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CALCULATION OF THE EIGENVALUES OF THE PROBLEMS GENERATED BY THE ARBITRARY EVEN ORDER STURM – LIOUVILLE OPERATORS

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The problem of calculating the eigenvalues of discrete semi-bounded differential operators is one of the important problems of numerical analysis. Despite of the simple formulation, for solving many problems that are encountered in practice, it is impossible to propose a single algorithm for calculating eigenvalues. Known methods of calculating the eigenvalues of linear differential operators are based on reducing problems to discrete models, using mainly grid methods or projection methods, which reduce the problem of finding the spectral characteristics of systems of linear algebraic equations. In view of the fact that poor separation of eigenvalues of matrices obtained from the corresponding systems of equations, the use of traditional methods of solving requires a very significant amount of calculations.This often leads to the need of solving the ill-posed problems. Generally, the choice of algorithms for approximate finding of eigenvalues of matrices is mainly determined by their type. This greatly narrows the possibilities of using computational methods to find their eigenvalues of matrices. It's important to note that the problem of finding the all points of the spectrum for high-order matrices does not have a satisfactory numerical solution yet.

Using the numerical method of regularized traces and the Galerkin method, we previously obtained the linear formulas for calculating approximate eigenvalues of discrete semi-bounded operators. They allow you to find approximate eigenvalues with an any ordinal number. However, there are no computational difficulties that occur in other methods. The comparison of the results of computational experiments has shown that the eigenvalues found by linear formulas using the Galerkin method, as well as the known eigenvalues of spectral problems, are in line with each other.

The article examines the possibility of using linear formulas obtained in the authors' articles to find the eigenvalues of the Sturm – Liouville operators of arbitrary even order. The considered examples show that the eigenvalues found from linear formulas and known asymptotic formulas are computationally the same.

Keywords: eigenvalues and eigenfunctions; discrete, self-adjoint, semibounded operators; the Galerkin method; asymptotic formula.

Introduction

Modern methods for calculating eigenvalues of linear differential operators are based on reducing problems to discrete models, using mainly the grid method or projection methods. As a result, the problems are reduced to finding the spectral characteristics of matrices. The use of traditional methods requires a very significant amount of calculations, due to the poor separation of eigenvalues of matrices obtained from the corresponding systems of equations. Despite the simplicity of the wording, it is impossible to offer a single calculation algorithm for solving problems encountered in applications. The choice of the algorithm for calculating the eigenvalues of matrices is determined by the type of matrices. It's important to note that the problem of finding an all points of the spectrum for high-order matrices does not have a satisfactory numerical solution yet. Therefore, the development of new methods for calculating the eigenvalues of boundary value problems that do not use the process of reducing them to finding the eigenvalues of the corresponding matrices is a matter of great scientific interest.

In papers [1] – [11] the numerical method for calculating the eigenvalues of discrete semi-bounded operators, which uses linear formulas to find approximate eigenvalues with any ordinal numbers, is developed.

Consider a discrete semi-bounded operator L on a separable Hilbert space H with domain $D_L \in H$, generating a boundary value problem

$$Lu = \mu u, \quad Gu|_\Gamma = 0, \tag{1}$$

here Γ is a boundary of domain D_L.

To discretize the D_L we construct a sequence of finite-dimensional spaces $\{H_n\}_{n=1}^{\infty}$ which is complete in H. Let be an orthonormal basis $\{\varphi_k\}_{k=1}^{n}$ of spaces $H_n \subseteq H$ satisfies homogeneous boundary conditions of the problem (1). To find an approximate solution to a boundary value problem (1), use the Galerkin method, and the solution of the boundary value problem (1) is found as a finite sum of the form

$$u_n = \sum_{k=1}^{n} a_k(n)\varphi_k. \tag{2}$$

In this case, the constants $a_k(n)$ in (2) are found from the requirement that the expression that is obtained after substituting u_n instead of u in the equation (1),becomes orthonormal to the system of functions $\{\varphi_k\}_{k=1}^{n}$.

In work [11] the following theorems are proved.

