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TABULATION OF PRIME LINKS IN THE THICKENED SURFACE OF GENUS 2 HAVING DIAGRAMS WITH AT MOST 4 CROSSINGS

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The aim of this paper is to tabulate all prime not oriented links in the thickened surface of genus 2 having diagrams with no more than 4 crossings. A preliminary set of diagrams is constructed based on the table of prime link projections in the surface of genus 2. In order to remove duplicates and prove that all the rest links are not equivalent, as well as to prove that all tabulated links admit no destabilisations, we use an invariant called the Kauffman bracket frame, which is a simplification of the generalized Kauffman bracket polynomial. The idea of the invariant is to consider only the values and order of coefficients and do not take into account the powers of one of the variables. Finally, we prove that each tabulated link can not be given by a connected sum under the hypothesis that the sum of complexities of the terms that form the connected sum is not more than the complexity of the connected sum.

Keywords: prime link; thickened surface of genus 2; tabulation; generalised Kauffman bracket polynomial; Kauffman bracket frame.

Introduction

In the knot theory, one of the oldest and the most important problems is to recognize a knot (or a link), i. e., to associate the considered object with a unique tabulated one. This problem involves the problem on complete classification of knots and links ordered taking into account some their properties. Most of the classifications obtained during last 150 years are devoted to classical knots and links, see [12, 19, 6]. Today, developing of the theory of global knots and links leads to tabulation of knots and links in manifolds different from the 3-dimensional sphere. However, in contrast to the case of knots and links in the 3-dimensional sphere, there is a gap between global knots and links in the sense of tabulation. In order to show this gap, compare presence of classifications of global knots and links.

As regards tabulation of global knots, note that knots in the solid torus [9] and the thickened Klein bottle [18], as well as prime knots in the lens spaces [10] are tabulated. In the knot theory, recent classifications consider only the so-called prime objects, which can not be obtained by some known operations from already tabulated objects. Knots in the thickened surfaces and virtual knots have been of particular interest during last 20 years. Hence, some classifications of such knots were also obtained. In particular, the works [11, 20] present perfect classifications of virtual knots ordered taking into account the number of classical crossings and obtain a list of some characteristics of each knot. However, in these classifications, such important properties of a knot as primality and genus are not taken into account. Recall that genus of a virtual knot is the minimal genus of the thickened surface which can contain the considered knot. We propose to tabulate virtual knots with respect to both numerical characteristics, i. e. not only the number of classical crossings as usual, but also the genus of a knot, see the articles [1, 5] for classifications of prime knots in the thickened torus and the thickened surface of genus 2, respectively. In a sense, such classifications can be considered as classifications of prime virtual knots of genus 1 and 2, respectively.

As regards tabulation of global links, note classifications of links in the projective space [7] and prime links in the thickened torus [2, 3]. Also, note a classification of virtual links of special type, namely, alternating virtual links [21], see also [22] for the associated database, which include alternating virtual knots as well.

In this paper, we tabulate prime not oriented links in the thickened surface of genus 2. Namely, we obtain a table of prime diagrams, i.e. classification of prime links, based on the result of the first step [4], i.e. a classification of prime link projections in the surface of genus 2 having no more than 4 crossings. Following [5], we apply the Kauffman bracket frame $\mathfrak{F}(\cdot)$ at the steps devoted to cancellation of duplicates and proof of the fact that all the rest links are not equivalent and admit no destabilisations and representations as connected sums. Such an invariant is obtained as a simplification of the surface bracket polynomial $\langle\cdot\rangle$ [8], which generalises the Kauffman bracket [14] (see also [13] for the original version called the Jones polynomial). The idea of the invariant is to consider only the values and order of coefficients and do not take into account the powers of one of the variables. Finally, we show that each tabulated link can not be given by a connected sum under the hypothesis that the sum of complexities of the terms that form the connected sum is not more than the complexity of the connected sum.

The paper is organized as follows. In Section 1, we present some background material. Section 2 is devoted to a definition of the Kauffman bracket frame $\mathfrak{F}(\cdot)$ [5]. In Section 3, we present main steps of the tabulation of prime links in the thickened surface of genus 2 and prove the main theorem that there exist no more than 38 pairwise not equivalent prime such links having diagrams with no more than 4 crossings.

1. Background Material

Let T and T_2 be a 2-dimensional torus and a 2-dimensional surface of genus 2, respectively. Further, for shortness, we omit the words «a 2-dimensional».

We call a simple closed circle $C \subset T_2$ cut, if the complement $T_2 \backslash C$ consists of two components, and not cut, if the complement $T_2 \backslash C$ consists of the unique component.

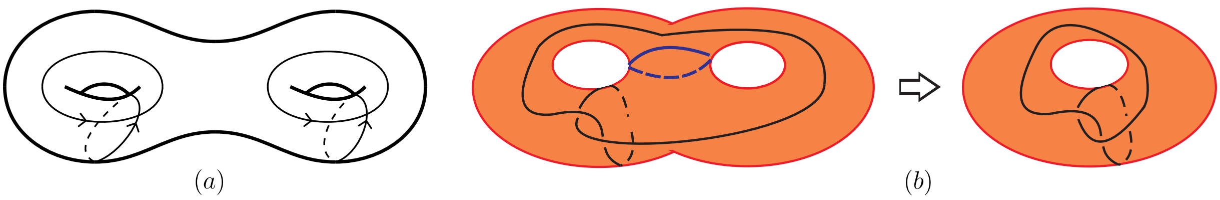

Fig. 1. (a) A surface T_2 endowed with oriented pairs «meridian-longitude» of its handles, (b) destabilization of the surface T_2 of genus 2

For any oriented not cut circle $C \subset T_2$ and two fixed oriented pairs «meridian-longitude» of handles of the surface T_2 (within the paper, see Fig. 1(a)), the numbers a and c (respectively, b and d) are calculated as intersection numbers of the circle C and the corresponding meridian (respectively, longitude) of the surface T_2. Then the circle C is associated with the ordered set of four numbers (a, b, c, d), where a, b, c, d are called the coordinates of the circle C. In a fix basis, i.e. for the fixed four circles, homology classes of which form a basis in the first homology group (for instance, see Fig. 1(a)), the homology class of the circle C is uniquely defined by the coordinates of C. Since direction of orientation of C is arbitrary, the coordinates (a, b, c, d) and $(-a, -b, -c, -d)$ are considered to be equal. The signs of the numbers a, b, c, d are positive, if the the direction of orientation of C coincides with the direction of the corresponding longitude or meridian. Note that, in contrast to the case of the torus T, where the greatest common divisor $gcd(a, b) = 1$, there exist not cut circles such that $gcd(a, b) \neq 1$ or $gcd(c, d) \neq 1$. For instance, we can consider the circle having the coordinates $(2, 1, 0, -2)$.

