in suitable functional spaces, equation (1) is reduced to the linear Sobolev type equation [3]

$$L\dot{u} = Mu, \tag{2}$$

where $L$ and $M$ are linear and continuous operators, and the operator $M$ is relatively sectorial. The paper [4] shows the existence of a unique solution to equation (1) in the spaces of differential forms defined on a compact smooth oriented manifold without boundary.

Next, the solvability of equation (1) is studied in the spaces of "noises" [5]. To this end, equation (1) is considered as the stochastic linear Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta. \tag{3}$$

Here $\eta = \eta(t)$ is a stochastic process, and $\overset{o}{\eta}$ is the Nelson – Gliklikh derivative of $\eta$ [6]. The paper [7] shows the existence of solutions to stochastic equation (1) in the spaces of differential forms.

The study of invariant spaces and exponential dichotomies of solutions to equation (2) was considered in [8]. Under certain conditions imposed on the parameters $\alpha,\ \beta,\ \lambda$, the paper [6] proves the existence of exponential dichotomies of solutions to equation (1). This paper is devoted to the study of the stability of solutions to equation (1) in the spaces of differential forms with "noises"on a three-dimensional torus.

The paper is organized as follows. In Section 1, we construct the spaces of **K**-variables and **K**-"noises" on a three-dimensional torus. Sufficient conditions for the existence of stable and unstable invariant spaces and dichotomies of solutions to stochastic equation (1) are presented. Section 2 is devoted to a computational experiment. Here, following the results of Section 1, we construct graphs of the stable and unstable invariant spaces of stochastic equation (1) on one of the maps of the three-dimensional torus. The conditions of "gluing"on the map are provided by choosing the basic functions.

## 1. Invariant Spaces on a 3-Torus

Consider a three-dimensional torus (3-torus) $T^3 = S^1 \times S^1 \times S^1$ defined as the direct product of three circles $S^1$. A three-dimensional torus is obtained from a three-dimensional cube by "gluing" opposite faces. Therefore, a cube is a map of a three-dimensional torus. As well as a two-dimensional torus, a three-dimensional torus is a smooth, compact, oriented manifold without boundary.

**Remark 1.** If we consider the «gluing» opposite faces of a parallelepiped with sides equal to $A,\ B$ and $C$, then we obtain a 3-torus $T^3 = [0, A] \times [0, B] \times [0, C]$.

Consider the spaces of **K**-variables and **K**-"noises"on the torus $T^3$. Let $\mathbf{L}_2$ be the space of random variables $\xi$ with zero mathematical expectation and finite variance, and $\mathbf{L}_2$ be the space of continuous stochastic processes $\eta$. Fix $\eta \in \mathbf{L}_2$ and $t \in \mathfrak{I}$, where $\mathfrak{I}$ is an interval, and denote $\sigma$-algebra generated by $\eta$ and $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$ by $\mathcal{N}_t^\eta$. Define the *Nelson – Gliklikh derivative* of the stochastic process $\eta$ at the point $t \in \mathfrak{I}$ as the limit

$$\overset{o}{\eta}(\cdot,\omega) = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t}\right)\right),$$

if the limit converges in the uniform metric on $\mathbb{R}$. Denote by $\mathbf{C}^l\mathbf{L}_2$ the space of stochastic processes whose Nelson – Gliklikh derivatives are a.s. (almost sure) continuous on $\mathfrak{I}$ up to the order $l$ inclusive.

Next, we define the spaces of differential forms on the 3-torus $E^q = E^q(T^3),\ q = 0, 1, 2, 3$, with the scalar products

$$(a, b)_0 = \int\limits_{T^3} a \wedge *b,\ (a, b)_1 = (a, b)_0 + (\Delta a, b)_0,$$

$$(a, b)_2 = (a, b)_1 + (\Delta a, \Delta b)_0,\ (a, b)_4 = (\Delta^2 a, \Delta^2 b)_2 + (a, b)_2.$$

Here $* : E^q \to E^{3-q}$ is the Hodge operator, $\Delta = d\delta + \delta d$ is the Laplace – Beltrami operator, $d : E^q \to E^{q+1}$. Denote by $H_l^q$ the completion of the linear $E^q$ in the norms $\|\cdot\|_l,\ l = 0, 1, 2, 3, 4$.

