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OPTIMAL CONTROL OF SOLUTIONS TO CAUCHY PROBLEM FOR SOBOLEV TYPE EQUATION OF HIGHER ORDER

O. N. Tsyplenkova, South Ural State University, Chelyabinsk, Russian Federation, Tsyplenkova_Olga@mail.ru

An optimal control problem for the higher order Sobolev type equation with a relatively polynomially bounded operator pencil is considered. The existence and uniqueness of a strong solution to the Cauchy problem for this equation are proved. Sufficient conditions for the existence and uniqueness of an optimal control of such solutions are obtained.

Keywords: Sobolev-type equations, relatively polynomially bounded operator pencil, strong solutions, optimal control.

Introduction

Consider the complete Sobolev type equation of higher order

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0x + y + Cu, \tag{1}$$

where the operators $A, B_{n-1}, \ldots, B_0 \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, the function $u : [0,\tau) \subset \mathbf{R}_+ \to \mathfrak{U}$, $y : [0,\tau) \subset \mathbf{R}_+ \to \mathfrak{Y}$ $(\tau < \infty)$, $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{U}$ are some Hilbert spaces.

Consider the Cauchy problem

$$x^{(m)}(0) = x_m, \quad m = \overline{0, n-1}. \tag{2}$$

We are interested in the optimal control problem of finding of a pair $(\hat{x}, \hat{u})$, where $\hat{x}$ is a solution to problem (1), (2), and the control $\hat{u}$ belongs to $\mathfrak{U}_{ad}$, and satisfies the relation

$$J(\hat{x}, \hat{u}) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u). \tag{3}$$

Here $J(x, u)$ is some specially constructed performance functional, where $\mathfrak{U}_{ad}$ is a closed convex set in the space $\mathfrak{U}$ of controls.

First the study of optimal control problems for linear Sobolev type equations was engaged by G.A. Sviridyuk and A.A. Efremov [1]. Optimal control of solutions to the Cauchy problem for linear Sobolev type equations was considered in [2]. This research was continued by disciples of G.A. Sviridyuk including N.A. Manakova [3], A.V. Keller [4] and etc. The results by A.A. Efremov initiated the study of controllability issues for Sobolev type equations [5]. Other aspects of controllability of Sobolev type equations were studied in [6].

1. Polynomially A-bounded operator pencils and projections

By $\vec{B}$ denote the pencil formed by the operators $B_{n-1}, \ldots, B_0$.

Definition 1. *The sets*

$$\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1}B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{Y};\mathfrak{X})\}$$

and $\sigma^A(\vec{B}) = \mathbb{C}\backslash\rho^A(\vec{B})$ *are called the A-resolvent set and the A-spectrum of pencil* $\vec{B}$*, respectively.*

Definition 2. *An operator function*

$$R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1}B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}$$

of a complex variable μ *with domain* $\rho^A(\vec{B})$ *is called A-resolvent of pencil* $\vec{B}$*.*

Definition 3. *An operator pencil* $\vec{B}$ *is said to be polynomially bounded with respect to operator A (or simply polynomially A-bounded), if there exists* $a \in \mathbb{R}_+$ *such that for each* $\mu \in \mathbb{C}$ *the inequality* $(|\mu| > a)$ *implies the inclusion* $(R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{Y};\mathfrak{X}))$*.*

Introduce the additional condition

$$\int\limits_\gamma \mu^m R^A_\mu(\vec{B})d\mu \equiv \mathbb{O}, m = \overline{0, n-2}, \tag{A}$$

where the contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1. *[2] Let a pencil* $\vec{B}$ *be polynomially A-bounded, and let condition* (A) *be satisfied. Then operators*

$$P = \frac{1}{2\pi i}\int\limits_\gamma R^A_\mu(\vec{B})\mu^{n-1}Ad\mu, \quad Q = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1}AR^A_\mu(\vec{B})d\mu$$

are projections in spaces $\mathfrak{X}$ *and* $\mathfrak{Y}$*, respectively.*

Set $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \text{im } P$, $\mathfrak{Y}^1 = \text{im } Q$. It follows from Lemma 1 that $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$. By A^k $((B^k_l)$, respectively,) denote the restriction of the operator A $((B_l)$, respectively,) to $\mathfrak{X}^k$, $k = 0, 1$; $l = \overline{0, n-1}$.