Consider a surface T_2 and an interval $I = [0, 1]$. A 3-dimensional manifold homeomorphic to the direct product $T_2 \times I$ is called a thickened surface of genus 2.

Denote by $L \subset T_2 \times I$ an m-component link in $T_2 \times I$, which is defined as a smooth embedding of m simple closed circles, which form a not connected 1-dimensional manifold, in the interior of $T_2 \times I$ such that the images of the circles do not intersect each other. Note that 1-component link is said to be a knot. Two links $L_1 \subset T \times I$ and $L_2 \subset T \times I$ are called equivalent, if there exists a homeomorphism of $T \times I$ onto itself that takes L_1 to L_2.

As in the classical case, links in $T_2 \times I$ can be presented by their diagrams. A diagram $D \subset T_2$ of a link $L \subset T_2 \times I$ is defined by analogy with the diagram of the classical link except that the link is projected into the surface T_2 instead of the plane. For each component of L, we refer to the part of D associated with this component as the component of D.

Assume that $D \subset T_2$ is a link diagram. A not cut circle $C \subset T_2$ is called a cancellation circle for the pair (D, T_2), if C and D do not intersect each other. In order to perform destabilization of the surface T_2, it is sufficient to cut T_2 along a cancellation circle C and glue each obtained component of the boundary by a disk D^2. Fig. 1(b) presents a torus T as a result of destabilization of the surface T_2 of genus 2.

Let us describe the following types of links in $T_2 \times I$ (compare with the types of link projections in the surface T_2 presented in [4] and types of knots in $T_2 \times I$ presented in [5]).

A link $L \subset T_2 \times I$ is called essential, if any diagram of L admits no destabilization. In other words, any annulus $\mathcal{A}$, which is isotopic to $C \times I \subset T_2 \times I$, where $C \subset T_2$ is a not cut circle, has nonempty intersection with L.

A link $L \subset T_2 \times I$ is called trivial, if L admits a diagram without crossings.

A link $L \subset T_2 \times I$ is said to be composite, if at least one of the following three conditions (a), (b), or (c) is satisfied.

(a) L is a connected sum of an essential link $L_1 \subset T_2 \times I$ having m_1 components and a nontrivial link $L_2 \subset S^3$ having m_2 components, which is defined by analogy with the classical connected sum of two classical links in the 3-dimensional sphere S^3. Namely, in $T_2 \times I$ (respectively, S^3), remove an open 3-dimensional ball B^3 that intersects L_1 (respectively, L_2) by an unknotted arc. As a result, the link L_i is transformed to the union of a knotted arc and $m_i - 1$ closed circles, $i = 1, 2$. Then, glue the resulting 3-dimensional manifolds into one new $T_2 \times I$ by a homeomorphism that identifies the obtained spherical boundaries such that endpoints of different knotted arcs are glued pairwise.

(b) L is a circular connected sum of essential links $L_1 \subset T_2 \times I$ having m_1 components and $L_2 \subset T \times I$ having m_2 components, which is defined by analogy with the circular connected sum introduced by S.V. Matveev in [17]. Namely, consider L_1 and L_2 to be such that there exist annuli $\mathcal{A}_1 \subset T_2 \times I$ and $\mathcal{A}_2 \subset T \times I$, where $\mathcal{A}_i$ is isotopic to $C_i \times I$ (here C_i is a not cut circle in T_2 or T, respectively), and $\mathcal{A}_i$ intersects L_i transversally at exactly one point, $i = 1, 2$. Cut $T_2 \times I$ along $\mathcal{A}_1$ and $T \times I$ along $\mathcal{A}_2$. As a result, the link L_i is transformed to the union of a knotted arc and $m_i - 1$ closed circles, $i = 1, 2$. Then, glue the resulting thickened surfaces (a thickened torus $T^{oo} \times I$ with two holes and a thickened annulus $\mathcal{A} \times I$) into one new $T_2 \times I$ by a homeomorphism that identifies the obtained annular boundaries such that endpoints of different knotted arcs are glued pairwise.

(c) L is a connected sum of two nontrivial links $L_i \subset T \times I$ having m_i components defined as follows. In each $T \times I$, remove a thickened disk $D^2 \times I$, where $D^2 \subset T$, that intersects a link by an unknotted arc. As a result, the link L_i is transformed to the union of a knotted arc and $m_i - 1$ closed circles, $i = 1, 2$. Then, glue the resulting thickened surfaces (two copies of a thickened torus $T^o \times I$ with a hole) into one new $T_2 \times I$ by a homeomorphism that identifies the obtained annular boundaries such that endpoints of different knotted arcs are glued pairwise.

For all three cases (a), (b), and (c), note that one of two terms in the sum can be a knot, since the result is a link anyway.

A link $L \subset T_2 \times I$ is called split, if there exists an embedded surface in the thickened surface T_2 (a 2-dimensional sphere, a torus T, or a surface T_2, which is parallel to the boundary of $T \times I$), which does not intersect L and cuts the thickened surface T_2 into two parts such that each part contains at least one component of L.

A link $L \subset T_2 \times I$ is called prime, if L is essential, not composite, not split and contains more than one component.

Let us explain the interest to tabulation of the prime links only. Indeed, nonessential links correspond to links that are presented in already existing tables of links in the 3-dimensional sphere S^3 [19], [6], thickened annulus $\mathcal{A} \times I$ (solid torus), or thickened torus $T \times I$ [?]. Here we note that today there exist no classification of links in the solid torus, but, as well as in the case of knots, we consider the construction of such a classification as an independent problem, which is beyond the scope of our interests in this article. In their turn, composite links correspond to links, which can be obtained using already known knots and links by connected sums described in types (a) – (c). Finally, a split link can be considered as a trivial union of already tabulated knots and links, while a link having the unique component is a knot.

2. Kauffman Bracket Frame $\mathfrak{F}(\cdot)$

Recall a definition of the Kauffman bracket frame $\mathfrak{F}(\cdot)$ [5] obtained as a simplification of the surface bracket polynomial $\langle\cdot\rangle$ proposed in [8], which generalises the Kauffman bracket [14] (see also [13] for the original version called the Jones polynomial). The Kauffman bracket frame $\mathfrak{F}(\cdot)$ is sufficient to prove that almost all tabulated links are not equivalent, see Subsection 3.2. Moreover, in Subsection 3.3, we apply this invariant as a tool to show both impossibility to realize any of tabulated links as a link in the thickened surface having smaller genus and impossibility to represent any of tabulated links as a connected sum.

Assume that $D \subset T_2$ is a diagram in the surface T_2 of a link L in the thickened surface $T_2 \times I$.