The basis of the spaces $H_l^q$ is formed by the eigenvalues of the Laplace – Beltrami operator $\mathbf{K} = \{\lambda_k\}$. Denote by $\{\varphi_k\}$ the corresponding eigenfunctions. Let the sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ be such that the variances $\mathbf{D}\xi_k \leq const$. Define the spaces $\mathbf{H}_l^q$ as the completion of the linear span of *random* $\mathbf{K}$*-variables*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k \tag{4}$$

in the norm

$$\|\xi\|^2_{\mathbf{H}_l^q} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k. \tag{5}$$

By *a continuous stochastic* $\mathbf{K}$*-process* we mean the map $\eta : \mathfrak{I} \to \mathbf{H}_l^q$ given by the formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t) \varphi_k, \tag{6}$$

if the series converges uniformly on any compactum in $\mathfrak{I}$, where $\mathfrak{I}$ is an interval, and $\{\eta_k\} \subset \mathbf{CL}_2$. If the series

$$\overset{\circ}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{o}{\eta}_k(t) \varphi_k \tag{7}$$

converges uniformly on any compactum in $\mathfrak{I}$ and $\{\eta_k\} \subset \mathbf{C}^1\mathbf{L}_2$, then the stochastic $\mathbf{K}$-process is called *continuously differentiable by Nelson – Gliklikh*. Let $\mathbf{C}(\mathfrak{I}; \mathbf{H}_l^q)$ be the set of continuous processes, and $\mathbf{C}^1(\mathfrak{I}; \mathbf{H}_l^q)$ be processes that are continuously differentiable by Nelson – Gliklikh.

Let the operators $L, M \in \mathcal{L}(\mathbf{H}_0^q; \mathbf{H}_4^q)$. Consider the linear stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta. \tag{8}$$

The stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathfrak{I}; \mathbf{H}_0^q)$ is called a *solution to equation (8)*, if the process a.s. converts equation (8) into identity.

**Definition 1.** *The set $\mathfrak{P} \subset \mathbf{H}_0^q$ is called a phase space of equation (8), if*

*(i) each trajectory of the solution $\eta = \eta(t)$ to equation (8) a.s. belongs to $\mathfrak{P}$;*

*(ii) for a.a. (almost all) $\eta_0 \in \mathfrak{P}$, there exists a solution to equation (8) that satisfies the condition $\eta(0) = \eta_0$.*

**Definition 2.** *The subspace $\mathbf{I_K} \subset \mathbf{H}_0^q$ is said to be the invariant space of equation (8), if solution to problem $\eta(0) = \eta_0$ for equation (8) is $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{I_K})$ for any $\eta_0 \in \mathbf{I_K}$.*

**Definition 3.** *(i) The space $\mathbf{I}^+_\mathbf{K} \subset \mathfrak{P}$ is said to be stable invariant space of equation (8), if there exist the constants $N \in \mathbb{R}_+$ and $\nu_k \in \mathbb{R}_+$ such that*

$$\|\eta^1(t)\|_{\mathbf{H}_0^q} \leq N_1 e^{-\nu_1(s-t)} \|\eta^1(s)\|_{\mathbf{H}_0^q} \qquad \texttt{for } s \geq t,$$

*where $\eta^1 = \eta^1(t) \in \mathbf{I}^+_\mathbf{K}$ for all $t \in \mathbb{R}$.*

*(ii) The space $\mathbf{I}^+_\mathbf{K} \subset \mathfrak{P}$ is said to be unstable invariant space of equation (8), if there exist the constants $N \in \mathbb{R}_+$ and $\nu_k \in \mathbb{R}_+$ such that*

$$\|\eta^2(t)\|_{\mathbf{H}_0^q} \leq N_2 e^{-\nu_2(t-s)} \|\eta^2(s)\|_{\mathbf{H}_0^q} \qquad fort \geq s,$$

*where $\eta^2 = \eta^2(t) \in \mathbf{I}^-_{\mathbf{K}}$ for all $t \in \mathbb{R}$. If the phase space splits into the direct sum $\mathfrak{P} = \mathbf{I}^+ \oplus \mathbf{I}^-$, then the solutions $\eta = \eta(t)$ to equation (8) have an exponential dichotomy.*

Denote $\sigma^L_+(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\mu < 0\}$ and $\sigma^L_-(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\mu > 0\}$.