Theorem 1. *[7] Let the assumptions of Lemma 1 be satisfied. Then*
(i) $A^k \in \mathcal{L}(\mathfrak{X}^k;\mathfrak{Y}^k)$, $k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{X}^k;\mathfrak{Y}^k)$, $k = 0, 1$, $l = 0, 1, \ldots, n-1$;
(iii) there exists an operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{Y}^1;\mathfrak{X}^1)$;
(iv) there exists an operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{Y}^0;\mathfrak{X}^0)$.

Let us construct the operators $H_0 = (B^0_0)^{-1}A^0$, $H_m = (B^0_0)^{-1}B^0_{n-m}$, $m = \overline{1, n-1}$, $S_m = (A^1)^{-1}B^1_m$, $m = \overline{0, n-1}$.

Definition 4. *Introduce the family of operators* $\{K^1_q, K^2_q, \ldots, K^n_q\}$ *as follows:*

$$K^s_0 = \mathbb{O}, s \neq n, K^n_0 = \mathbb{I},\ K^1_1 = H_0,\ K^2_1 = -H_{n-1}, \ldots, K^s_1 = -H_{n+1-s}, \ldots, K^n_1 = -H_1,$$
$$K^1_q = K^n_{q-1}H_0,\ K^2_q = K^1_{q-1} - K^n_{q-1}H_{n-1}, \ldots,\ K^s_q = K^{s-1}_{q-1} - K^n_{q-1}H_{n+1-s}, \ldots,$$
$$K^n_q = K^{n-1}_{q-1} - K^n_{q-1}H_1, q = 2, 3, \ldots$$

Definition 5. *A point* ∞ *is called:*
(i) a removable singularity of A-resolvent of pencil $\vec{B}$*, if* $K^1_1 = K^2_1 = \ldots = K^n_1 \equiv \mathbb{O}$;
(ii) a pole of order $p \in \mathbb{N}$ *of A-resolvent of pencil* $\vec{B}$*, if* $K^s_p \not\equiv \mathbb{O}$*, for some s, but* $K^s_{p+1} \equiv \mathbb{O}$ *for any s;*
(iii) an essentially singular point of A-resolvent of pencil $\vec{B}$ *if* $K^n_p \not\equiv \mathbb{O}$ *for any* $p \in \mathbb{N}$*.*

2. Strong solutions

Consider linear homogeneous Sobolev type equation

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x. \tag{4}$$

Let a pencil $\vec{B}$ be polynomially A-bounded, and let condition (A) be satisfied. Fix a contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$ and consider family of operators for all $t \in \mathbb{R}$

$$X_m^t = \frac{1}{2\pi i} \int\limits_{\gamma} R_\mu^A(\vec{B})(\mu^{n-m-1}A - \mu^{n-m-2}B_{n-1} - \ldots - B_{m+1})e^{\mu t} d\mu,$$

where $m = \overline{0, n-1}$.

Lemma 2. *[7] (i) For every $m = \overline{0, n-1}$ operator function X_m^t is a propagator of equation (4).*

(ii) For every $m = \overline{0, n-1}$ operator function X_m^t is an integer function.

(iii)

$$\left. \frac{d^l}{dt^l} X_m^t \right|_{t=0} = \begin{cases} P, \ l = m; \\ \mathbb{O}, \ l \neq m; \end{cases} \quad \text{for all } m = \overline{0, n-1}, \ l = 0, 1, \ldots .$$

Definition 6. *A set $\mathcal{P} \subset \mathfrak{X}$ is called a phase space of equation (4), if*
(i) every solution $x = x(t)$ of equation (4) lies in $\mathcal{P}$, i.e. $x(t) \in \mathcal{P} \quad \forall t \in \mathbb{R}$.
(ii) for an arbitrary $x_m, m = \overline{0, n-1} \in \mathcal{P}$ there exists a unique solution to problem (2), (4).