Following the rule presented in the center of Fig. 2(a), each angle of each crossing of D is endowed with a marker A or B. Define each state s of the diagram D by a combination of ways to smooth each crossing of D such as to join together two angles endowed with the same markers, A or B, see Fig. 2(a) on the left and right, respectively. Obviously, for the diagram D having n crossings, there exist exactly 2^n states of D.

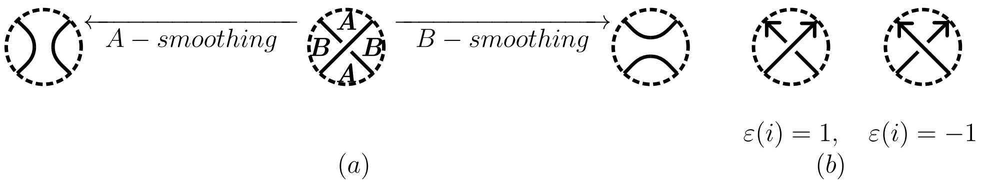

Fig. 2. (a) A- and B-smoothings of a crossing, (b) rules to define the sign $\varepsilon(i)$ of the i-th crossing

Associate the union of disjoint not cut circles in each state s_i, $i = 1, 2, 3, \ldots, 2^n$, with a product of the corresponding variables y_j, which take values in the coordinates (a_j, b_j, c_j, d_j) of the not cut circles that form the union associated with s_i, see Table 1. Note that Table 1 includes only the coordinates (a_j, b_j, c_j, d_j) that are sufficient for the links and basis considered in the present paper, while some new values of the variables y_j can be obtained when considering other links and / or basis.

Table 1

Values of the variables y_j in terms of the coordinates (a_j, b_j, c_j, d_j) of not cut circles in T_2

$y_1 = (0,0,0,1)$	$y_{16} = (0,1,2,0)$	$y_{31} = (1,1,0,-1)$	$y_{46} = (2,0,0,-1)$
$y_2 = (0,0,1,0)$	$y_{17} = (0,1,2,1)$	$y_{32} = (1,-1,0,1)$	$y_{47} = (2,0,1,-1)$
$y_3 = (0,0,1,1)$	$y_{18} = (0,-1,2,1)$	$y_{33} = (-1,1,0,1)$	$y_{48} = (2,1,0,0)$
$y_4 = (0,0,1,-1)$	$y_{19} = (0,1,2,-2)$	$y_{34} = (1,1,0,-2)$	$y_{49} = (2,-1,0,0)$
$y_5 = (0,0,2,1)$	$y_{20} = (0,2,0,-1)$	$y_{35} = (1,1,1,0)$	$y_{50} = (2,-1,0,1)$
$y_6 = (0,1,0,0)$	$y_{21} = (0,2,1,-1)$	$y_{36} = (1,-1,1,0)$	$y_{51} = (-2,1,0,1)$
$y_7 = (0,1,0,1)$	$y_{22} = (0,-2,2,1)$	$y_{37} = (-1,1,1,0)$	$y_{52} = (2,1,0,2)$
$y_8 = (0,1,0,-1)$	$y_{23} = (1,0,0,0)$	$y_{38} = (1,1,1,1)$	$y_{53} = (2,-1,0,2)$
$y_9 = (0,1,0,-2)$	$y_{24} = (1,0,0,1)$	$y_{39} = (1,1,1,-1)$	$y_{54} = (2,1,1,1)$
$y_{10} = (0,1,1,0)$	$y_{25} = (1,0,0,-1)$	$y_{40} = (1,1,-1,-1)$	$y_{55} = (2,1,1,-1)$
$y_{11} = (0,1,-1,0)$	$y_{26} = (1,0,1,0)$	$y_{41} = (1,-1,1,1)$	$y_{56} = (2,-1,1,1)$
$y_{12} = (0,1,1,1)$	$y_{27} = (1,0,1,1)$	$y_{42} = (1,-1,1,-1)$	$y_{57} = (2,-1,1,-1)$
$y_{13} = (0,1,1,-1)$	$y_{28} = (-1,0,1,1)$	$y_{43} = (1,-1,-1,1)$	$y_{58} = (2,-1,2,2)$
$y_{14} = (0,1,-1,1)$	$y_{29} = (1,1,0,0)$	$y_{44} = (1,-1,2,0)$	$y_{59} = (2,-2,1,1)$
$y_{15} = (0,-1,1,1)$	$y_{30} = (1,-1,0,0)$	$y_{45} = (1,-1,2,1)$	$y_{60} = (2,-2,2,1)$

In order to calculate the generalised Kauffman bracket polynomial, it is necessary to use the so-called writhe, which is defined for the oriented diagram only and is aimed to catch the first Reidemeister move Ω_1, i.e. addition and cancellation of a loop. Since our aim is to tabulate not oriented link diagrams, then we consider orientation just as a tool to calculate the writhe. Therefore, we orient each component of the link diagram D in any of two possible ways and define the writhe $w(D)$ to be the sum $w(D) = \sum_{i=1}^{\mathfrak{n}} \varepsilon(i)$ over all $\mathfrak{n}$ crossings of D that are self-crossings of the components of D, where $\varepsilon(i)$ is the sign of the i-th crossing of D defined by the rules presented in Fig. 2(b). Note that, in order to calculate the writhe $w(D)$, we use an arbitrary orientation and consider only self-crossings of the components, because Ω_1 is not defined for crossings that are intersections of different components.

The formula of the generalised Kauffman bracket polynomial is the following:

$$\widetilde{\mathcal{X}}\,(D) = (-a)^{-3w(D)} \sum_{i=1}^{2^n} a^{\alpha(s_i)-\beta(s_i)}(-a^2 - a^{-2})^{\gamma(s_i)} \prod_j y_j^{\delta_j(s_i)}. \tag{1}$$

Here $\alpha(s_i)$ and $\beta(s_i)$ are the numbers of markers A and B in the given state s_i, while $\gamma(s_i)$ is the number of cut circles in the surface obtained by smoothing according to the state s_i, and $\delta_j(s_i)$ is the number of not cut circles having the coordinates (a_j, b_j, c_j, d_j) associated with the variable y_j, see Table 1. The sum is taken over all 2^n states.

We order terms of (1) in nondecreasing order of the powers of the variable a and collect terms having the same power of the variable a, i.e. represent (1) as $\sum_m P_m a^m$, where P_m is a polynomial in the variables y_j. Then, we associate the polynomial (1) with an ordered set of nonzero polynomials P_m in the variables y_j, which is called the Kauffman bracket frame $\mathfrak{F}\,(\cdot)$ [5]. For instance, $\widetilde{\mathcal{X}}\,(D) = -a^{-12}y_{62} - 2a^{-8}y_{12} - a^{-8}y_3y_7 - a^{-6}y_4y_7$ is associated with $\mathfrak{F}\,(D) = (-y_{62}, -2y_{12} - y_3y_7, -y_4y_7)$.