**Theorem 1.** *Let the operator $M$ be $(L,p)$-sectorial, then*

*(i) if $\sigma^L(M) = \sigma^L_+(M) \bigcup \sigma^L_-(M)$, then the solutions to equation (8) have an exponential dichotomy;*

*(ii) if $\sigma^L(M) = \sigma^L_+(M)$, then the phase space of equation (8) coincides with a stable invariant space;*

*(iii) if $\sigma^L(M) = \sigma^L_-(M)$, then the phase space of equation (8) coincides with an unstable invariant space.*

In order to study the existence of stable and unstable invariant spaces of equation (1) in the spaces $\mathbf{H}_0^q$, consider

$$L = (\lambda + \Delta),\ M = -\alpha\Delta + \beta\Delta^2, \tag{9}$$

where $\Delta$ is the Laplace – Beltrami operator.

**Theorem 2.** *[7] For any $\alpha, \beta, \lambda \in \mathbf{R} \setminus \{0\}$ and $\lambda \neq \frac{\alpha}{\beta}$, there exists a solution $\eta = \eta(t)$ to the Cauchy problem $\eta(0) = \eta_0 \in \mathcal{P}$ for equation (1) given by*

$$\eta(t) = {\sum_{l=1}^{\infty}}' \left[ exp\left(\frac{-\alpha\lambda_l - \beta\lambda_l^2}{\lambda - \lambda_l} t\right) \left(\sum_{k=1}^{\infty} \lambda_k \xi_k (\varphi_k, \varphi_l)_0 \varphi_l\right)\right]. \tag{10}$$

**Theorem 3.** *Let $\lambda \neq \frac{\alpha}{\beta}$ and $\alpha,\ \beta,\ \lambda \in \mathbb{R}_+$. Then*

*(i) if $\frac{\alpha}{\beta} > \lambda_1$, then the solutions to equation (1) have an exponential dichotomy;*

*(ii) if $\frac{\alpha}{\beta} < \lambda_1$, then there exists only a stable invariant space of equation (1).*

## 2. Computational Experiment

Consider the 3-torus $T^3 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. As a map, consider the cube with the side equal to $2\pi$. If the solutions to equation (1) are stable on the map of the manifold, then the solutions are also stable on the manifold. The converse is also true. In view of the foregoing, we consider a computational experiment on the map. The «gluing» conditions are satisfied due to the choice of the functions $\varphi(x, y, z)$. Let us describe the algorithm of a computational experiment to study the stability of solutions to Dzektser equation (1) (see Fig. 1).

**Step 1.** Introduce the parameters $\lambda,\ \alpha,\ \beta$ of the equation, the number $K$ of random numbers $\xi_k$, the lengths $A,\ B,\ C$ of the sides of the parallelepiped, which is a map of the 3-torus.