Theorem 1. *The Galerkin method, when applied to the problem of finding eigenvalues of the spectral problem* (1)*, built on the system of functions* $\{\varphi_k\}_{k=1}^{\infty}$*, converges.*

Theorem 2. *Approximate eigenvalues of* $\widetilde{\mu}_n$ *of the spectral problem* (1) *are found by linear formulas*

$$\widetilde{\mu}_n(n) = (L\varphi_n, \varphi_n) + \widetilde{\delta}_n, \ \ n \in \mathbb{N}, \tag{3}$$

where $\widetilde{\delta}_n = \sum\limits_{k=1}^{n-1} [\widetilde{\mu}_k(n-1) - \widetilde{\mu}_k(n)]$*,* $\widetilde{\mu}_k(n)$ *– n-th Galerkin approximations to the corresponding eigenvalue* μ_k *of the spectral problem* (1)*.*

Using Theorems 1 and 2, it is not difficult to show that $\lim\limits_{n\to\infty} \widetilde{\delta}_n = 0$.

It should be noted that to calculate the approximate eigenvalue of $\widetilde{\mu}_{n_0}(n_0)$ with the sequence number n_0, using the formulas (3), the sum of (2) consists of n_0 members. Therefore, for small n_0, the accuracy of calculating the eigenvalues of $\widetilde{\mu}_{n_0}(n_0)$ by linear formulas (3) is not large. As n_0 increases the number of members in the sum of (2), the accuracy of calculating $\widetilde{\mu}_{n_0}(n_0)$ by the formulas (3), which is confirmed by numerous computational experiments.

Algorithms for finding approximate eigenvalues of the boundary value problem (1) using the formulas (3) are simple and computationally efficient. In this case, the calculation of eigenvalues using the (3) formulas can start with any ordinal number, since eigenvalues with smaller ordinal numbers are not used for their calculations. The method of finding eigenvalues using the formulas (3) was called by the authors *a modified Galerkin method.*

To use the formulas (3) when calculating approximate eigenvalues of the boundary value problem (1), you must have a system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$, which would be an orthonormal basis H and satisfy the boundary conditions (1). This can be done if the operator L is represented as $L = T + P$, where T is a self-adjoint differential operator of the same order as the operator L with the domain of definition D_L. In this case, the system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ can be taken as the system of proper functions $\{v_k\}_{k=1}^{\infty}$ of the spectral problem

$$Tv = \lambda v, \quad Gv|_{\Gamma} = 0. \tag{4}$$

Finding the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and the orthonormal eigenfunctions $\{v_k\}_{k=1}^{\infty}$ of the operator T, we write the formulas (3) as

$$\widetilde{\mu}_n(n) = \lambda_n + (Pv_n, v_n) + \widetilde{\delta}_n, \quad \forall n \in N. \tag{5}$$

The question of whether the system of eigenfunctions $\{v_k\}_{k=1}^{\infty}$ of the problem (4) is the basis of the Hilbert space H must be solved separately in each case.

Based on the formulas (3) in the works [12] – [16] was developed a new method for solving inverse spectral problems generated by discrete semi-bounded operators. When discussing the results of research on this method at the scientific seminar Of the Institute for mathematical research of complex systems of The Moscow state University "Time, chaos and mathematical problems which is headed by academician of Russian Academy of Sciences V. A. Sadovnichy, it was suggested to compare the results of calculating eigenvalues using the formulas (3) and known asymptotic formulas. This work presents the results of this comparison for spectral problems generated by Sturm-Liouville operators of arbitrary even order, which are often used in solving applied problems.