We call the Kauffman bracket frames $\mathfrak{F}\,(D_1)$ and $\mathfrak{F}\,(D_2)$ inverted to each other, if the elements of $\mathfrak{F}\,(D_1)$ are the corresponding elements of $\mathfrak{F}\,(D_2)$, where the polynomials P_m are taken in reverse order. Such a transformation of the Kauffman bracket frames is said to be an inversion. For instance, $\mathfrak{F}\,(D_1) = (-y_{62}, -2y_{12} - y_3y_7, -y_4y_7)$ is inverted to $\mathfrak{F}\,(D_2) = (-y_4y_7, -2y_{12} - y_3y_7, -y_{62})$.

By analogy with Lemma 1 [5], the following statement can be proved.

Lemma 1. *The Kauffman bracket frame $\mathfrak{F}\,(\cdot)$ considered up to inversion, multiplication by -1, and changes of variables y_j (in the last case, some new values of the variables y_j are available) associated with changes of the corresponding the coordinates of not cut circles in the surface T_2 generated by orientation preserving homeomorphisms of T_2 is an invariant of links in the thickened surface $T_2 \times I$ of genus 2.*

3. Table of Prime Links

Theorem 1. *In the thickened surface of genus 2, there exist no more than 38 pairwise not equivalent prime links having diagrams with no more than 4 crossings, see Fig. 3.*

Subsections 3.1 – 3.3 describe three steps of the proof of Theorem 1.

3.1. Construction of a Preliminary List of Diagrams on Prime Projections

Each projection obtained in [4] is converted to the set of corresponding diagrams by enumeration of all possible ways to replace each crossing of a projection by either an

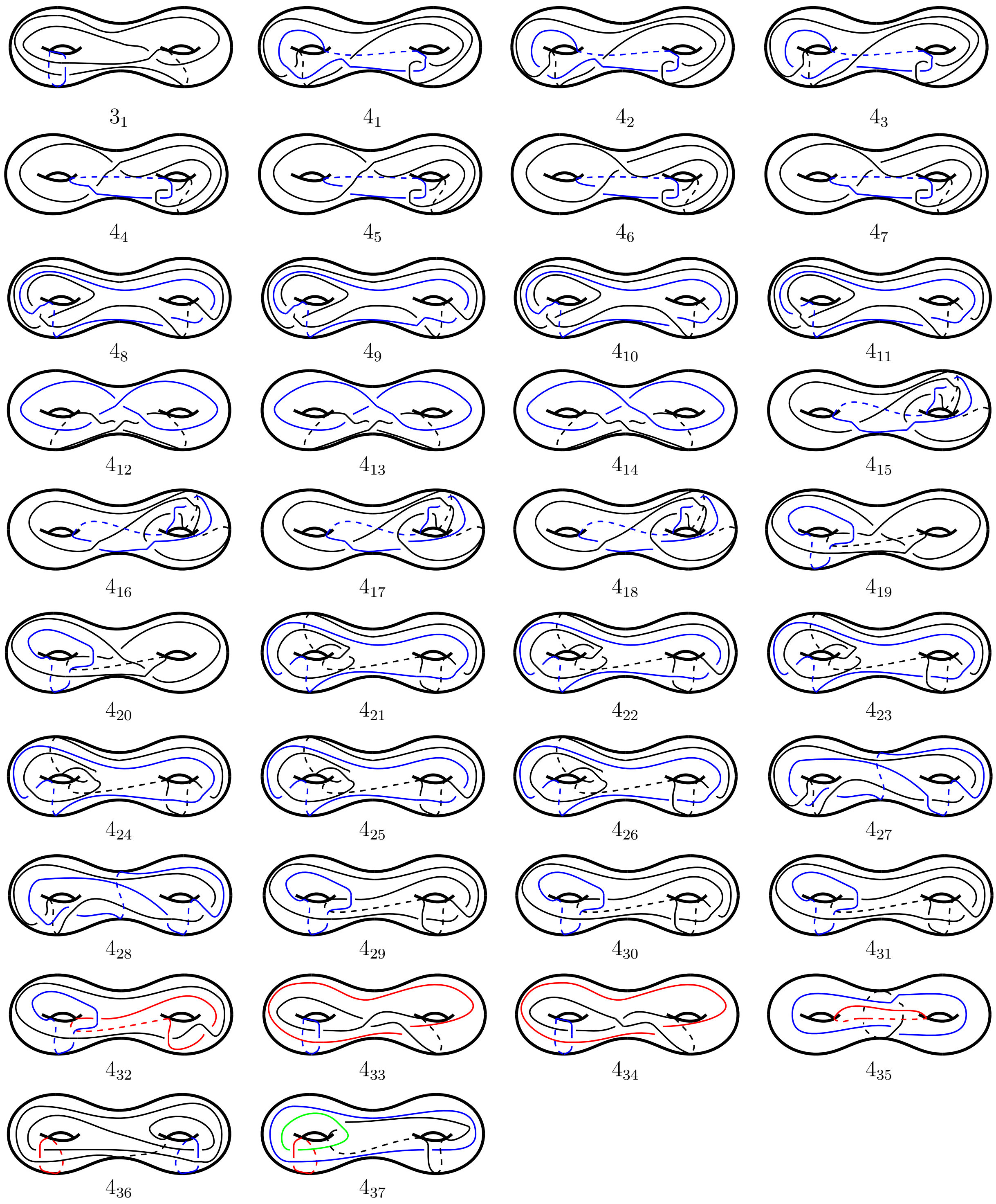

Fig. 3. Diagrams 3_1, 4_1 – 4_{37} of prime links in the thickened surface $T_2 \times I$

over- or undercrossing of a diagram. It is easy to see that each projection with n crossings leads to 2^n diagrams. Therefore, direct construction by tabulated 14 projections [4] leads to $2^3 + 14 \cdot 2^4 = 232$ diagrams. However, such enumeration is significantly reduced by the following ideas, see [1] and [2].

First, any diagram is converted to the equivalent one by simultaneous switching of all crossings. Therefore, the set of diagrams on the projection is halved by fixing the type of a crossing of each projection.

Second, both crossings of a bigon should have the same type. Otherwise, the number of crossings is reduced by the second Reidemeister move Ω_2.

Third, each component of a diagram contains both types of crossings (under- and over-crossings) with the union of other components, otherwise we have a split diagram.

3.2. Formation of Equivalence Classes of the Constructed Diagrams

In order to compare the constructed diagrams, we use the software «Wolfram Mathematica» to calculate the Kauffman bracket frames for each diagram obtained in Subsection 3.1. Therefore, we find more than 17 groups formed by diagrams with the same Kauffman bracket frames. Each group contains from 2 to 4 diagrams, and there exist no groups having diagrams on not equivalent projections. Then, by hand, we construct sequences of simultaneous switching of all crossings, homeomorphisms of the surface of genus 2 onto itself and Reidemeister moves in order to prove that diagrams with the same Kauffman bracket frame are equivalent. At this step, we also use another representation of the considered links called virtual link diagrams [16]. Fortunately, with the exclusion of the pair (4_{30}, 4_{31}) discussed below, the list of the Kauffman bracket frames presented below is sufficient to prove that all tabulated links are pairwise not equivalent.