**Step 2.** Define the basis function $\varphi(x, y, z) = \sin(mx)\sin(ny)\sin(lz)$.

**Step 3.** Write the procedure for finding the eigenvalues $\lambda_k$ of the Laplace – Beltrami operator.

**Step 4.** Write the procedure for finding the relative spectrum of the operator $M$

$$\mu_k = \frac{-\alpha\lambda_k - \beta\lambda_k^2}{\lambda + \lambda_k}.$$

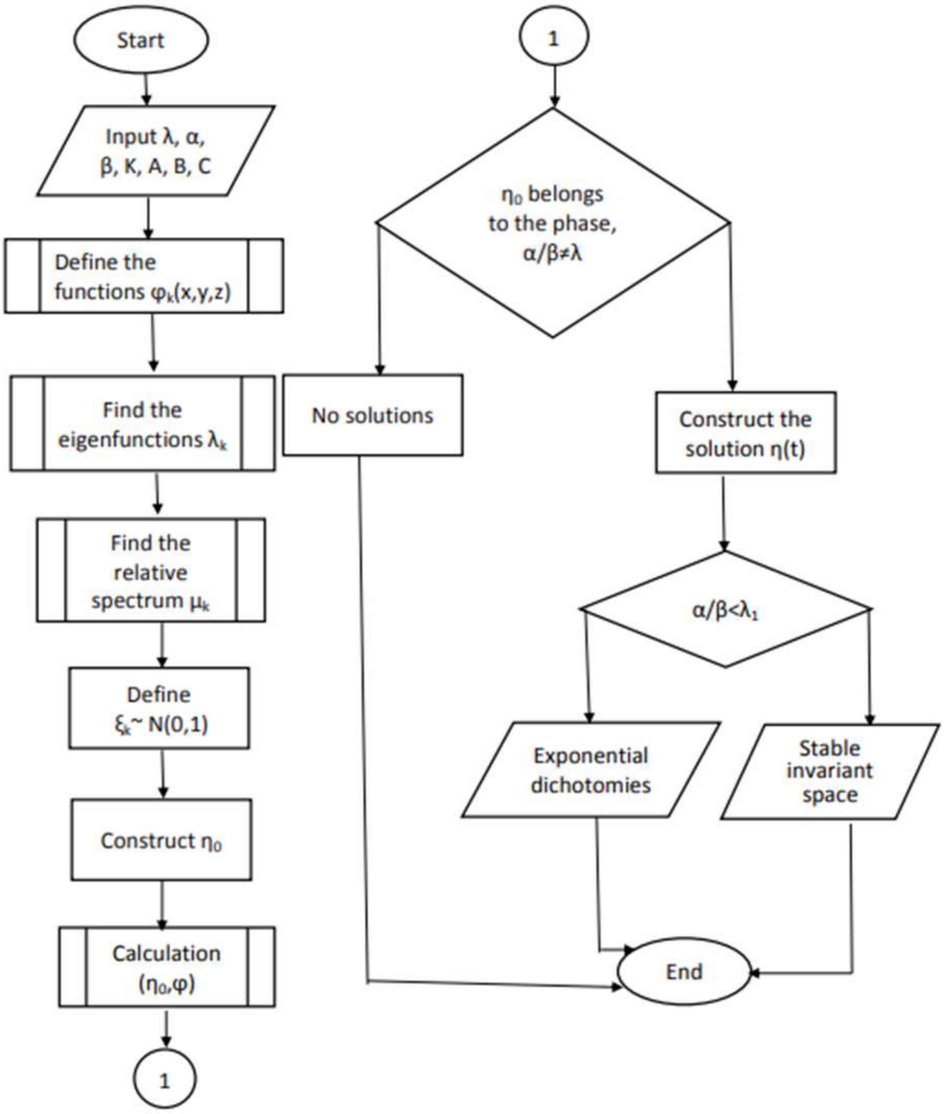


**Fig. 1**. The block diagram of the program

**Step 5.** Define the initial condition $\eta_0$ as follows: $\eta_0 = \sum\limits_{k=1}^{K} \lambda_k \xi_k \varphi_k$.

**Step 6.** Write the procedure for finding the scalar product

$$(\eta_0, \varphi) = \int\limits_0^A \int\limits_0^B \int\limits_0^C \eta_0 \cdot \varphi(x, y, z) dx dy dz.$$

**Step 7.** If the initial condition $\eta_0$ belongs to the phase space and the condition $\lambda \neq \frac{\alpha}{\beta}$ is satisfied, then go to Step 8. Otherwise, display a message stating that a solution to equation (1) does not exist and stop the program.

**Step 8.** Find the solution $\eta(t, x, y, z)$ to equation (1) by formula (10).

**Step 9.** If the condition $\frac{\alpha}{\beta} < \lambda_1$ (Example 1) is satisfied, then there exist stable invariant spaces of equation (1). Otherwise, go to Step 10.

**Step 10.** The solutions to equation (1) have an exponential dichotomy (Example 2).

**Example 1.** Let $\lambda = 0,5$, $\alpha = 0,3$, $\beta = 0,2$. Fig. 2 shows the stable solution to equation (1) in the section $z = 5$, $y = 5$ when $t$ takes values from 0 to 3.