1. The Sturm – Liouville Operator of Arbitrary Even Order

We investigate the possibility of using the formulas (5) to find eigenvalues in the entire range of their ordinal numbers of the following boundary value problems generated by differential operators of arbitrary even order, given in $L_2[0,\pi]$, of the form

$$(T_m + P_m)u_m(s) = \mu_m u_m(s), \quad 0 < s < \pi, \tag{6}$$

$$u_m^{(2\nu-1)}(0) = u_m^{(2\nu-1)}(\pi) = 0, \quad \nu = \overline{1, m}, \tag{7}$$

where $T_m u_m(s) = (-1)^m \dfrac{d^{2m} u_m(s)}{ds^{2m}}$, $P_m u_m(s) = \sum\limits_{k=0}^{m_v} p_{m_k}(s) \dfrac{d^k u_m(s)}{ds^k}$, $m \geq 2$, $m_v = \overline{0, 2m-1}$.

To construct an orthonormal system of functions that is the basis of the space $L_2[0,\pi]$ and satisfies the boundary conditions (7), we consider spectral problems

$$T_m v_m(s) = \lambda_m v_m(s), \quad 0 < s < \pi, \tag{8}$$

$$v_m^{(2\nu-1)}(0) = v_m^{(2\nu-1)}(\pi) = 0, \quad \nu = \overline{1, m}. \tag{9}$$

In [17], it is shown that (8) and (9) are self-adjoint problems whose eigenvalues λ_{m_n} and orthonormal eigenfunctions v_n will be

$$\lambda_{m_n} = n^{2m}, \quad v_n = \sigma_n cos(ns), \quad \sigma_n = \begin{cases} \sqrt{\dfrac{1}{\pi}}, & n = 0, \\ \sqrt{\dfrac{2}{\pi}}, & n > 0. \end{cases} \tag{10}$$

It is known that the system of functions $\{\sigma_n cos(ns)\}_{n=1}^{\infty}$ is an orthonormal basis of the space $L_2[0, \pi]$.

We write the formulas (5) for calculating the eigenvalues of spectral problems (6), (7). To do this, use the eigenvalues λ_{m_n} and the orthonormal eigenfunctions v_n using (10)

$$\widetilde{\mu}_{m_n}(n) = \lambda_{m_n} + (P_m v_n, v_n) + \widetilde{\delta}_{m_n} = \\ = n^{2m} + \sigma_n^2 \int_0^{\pi} cos(ns) \sum_{k=0}^{2m-1} p_{m_k}(s) \frac{d^k cos(ns)}{ds^k} ds + \widetilde{\delta}_{m_n}. \tag{11}$$

If the equations (6) take $P_m = p_{m_0}(s)$, that is, $p_{m_k}(s) = 0$ for $k = \overline{1, 2m-1}$, then for the resulting differential equations

$$(T_m + p_{m_0}) u_m(s) = \mu_m u_m(s) \tag{12}$$

with boundary conditions (7), the following theorem holds [17].

Theorem 3. *Asymptotic formulas for eigenvalues μ_m of boundary value problems* (12), (7) *have the form*

$$\mu_{m_n} = n^{2m} + a_{m_0} + a_{m_{2n}} + O\Big(\frac{1}{n^{2m-1}}\Big), \quad n \to \infty, \tag{13}$$

where $\quad a_{m_n} = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} p_{m_0}(s) \cos(ns) ds, \quad n = \overline{0, \infty}.$

Let's write the formulas (11), which are obtained from the linear formulas (5), for the case when $P_m = p_{m_0}(s)$

$$\widetilde{\mu}_{m_n}(n) = \lambda_{m_n} + (P_m v_n, v_n) + \widetilde{\delta}_{m_n} = n^{2m} + \sigma_n^2 \int_0^{\pi} \cos^2(ns) p_{m_0}(s) ds + \widetilde{\delta}_{m_n} = \\ = n^{2m} + \frac{1}{\pi} \int_0^{\pi} [1 + \cos(2ns)] p_{m_0}(s) ds + \widetilde{\delta}_{m_n}.$$

or using the notation (3), write

$$\widetilde{\mu}_{m_n}(n) = n^{2m} + a_{m_0} + a_{m_{2n}} + \widetilde{\delta}_{m_n}, \quad \forall n \in N. \tag{14}$$

If we compare the asymptotic formulas (13) to the formulas (14), they are different in the order of errors only. Therefore, the results of calculating the eigenvalues of (12),

(7) using the formulas (13) and (14) for any ordinal numbers using these formulas will be computationally the same. We can assume that $\widetilde{\delta}_{m_n} = O\left(\frac{1}{n^{2m-1}}\right)$ when $n \to \infty$. Therefore, the formulas (14) allow us to calculate approximate values of μ_{m_n} of boundary value problems (12), (7) for any ordinal numbers n at $m \geq 2$.