Indeed, represent each y_j by the same value y and take into account only the first and the last elements of each Kauffman bracket frame with respect to both inversion and simultaneous change of the sign. As a result, all the Kauffman bracket frames presented in the list above are divided into 15 groups having from 1 to 16 elements. In the framework of each of the obtained group, it is easy to see that all the Kauffman bracket frames are pairwise not equivalent in the sense of powers and coefficients of y in the inner elements.

Moreover, the list of the Kauffman bracket frames is sufficient to prove that all tabulated links admit no destabilisations, see Subsection 3.3.

As mentioned above, there exists a pair of the tabulated diagrams having the same Kauffman bracket frames: (4_{30}, 4_{31}). In order to show that the diagrams are not equivalent, it is sufficient to calculate and compare their generalised Alexander polynomials [15], which takes zero value for the diagram 4_{31} only.

$\mathfrak{F}(3_1) = (-y_{13}, -y_{15} - y_{26}y_{32} - y_{24}y_{35}, -y_{13} - y_{54} - y_{56}, -y_{12})$

$\mathfrak{F}(4_1) = (y_6y_8, 2y_1 + y_{18}y_6, y_{22} + y_{23}y_{25} + y_5, y_{46} - y_{18}y_6, -y_{22} - y_5, -y_{46})$

$\mathfrak{F}(4_2) = (y_1, y_{23}y_{25} + y_5 + y_6y_8, y_1 + y_{46} + y_{18}y_6, y_{22} - y_5, -y_{46} - y_{18}y_6, -y_{22})$

$\mathfrak{F}(4_3) = (y_{23}y_{25}, 2y_1 + y_{46}, y_{22} + y_5 + y_6y_8, -y_{46} + y_{18}y_6, -y_{22} - y_5, -y_{18}y_6)$

$\mathfrak{F}(4_4) = (-y_3 - y_1y_{32} - y_{34} - y_2y_{36}, -y_{30} - y_1y_{31} - y_{44}, -y_{29} + y_2y_{36}, y_{30} + y_{44}, y_{29})$

$\mathfrak{F}(4_5) = (-y_{34}, -y_{30} - y_1y_{31} - y_{44}, -y_{29} - y_1y_{32} - y_2y_{36}, -y_{30} + y_{44}, y_{29} + y_2y_{36}, y_{30})$

$\mathfrak{F}(4_6) = (-y_1y_{31}, -y_{29} - y_1y_{32} - y_{34}, -2y_{30} - y_{44}, y_{29} - y_2y_{36}, y_{30} + y_{44}, y_2y_{36})$

$\mathfrak{F}(4_7) = (-y_1y_{32}, -2y_{30} - y_1y_{31}, -y_{29} - y_{34} - y_2y_{36}, y_{30} - y_{44}, y_{29} + y_2y_{36}, y_{44})$

$\mathfrak{F}(4_8) = (y_1y_3y_{32}, -2y_1y_3 - y_1y_{29} - 3y_{32} - y_{33}, -2y_{31} - y_{45} - y_{26}y_{56}, -y_{15}y_{26})$

$\mathfrak{F}(4_9) = (y_{32}, y_1y_3 - 2y_{32}, -2y_1y_{30} - y_{31} - y_{45} - y_{26}y_{56}, -y_{15}y_{26} - y_1y_{29} - y_{32} - y_{33}, -y_{31})$

$\mathfrak{F}(4_{10}) = (-y_{32} - y_{33}, -y_1y_{30} - y_{31} - y_{45} - y_{26}y_{56}, -y_{15}y_{26} - y_1y_{29} - y_{32}, -y_{31})$

$\mathfrak{F}(4_{11}) = (-y_{33}, -y_1y_{30} - y_{31} - y_{45}, -y_{15}y_{26} - y_1y_{29} - 2y_{32}, -y_{31} - y_{26}y_{56})$

$\mathfrak{F}(4_{12}) = (y_{41}, 4y_{41}, y_3y_{30} + y_{38} + y_{40} + y_{42} + y_{43} + y_{26}y_8, y_{29}y_3 + 2y_{39} + y_{30}y_4, y_{29}y_4)$

$\mathfrak{F}(4_{13}) = (y_{41}, y_3y_{30} + y_{40} + y_{41} + y_{43}, y_{29}y_3 + y_{39} + y_{30}y_4 + 3y_{41}, y_{38} + y_{29}y_4 + y_{42} + y_{26}y_8, y_{39})$

$\mathfrak{F}(4_{14}) = (y_3y_{30}, y_{29}y_3 + y_{30}y_4 + 2y_{41}, y_{38} + y_{29}y_4 + y_{40} + y_{41} + y_{42} + y_{43}, 2y_{39} + 2y_{41}, y_{26}y_8)$