Theorem 2. *[7] Let a pencil $\vec{B}$ be polynomially A-bounded, condition (A) be satisfied and ∞ be a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent. Then the phase space of equation (4) coincides with the image of projector P.*

Proceed to linear inhomogeneous Sobolev type equation

$$Ax^{(n)} = B_{n-1}x^{(n-1)} + \ldots + B_0 x + y. \tag{5}$$

Consider sets

$$\mathcal{M}_f^m = \{x \in \mathfrak{X} : \ (\mathbb{I} - P)x = -\sum_{l=0}^{p} K_l^n (B_0^0)^{-1} \frac{d^{l+m}}{dt^{l+m}} (\mathbb{I} - Q)y(0)\},$$

where $m = \overline{0, n-1}$.

Theorem 3. *[7] Let a pencil $\vec{B}$ be polynomially A-bounded, condition (A) be satisfied and ∞ be a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent. Let a vector function $y : (-\tau, \tau) \to \mathfrak{Y}$ be such that $y^0 = (\mathbb{I} - Q)y \in C^{p+n}((-\tau, \tau); \mathfrak{Y}^0)$ and $y^1 = Qy \in C((-\tau, \tau); \mathfrak{Y}^1)$. Then for an arbitrary $x_m \in \mathcal{M}_f^m, \ m = \overline{0, n-1}$ there exists a unique solution to problem (2), (5) for $t \in (-\tau, \tau)$ given by*

$$x(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \frac{d^q}{dt^q} y^0(t) + \sum_{m=0}^{n-1} X_m^t x_m^1 + \int\limits_0^t X_{n-1}^{t-s} (A^1)^{-1} y^1(s) ds. \tag{6}$$

Definition 7. *A vector function* $x \in H^n(\mathfrak{X}) = \{x \in L_2(0,\tau;\mathfrak{X}) : \ x^{(n)} \in L_2(0,\tau;\mathfrak{X})\}$ *is called a strong solution to equation (5), if it makes the equation an identity almost everywhere on interval* $(0,\tau)$. *A strong solution* $x = x(t)$ *of equation (5) is called a strong solution to problem (2),(5) if condition (2) holds.*

This is well defined by virtue of the continuity of the embedding $H^n(\mathfrak{X}) \hookrightarrow C^{n-1}([0,\tau];\mathfrak{X})$. The term "strong solution" has been introduced to distinguish a solution to equation (5) in this sense from solution (6), which is usually said to be "classical". Note that classical solution (6) is also a strong solution to problem (2), (5).

Let us construct the spaces

$$H^{p+n}(\mathfrak{Y}) = \{v \in L_2(0,\tau;\mathfrak{Y}) : v^{(p+n)} \in L_2(0,\tau;\mathfrak{Y}), p \in \{0\} \cup \mathbb{N}\}.$$

The space $H^{p+n}(\mathfrak{Y})$ is a Hilbert space with inner product

$$[v,w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{Y}} dt.$$

Let $y \in H^{p+n}(\mathfrak{Y})$. Introduce the operators

$$A_1 y(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \tfrac{d^q}{dt^q} y^0(t),$$

$$A_2 y(t) = \int_0^t X_{n-1}^{t-s} (A^1)^{-1} y^1(s) ds, t \in (-\tau,\tau)$$

and the functions

$$k(t) = \sum_{m=0}^{n-1} X_m^t x_m^1.$$

Lemma 3. *Let a pencil* $\vec{B}$ *be polynomially A-bounded, and condition* (A) *be satisfied. Then*

(i) $A_1 \in \mathcal{L}(H^{p+n}(\mathfrak{Y}); H^n(\mathfrak{X}))$*;*

(ii) for an arbitrary $x_m \in \mathcal{M}_f^m$, $m = \overline{0,n-1}$, *vector-function* $k \in C^n([0,\tau);\mathfrak{X})$*;*

(iii) $A_2 \in \mathcal{L}(H^{p+n}(\mathfrak{Y}); H^n(\mathfrak{X}))$.

Theorem 4. *Let a pencil* $\vec{B}$ *be polynomially A-bounded, let condition* (A) *be satisfied. Then, for an arbitrary* $x_m \in \mathcal{M}_f^m$, $m = \overline{0,n-1}$ *and* $y \in H^{p+n}(\mathfrak{Y})$ *there exists a unique strong solution to problem (2), (5).*

3. Optimal control

Consider the Cauchy problem (2) for linear inhomogeneous Sobolev type equation (1), where functions x, y, u lie in the Hilbert spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{U}$, respectively.