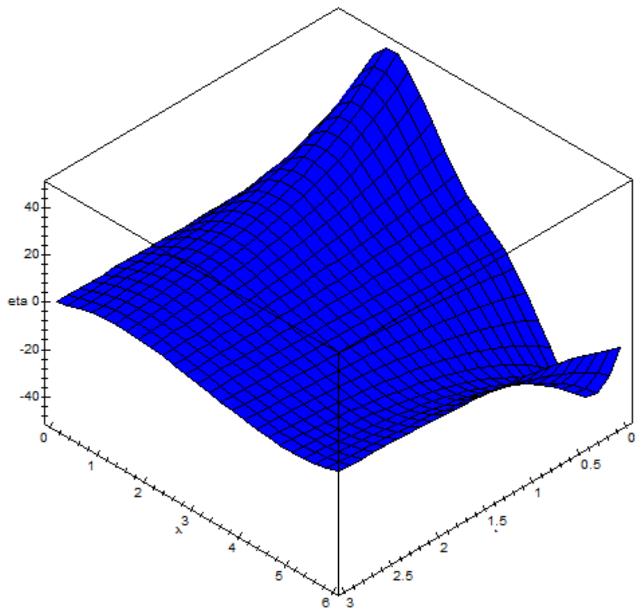


**Fig. 2.** $\lambda = 0,5,\ \alpha = 0,3,\ \beta = 0,2,\ z = 5,\ y = 5,\ t = 0,\dots,3$

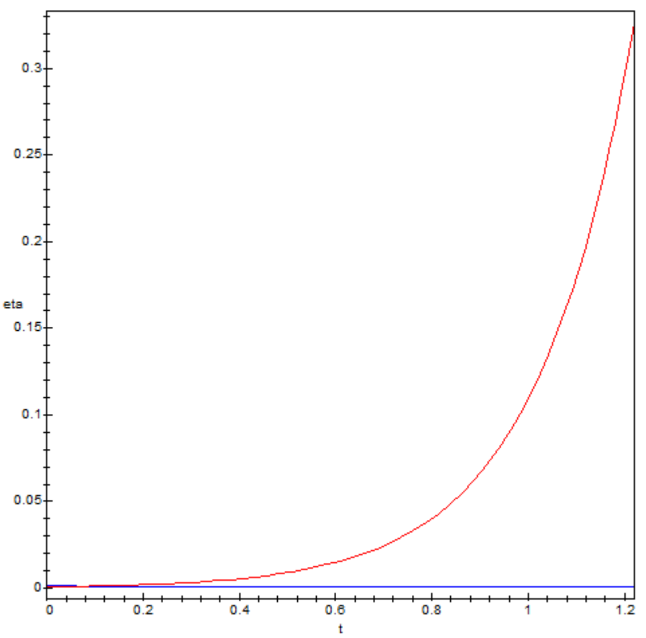


**Fig. 3.** $\lambda = 0,2,\ \alpha = 8,\ \beta = 2,\ x,\ y,\ z = 3,1,\ t = 0,\dots,3$

**Example 2.** Let $\lambda = 0,2$, $\alpha = 8$, $\beta = 2$. Fig. 3 shows the dichotomous behavior of the solution to equation (1) in the section $x,\ y,\ z = 3,1$ when $t$ takes values from 0 to 1,22. In the section $y,\ z = 3,1$, when $t$ takes values from 0 to 3, the stable solution is constructed for all $\lambda_k < \frac{\alpha}{\beta}$ (Fig. 4), while the unstable solution is constructed for all $\lambda_k > \frac{\alpha}{\beta}$ (Fig. 5).