$$\mathfrak{F}(4_{15}) = (y_{32} + y_{33} + y_{45}, 2y_1y_{30} + y_{31} + 2y_{45}, y_{15}y_{26} + y_{32} + y_{33} + y_1y_{34}, y_{31})$$
$$\mathfrak{F}(4_{16}) = (y_{32}, y_1y_{30} + y_{31} + y_{45}, y_{15}y_{26} + 2y_{33} + y_1y_{34} + y_{45}, y_1y_{30} + y_{31} + y_{45}, y_{32})$$
$$\mathfrak{F}(4_{17}) = (-y_1y_3 3y_1y_3 2y_{32} + 2y_{33} + y_1y_{34} + y_{45}, 2y_{31} + 2y_{45}, y_{15}y_{26})$$
$$\mathfrak{F}(4_{18}) = (y_{32} + y_{33} + y_1y_{34}, 2y_1y_{30} + 2y_{31} + y_{45}, y_{15}y_{26} + y_{32} + y_{33} + y_{45}, y_{45})$$
$$\mathfrak{F}(4_{19}) = (y_3, y_{21} + 2y_3 + y_{47}, y_{21} + y_{23}y_{28} + y_3 + y_{47} + y_{59} + y_{15}y_6, y_{23}y_{28} + y_3 + y_{59} + y_{15}y_6, y_3)$$
$$\mathfrak{F}(4_{20}) = (y_3, y_{23}y_{28} + 2y_3 + y_{15}y_6, y_{21} + y_{23}y_{28} + 2y_3 + y_{47} + y_{15}y_6, y_{21} + y_3 + y_{47} + y_{59}, y_{59})$$
$$\mathfrak{F}(4_{21}) = (-y_{15}y_3 - y_{58} - y_1y_8, -y_3y_{56} - 2y_6 - y_1^2y_6 - y_9, -2y_{49} - y_1y_7 - y_1y_8, -y_1y_{51})$$
$$\mathfrak{F}(4_{22}) = (y_6, -2y_6 - y_1^2y_6, -y_{15}y_3 - y_{49} - y_{58} - y_1y_7 - 2y_1y_8, -y_1y_{51} - y_3y_{56} - y_6 - y_9, -y_{49})$$
$$\mathfrak{F}(4_{23}) = (-y_1y_8, -y_6 - y_1^2y_6 - y_9, -y_{15}y_3 - y_{49} - y_{58} - y_1y_7 - y_1y_8, -y_1y_{51} - y_3y_{56} - y_6, -y_{49})$$
$$\mathfrak{F}(4_{24}) = (-y_{15}y_3, -y_3y_{56} - 2y_6, -2y_{49} - y_{58} - y_1y_7 - y_1y_8, -y_1y_{51} - y_1^2y_6 - y_9, -y_1y_8)$$
$$\mathfrak{F}(4_{25}) = (-y_{15}y_3 - y_{49} - y_1y_7, -y_1y_{51} - y_3y_{56} - 2y_6 - y_1^2y_6, -y_{49} - y_{58} - 2y_1y_8, -y_9)$$
$$\mathfrak{F}(4_{26}) = (-y_6, -y_{15}y_3 - y_{49} - y_1y_7 - y_1y_8, -y_1y_{51} - y_3y_{56} - y_6 - y_1^2y_6 - y_9, -y_{49} - y_{58} - y_1y_8)$$
$$\mathfrak{F}(4_{27}) = (y_1y_{49}, 2y_{50} + y_1y_6, y_{27}y_{37} + 2y_7 + 2y_8, 2y_{17} + y_1y_6, y_5y_6)$$
$$\mathfrak{F}(4_{28}) = (y_7, y_{17} + y_{50} + y_1y_6, y_{27}y_{37} + y_1y_{49} + y_5y_6 + 2y_8, y_{17} + y_{50} + y_1y_6, y_7)$$
$$\mathfrak{F}(4_{29}) = (-2 + y_1^2, y_1y_{23}y_{25} + y_1y_6y_8, -6 + y_1^2 + y_{23}^2 + y_{25}^2 + y_3^2 + y_6^2 + y_8^2, \; -4 + y_{41}^2)$$
$$\mathfrak{F}(4_{30}) = (-1, -5 + y_1^2 + y_{25}^2 + y_3^2 + y_8^2, y_1y_{23}y_{25} + y_1y_6y_8, -5 + y_1^2 + y_{23}^2 + y_{41}^2 + y_6^2, -1)$$
$$\mathfrak{F}(4_{31}) = (-1, -5 + y_1^2 + y_{23}^2 + y_3^2 + y_6^2, y_1y_{23}y_{25} + y_1y_6y_8, -5 + y_1^2 + y_{25}^2 + y_{41}^2 + y_8^2, -1)$$
$$\mathfrak{F}(4_{32}) = (-y_1, -y_{23}y_{25} - y_6y_8, y_{23}y_{25} + y_5 + y_6y_8, y_1 + y_{20} + y_{46}, y_{60})$$
$$\mathfrak{F}(4_{33}) = (y_{13}, -3y_{13}, -y_{12} - y_{26}y_{32} - y_{24}y_{35} - y_{55} - y_{57} - y_4y_6, -y_{13} - y_{14} - y_{54} - y_{56}, -y_{15})$$
$$\mathfrak{F}(4_{34}) = (-y_{12} - y_{55} - y_{57}, -2y_{13} - y_{14} - y_{54} - y_{56}, -y_{15} - y_{26}y_{32} - y_{24}y_{35} - y_4y_6, -y_{13})$$
$$\mathfrak{F}(4_{35}) = (y_{41}, y_{38} + y_{40} + y_{42} + y_{43}, y_{10}y_{24} + y_{11}y_{25} + y_{29}y_3 + 2y_{39} + y_{30}y_4, y_{38} + y_{40} + y_{42} + y_{43}, y_{41})$$
$$\mathfrak{F}(4_{36}) = (y_6, y_1y_{10} + y_{13}y_4 + y_{52} + y_{53}, 2y_{16} + 2y_{19} + 2y_6, y_{10}y_2 + y_{13}y_4 + y_{48} + y_{49}, y_6)$$
$$\mathfrak{F}(4_{37}) = (-1, -7 + y_{25}^2 + y_3^2 + y_{31}^2 + y_8^2, -7 + y_1^2 + y_{15}^2 + y_{27}^2 + y_{41}^2, -1)$$

3.3. On Primality of the Tabulated Links

In order to prove that a link is prime, it is sufficient to prove that the link is essential, not composite and not split. As regards to the latter property, we note that when constructing the table, we remove obviously split links, while the first two properties can be shown as follows.

First, we use the following obvious statement in order to show that each of the links presented in Fig. 3 admits no destabilisation, i.e. is essential.

Lemma 2. *Assume that the Kauffman bracket frame $\mathfrak{F}(D)$ of a connected link diagram $D \subset T_2$ contains terms corresponding to 4 not cut circles having not equivalent coordinates (a_k, b_k, c_k, d_k), $k \in \{1, 2, 3, 4\}$, such that the system of 4 linear equations of the form*

$$b_k \cdot a - a_k \cdot b + d_k \cdot c - c_k \cdot d = 0, \quad k \in \{1, 2, 3, 4\},$$

where a, b, c, d are the variables and a_k, b_k, c_k, d_k are known coefficients, has only zero solution. Then the link diagram D admits no destabilisation.

Lemma 2 is proved by analogy with Lemma 3 [5].

Following Lemma 2, we associate each tabulated diagram D with a set of 4 not cut circles involved in the Kauffman bracket frame $\mathfrak{F}(D)$, which is sufficient to prove that there exists no cancellation circle for the corresponding link $L \subset T_2 \times I$. To this end, we use the fact that the Kauffman bracket frames of the tabulated diagrams based on the same projection are formed by the same set of the variables y_j. In addition, it turned out that the Kauffman bracket frames of some tabulated diagrams based on different projections (for instance, 3_1 and 4_{11} [4]) involve the same sets of the variables y_j, which are sufficient to prove that the corresponding links are essential. More precisely, Table 2 gives the sets of the variables y_j involved in the Kauffman bracket frames that are sufficient to prove that the corresponding tabulated links are essential.