Introduce the control space

$$\overset{\circ}{H}{}^{p+n}(\mathfrak{U}) = \{u \in L_2(0,\tau;\mathfrak{U}) : u^{(p+n)} \in L_2(0,\tau;\mathfrak{U}), u^{(q)}(0) = 0, q = \overline{0,p}\},$$

$p \in \{0\} \cup \mathbb{N}$. It is a Hilbert space with inner product

$$[v,w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{Y}} dt.$$

In the space $\overset{\circ}{H}{}^{p+n}(\mathfrak{U})$ we single out a closed convex subset $\overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, which is called *the set of admissible controls.*

Definition 8. *A vector function $\hat{u} \in \overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$ is called an optimal control of solutions to problem (1), (2), if relation (3) holds.*

Our aim is to prove the existence of a unique control $\hat{u} \in \overset{\circ}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, minimizing the performance functional

$$J(x,u) = \mu \sum_{q=0}^{n} \int_0^{\tau} ||x^{(q)} - \tilde{x}^{(q)}||^2 dt + \nu \sum_{q=0}^{p+n} \int_0^{\tau} \left\langle N_q u^{(q)}, u^{(q)} \right\rangle_{\mathfrak{U}} dt. \tag{7}$$

Here $\mu, \nu > 0$, $\mu + \nu = 1$, $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0,\ 1,\ \ldots,\ p+n$, are self-adjoint positively defined operators, and $\tilde{x}(t)$ is the target state of the system.

Theorem 5. *Let the assumptions of Theorem 4 be satisfied. Then for an arbitrary $x_m \in \mathcal{M}_f^m$, $m = \overline{0, n-1}$ and $y \in H^{p+n}(\mathfrak{Y})$ there exists a unique optimal control of solutions to problem (1), (2).*

Proof.
By Theorem 4, for an arbitrary $y \in H^{p+n}(\mathfrak{Y})$, $x_m \in \mathfrak{X}$, and $u \in H^{p+n}(\mathfrak{U})$ there exists a unique strong solution $x \in H^n(\mathfrak{X})$ to problem (1), (2), given by

$$x(t) = (A_1 + A_2)(y + Cu)(t) + k(t), \tag{8}$$

where the operators A_1, A_2 and the vector function k are defined in Lemma 3.

Fix $y \in H^{p+n}(\mathfrak{Y})$ and $x_m \in \mathfrak{X}$, and consider function (8) as a map $D : u \mapsto x(u)$. The map $D : H^{p+n}(\mathfrak{U}) \to H^n(\mathfrak{X})$ is continuous. Therefore, the performance functional depends only on $u : J(x,u) = J(u)$.

We write out functional (7) in the form

$$J(u) = \|x(t,u) - \tilde{x}\|^2_{H^n(\mathfrak{X})} + [v, u],$$

where $v^{(q)}(t) = N_q u^{(q)}(t),\ q = 0, \ldots, p+n$. Hence it follows that

$$J(u) = \pi(u,u) - 2\lambda(u) + \|\tilde{x} - x(t,0)\|^2_{H^n(\mathfrak{X})},$$

where

$$\pi(u,u) = \|x(t,u) - x(t,0)\|^2_{H^n(\mathfrak{X})} + [v, u]$$

is a bilinear continuous coercive form on $H^{p+n}(\mathfrak{U})$ and

$$\lambda(u) = \langle \tilde{x} - x(t,0), x(t,u) - x(t,0) \rangle_{H^n(\mathfrak{X})}$$

is a linear continuous form on $H^{p+n}(\mathfrak{U})$. Therefore, the assumptions of theorem in [8, p. 13] are satisfied. The proof of the theorem is complete.

□

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Olga N. Tsyplenkova, Candidate of Physics and Mathematics Science, Associate Professor, Department of Equation of Mathematical Physics, South Ural State University, Chelyabinsk, Russian Federation, Tsyplenkova_ Olga@mail.ru

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