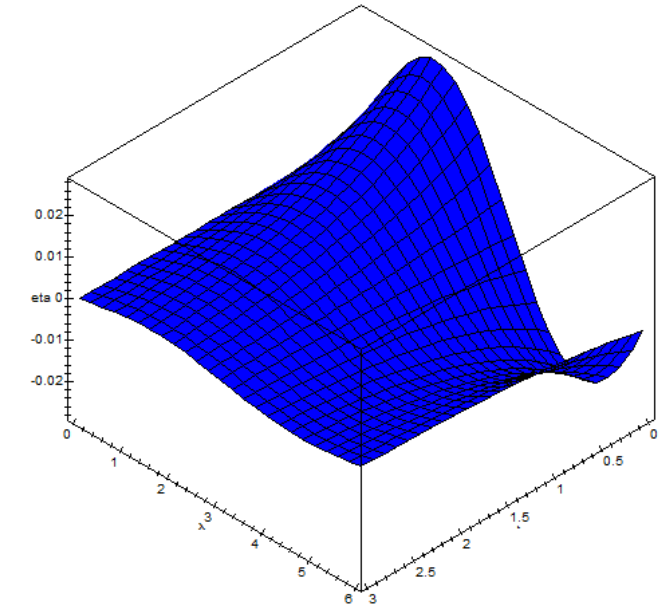


**Fig. 4**. $\lambda = 0, 2,\ \alpha = 8,\ \beta = 2,\ y,\ z = 3, 1,\ t = 0, \ldots, 3$

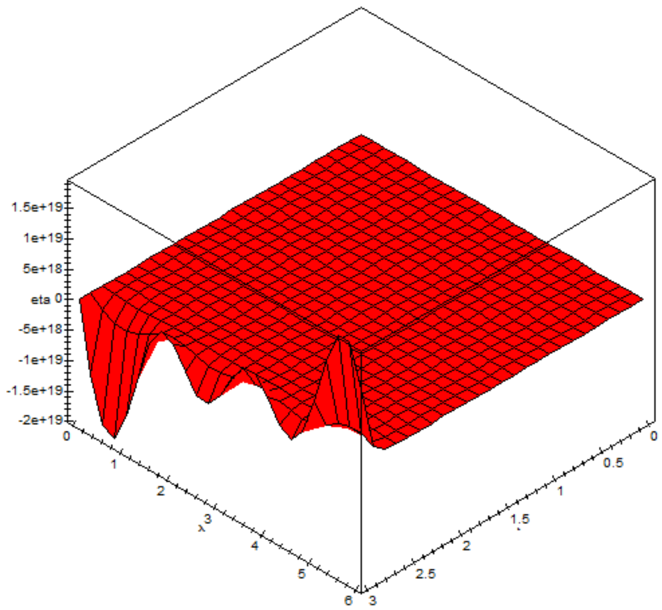


**Fig. 5**. $\lambda = 0, 2,\ \alpha = 8,\ \beta = 2,\ y,\ z = 3, 1,\ t = 0, \ldots, 3$

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# УСТОЙЧИВЫЕ И НЕУСТОЙЧИВЫЕ ИНВАРИАНТНЫЕ ПРОСТРАНСТВА ОДНОГО СТОХАСТИЧЕСКОГО НЕКЛАССИЧЕСКОГО УРАВНЕНИЯ С ОТНОСИТЕЛЬНО РАДИАЛЬНЫМ ОПЕРАТОРОМ НА 3-ТОРЕ

***О. Г. Китаева***

В работе рассматривается стохастический аналог уравнения Дзекцера, которое является моделью эволюции свободной поверхности фильтрующейся жидкости, в пространствах дифференциальных форм, определенных на гладком компактном ориентированном многообразии без края. В качестве такого многообразия был выбран трехмерный тор (3-тор). Рассмотрен вопрос об устойчивости решений уравнения Дзекцера в пространствах "шумов" на данном многообразии в терминах инвариантных пространств. Для этого стохастическое уравнение Дзекцера было сведено к линейному стохастическому уравнению соболевского типа. Показано существование устойчивого и неустойчивого инвариантных пространств и дихотомий решений стохастического уравнения Дзекцера на трехмерном торе. Проведен численный эксперимент. Разработан алгоритм в виде программы в среде Maple. В результате реализации данного алгоритма, во-первых построен график решений, когда коэффициенты уравнения Дзекцера удовлетворяют достаточным условиям существования только устойчивого инвариантного пространства данного уравнения. Во-вторых построены графики решений в случае существования экспоненциальных дихотомий решений. Показано, что в данном случае пространство решений расщепляется на устойчивое и неустойчивое инвариантные пространства, одном из которых решения растут, а в другом убывают.

*Ключевые слова: уравнения соболевского типа; стохастические уравнения; трехмерный тор; инвариантные пространства, экспоненциальные дихотомии.*

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