Table 2

Sets of tabulated links associated with sets of 4 variables y_j

3_1, 4_{33}, 4_{34}: $y_{12}, y_{32}, y_{35}, y_{54}$	$4_8 - 4_{11}$, $4_{15} - 4_{18}$: $y_1, y_{30}, y_{31}, y_{45}$
$4_{29} - 4_{31}$, 4_{37}: y_1, y_8, y_{25}, y_{41}	$4_4 - 4_7$: $y_1, y_{29}, y_{30}, y_{44}$
$4_{12} - 4_{14}$, 4_{35}: y_3, y_4, y_{30}, y_{40}	4_{19}, 4_{20}: $y_3, y_{21}, y_{23}, y_{47}$
$4_{21} - 4_{26}$: y_1, y_3, y_{15}, y_{58}	4_{27}, 4_{28}: y_1, y_6, y_{37}, y_{50}
$4_1 - 4_3$, 4_{32}: y_1, y_5, y_6, y_{25}	4_{36}: y_1, y_6, y_{16}, y_{49}

In order to show that all 38 tabulated links are not composite, it is sufficient to prove that each link can not be given by a connected sum of the type (a), (b), or (c) under the hypothesis that the sum of complexities of the terms that form the connected sum is not more than the complexity of the connected sum. More precisely, we assume that there exists no a pair of nontrivial links such that the connected sum of these links admits a diagram having number of crossings, which is smaller than a minimal sum of numbers of crossings of the diagrams corresponding to both links formed the pair. In the framework of the considered problem on classification of links that allow diagrams with either 3 or 4 crossings, the impossibility to represent each of the tabulated essential link as a connected sum of the types (a), (b), or (c) is obvious, even taking into account that exactly one of the terms in the sum can be a knot.

Indeed, for the connected sums of the types (a) and (b), we note that all essential knots and links in $T_2 \times I$ have diagrams with at least 3 crossings. Therefore, the second term in the sum should admit a diagram with no more than 1 crossing, while all nontrivial knots and links in S^3 have diagrams with at least 3 and 2 crossings, respectively, and all essential knots and links in $T \times I$ have diagrams with at least 2 crossings.

As regards to the connected sum of the type (c), we note that such a sum can be obtained only in the case then the terms are given by the unique essential link 2_1 [2] in $T \times I$ and / or knot 2_1 [1] having diagrams with no more than 2 crossings. Since at least one of two terms should be a link and taking into account the specific form of the link 2_1, we note that the Kauffman bracket frame $\mathfrak{F}(\cdot)$ of the obtained connected sum admits no terms associated with not cut circles having coordinates of the form (a, b, c, d), where both pairs (a, b) and (c, d) include odd numbers. As a result, all 38 tabulated links can not be given by the connected sum of the type (c), since the Kauffman bracket frame $\mathfrak{F}(\cdot)$ of each tabulated link contains a variable that belongs to the set $\{y_8,\ y_{10},\ y_{15},\ y_{32}\}$.

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References

1. Akimova A. A., Matveev S. V. Classification of Genus 1 Virtual Knots Having at Most Five Classical Crossings. *Journal of Knot Theory and Its Ramifications*, 2014, vol. 23, no. 6, art. no. 1450031. DOI: 10.1142/S021821651450031X

2. Akimova A. A., Matveev S. V., Tarkaev V. V. Classification of Links of Small Complexity in the Thickened Torus. *Proceedings of the Steklov Institute of Mathematics*, 2018, vol. 303 (suppl. 1), no. 12, pp. 12–24. DOI: 10.1134/S008154381809002X

3. Akimova A. A., Matveev S. V., Tarkaev V. V. Classification of Prime Links in the Thickened Torus Having Crossing Number 5. *Journal of Knot Theory and Its Ramifications*, 2020, vol. 29, no. 3, 27p. DOI: 10.1142/S0218216520500121

4. Akimova A. A. Tabulation of Prime Projections of Links in the Thickened Surface of Genus 2 with no More than 4 Crossings *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2020, vol. 12, no. 3, pp. 5–14.

5. Akimova A. A. Classification of Prime Knots in the Thickened Surface of Genus 2 Having Diagrams with at Most 4 Crossings. *Journal of Computational and Engineering Mathematics*, 2019, vol. 7, no. 1, pp. 32–46. DOI: 10.14529/jcem200103

6. Bar-Natan D. *The Hoste-Thistlethwaite Link Table*, available at: http://katlas.org /wiki/Main_Page (accessed on July 25, 2020).

7. Drobotukhina Yu. V. Classification of Links in $\mathbb{R}P^3$ with at most Six Crossings. *Zapiski Nauchnogo Seminara LOMI*, 1991, vol. 193, pp. 39–63. (in Russian)

8. Dye H. A., Kauffman L. H. Minimal Surface Representations of Virtual Knots and Links. *Algebraic and Geometric Topology*, 2005, vol. 5, pp. 509–535.

9. Gabrovšek B., Mroczkowski M. Knots in the Solid Torus up to 6 Crossings. *Journal of Knot Theory and Its Ramifications*, 2012, vol. 21, no. 11, 43 p. DOI: 10.1142/S0218216512501064

10. Gabrovšek B., Tabulation of Prime Knots in Lens Spaces, *Mediterranean Journal of Mathematics*, 2017, vol. 14, no. 2, art.no. 88, 24 p. DOI: 10.1007/s00009-016-0814-5

11. Green J. *A Table of Virtual Knots*, available at: http://katlas.math.toronto.edu/wiki/ (accessed on July 25, 2018).

12. Hoste J., Thistlethwaite M., Weeks J., The First 1,701,936 Knots. *The Mathematical Intelligencer*, 1998, vol. 20, no. 4, pp. 33–47.

13. Jones V. F. R. A Polynomial Invariant for Knots via von Neumann Algebra. *Bulletin of the American Mathematical Society*, 1985, vol. 12, no. 1, pp. 103–111.

14. Kauffman L. H. State Models and the Jones Polynomial. *Topology*, 1987, vol. 26, no. 3, pp. 395–407. DOI: 10.1016/0040-9383(87)90009-7

15. Kauffman L. H., Radford D.E. Bi-oriented Quantum Algebras, and a Generalized Alexander Polynomial for Virtual Links. arXiv:math/0112280

16. Kauffman L. H. Virtual Knot Theory. *European Journal of Combinatorics*, 1999, vol. 20, no. 7, pp. 663–691.

17. Matveev S. V. Prime Decompositions of Knots in $T \times I$. *Topology and its Applications*, 2011, vol. 159, no. 7, pp. 1820–1824.

18. Matveev S. V., Nabeeva L. R. Tabulating Knots in the Thickened Klein Bottle. *Siberian Mathematical Journal*, 2016, vol. 57, issue 3, pp. 542–548. DOI: 10.1134/S0037446616030174

19. Rolfsen D. Knots and Links. *Berkeley, CA: Publish or Perish*, 1976.

20. Stenlund E. *Classification of Virtual Knots*, available at: http://evertstenlund.se/knots/Virtual%20Knots.pdf (accessed on July 25, 2020).

21. Zinn-Justin P., Zuber J. B. Matrix Integrals and the Generation and Counting of Virtual Tangles and Links. *Journal of Knot Theory and Its Ramifications*, 2004, vol. 13, no. 3, pp. 325–355. DOI: 10.1142/S0218216504003172

22. Zinn-Justin P. *Alternating Virtual Link Database*, available at: https://www.lpthe.jussieu.fr/pzinn/virtlinks/ (accessed on July 25, 2020).

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ТАБУЛЯЦИЯ ПРИМАРНЫХ ЗАЦЕПЛЕНИЙ В УТОЛЩЕННОМ КРЕНДЕЛЕ РОДА 2, ИМЕЮЩИХ ДИАГРАММЫ С НЕ БОЛЕЕ ЧЕМ 4 ПЕРЕКРЕСТКАМИ

А. А. Акимова

В настоящей работе строится таблица примарных зацеплений в утолщенном кренделе рода 2, имеющих диаграммы с не более чем 4 перекрестками. Прежде всего, строится предварительный набор диаграмм на основе таблицы примарных проекций зацеплений на кренделе рода 2. Для того, чтобы удалить дубликаты и доказать, что все оставшиеся зацепления неэквивалентны, а также доказать, что все табличные зацепления не допускают дестабилизации, используется инвариант, называемый каркас скобки Кауфмана, который является упрощением обобщенного полинома скобки Кауфмана. Идея инварианта состоит в том, чтобы принимать во внимание только порядок и значения коэффициентов и игнорировать степени одной из переменных. На заключительном шаге доказывается, что ни одно из табулированных зацеплений не может быть представлено в виде связной суммы в рамках гипотезы, что наименьшее число перекрестков связной суммы зацеплений не меньше суммы наименьших чисел перекрестков слагаемых.

Ключевые слова: примарное зацепление; утолщенный крендель рода 2; табуляция; обобщенный полином скобки Кауфмана; каркас скобки Кауфмана.

Литература

1. Akimova, A. A. Classification of Genus 1 Virtual Knots Having at Most Five Classical Crossings / A. A. Akimova, S. V. Matveev // Journal of Knot Theory and Its Ramifications. – 2014. – V. 23, № 6. – Article: 1450031, 19 p.

2. Акимова, А. А. Классификация зацеплений малой сложности в утолщенном торе / А. А. Акимова, С. В. Матвеев, В. В. Таркаев // Труды института математики и механики УрО РАН. – 2017. – Т. 23, № 4. – С. 18–31.

3. Akimova, A. A. Classification of Prime Links in the Thickened Torus Having Crossing Number 5 / A. A. Akimova, S. V. Matveev, V. V. Tarkaev // Journal of Knot Theory and Its Ramifications. – 2020. – V. 29, № 3. – Article: 2050012, 27 p.

4. Akimova, A. A. Tabulation of Prime Projections of Links in the Thickened Surface of Genus 2 with no More than 4 Crossings / A. A. Akimova // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2020. – V. 12, №. 3. – P. 5–14.

5. Akimova, A. A. Classification of Prime Knots in the Thickened Surface of Genus 2 Having Diagrams with at Most 4 Crossings / A. A. Akimova // Journal of Computational and Engineering Mathematics. – 2020. – V. 7, № 1. – P. 32–46.

6. Bar-Natan, D. The Hoste-Thistlethwaite Link Table [Электронный ресурс]. – URL: http://katlas.org/wiki/Main_Page (дата обращения: 25.07.2018).

7. Дроботухина, Ю. В. Классификация зацеплений в $\mathbb{R}P^3$ с небольшим числом точек скрещивания / Ю. В. Дроботухина // Зап. научн. сем. ЛОМИ. – 1991. – Т. 193. – С. 39–63.

8. Dye, H. A. Minimal Surface Representations of Virtual Knots and Links / H. A. Dye, L. H. Kauffman // Algebraic and Geometric Topology. – 2005. – V. 5. – P. 509–535.

9. Gabrovšek, B. Knots in the Solid Torus up to 6 Crossings / B. Gabrovšek, M. Mroczkowski // Journal of Knot Theory and Its Ramifications. – 2012. – V. 21, № 11. – 43 p.

10. Gabrovšek, B. Tabulation of Prime Knots in Lens Spaces / B. Gabrovšek // Mediterranean Journal of Mathematics. – 2017. – V. 14, iss. 2. – Article: 88.

11. Green, J. A. Table of Virtual Knots [Электронный ресурс]. – URL: http://katlas.math.toronto.edu/wiki/ (дата обращения: 25.07.2018).

12. Hoste, J. The First 1,701,936 Knots / J. Hoste, M. Thistlethwaite, J. Weeks // The Mathematical Intelligencer. – 1998. – V. 20, №. 4. – P. 33–47.

13. Jones, V. F. R. A Polynomial Invariant for Knots via von Neumann Algebra / V. F. R. Jones // Bulletin of the American Mathematical Society. – 1985. V. 12, № 1. – P. 103–111.

14. Kauffman, L. H. State Models and the Jones Polynomial / L. H. Kauffman // Topology. – 1987. V. 26, № 3. – P. 395–407.

15. Kauffman, L. H., Radford D.E. Bi-oriented Quantum Algebras, and a Generalized Alexander Polynomial for Virtual Links / L. H. Kauffman // arXiv:math/0112280

16. Kauffman, L. H. Virtual Knot Theory / L. H. Kauffman // European Journal of Combinatorics. – 1999. V. 20, № 7. – P. 663–691.

17. Matveev, S. V. Prime Decompositions of Knots in $T \times I$ / S. V. Matveev // Topology and its Applications. – 2011. – V. 159, № 7. – P. 1820–1824.

18. Матвеев, С. В. Табулирование узлов в утолщенной бутылке Клейна / С. В. Матвеев, Л. Р. Набеева // Сибирский математический журнал. – 2016. – Т. 57, вып. 3. – С. 688–696.

19. Rolfsen, D. Knots and Links / D. Rolfsen. – Berkeley, CA: Publish or Perish, 1976.

20. Stenlund, E. Classification of Virtual Knots [Электронный ресурс]. – URL: http://evertstenlund.se/knots/Virtual%20Knots.pdf (дата обращения: 25.07.2020).

21. Zinn-Justin, P. Matrix Integrals and the Generation and Counting of Virtual Tangles and Links / P. Zinn-Justin, J. B. Zuber // Journal of Knot Theory and Its Ramifications. – 2004. – V. 13, № 3. – P. 325–355.

22. Zinn-Justin, P. Alternating Virtual Link Database [Электронный ресурс]. – URL: https://www.lpthe.jussieu.fr/pzinn/virtlinks/ (дата обращения: 25.07.2020).

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