We conduct computational experiments and investigate the possibility of using the formulas (5) to calculate eigenvalues for small ordinal numbers of spectral problems (6), (7). The results of the calculations were compared to the calculations performed using the Galerkin method.

2. Computational Experiment

In the mathematical environment of Maple, based on the theoretical material presented in the first section, computational experiments were performed to find eigenvalues with the first ordinal numbers of spectral problems (6), (7). In the tables below, approximate values calculated using the formulas (5) are denoted by $\widetilde{\mu}_{m_n}$, and by the Galerkin method – $\widehat{\mu}_{m_n}$. To define the functions $p_{m_k}(s)$ that define the perturbing operator P_m, equation (7) used the formula

$$p_{m_k}(s) = s^{st+2} - ks^{st+1} + 2m\sin((m_v+3)s) - m\cos(2(m_v+4)s) + e^{(st+1)s}, \quad st = \left\lceil \frac{m_v}{k+2} \right\rceil.$$

Here, the $\lceil\cdot\rceil : x \longmapsto \lceil x \rceil$ function is defined as the smallest integer.

Table 1 shows the results of calculating the eigenvalues of the problems (12), (7), which are special cases of the problems (6), (7) for $m_v = 0$. Calculations were performed at $m = \overline{1, 11}$.

Table 1

n	$m=2$ $\widetilde{\mu}_{m_n}$	$\lvert\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\rvert$	$m=3$ $\widetilde{\mu}_{m_n}$	$\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\rvert$	$m=4$ $\widetilde{\mu}_{m_n}$	$\lvert\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\rvert$
1	$9,34\cdot10^{0}$	$1,34\cdot10^{0}$	$7,39\cdot10^{0}$	$1,29\cdot10^{-1}$	$5,43\cdot10^{0}$	$7,24\cdot10^{-3}$
2	$2,04\cdot10^{1}$	$9,70\cdot10^{-1}$	$6,51\cdot10^{1}$	$1,53\cdot10^{-1}$	$2,54\cdot10^{2}$	$5,07\cdot10^{-3}$
3	$8,59\cdot10^{1}$	$2,31\cdot10^{-1}$	$7,31\cdot10^{2}$	$1,91\cdot10^{-2}$	$6,56\cdot10^{3}$	$1,79\cdot10^{-3}$
4	$2,60\cdot10^{2}$	$7,94\cdot10^{-2}$	$4,10\cdot10^{3}$	$4,73\cdot10^{-3}$	$6,55\cdot10^{4}$	$3,31\cdot10^{-4}$
5	$6,30\cdot10^{2}$	$2,00\cdot10^{-2}$	$1,56\cdot10^{4}$	$6,66\cdot10^{-4}$	$3,91\cdot10^{5}$	$2,83\cdot10^{-5}$
6	$1,30\cdot10^{3}$	$1,62\cdot10^{-2}$	$4,67\cdot10^{4}$	$5,66\cdot10^{-4}$	$1,68\cdot10^{6}$	$2,22\cdot10^{-5}$
7	$2,41\cdot10^{3}$	$8,17\cdot10^{-3}$	$1,18\cdot10^{5}$	$1,99\cdot10^{-4}$	$5,76\cdot10^{6}$	$5,53\cdot10^{-6}$
8	$4,10\cdot10^{3}$	$4,48\cdot10^{-3}$	$2,62\cdot10^{5}$	$7,71\cdot10^{-5}$	$1,68\cdot10^{7}$	$1,54\cdot10^{-6}$
9	$6,57\cdot10^{3}$	$2,92\cdot10^{-3}$	$5,31\cdot10^{5}$	$4,26\cdot10^{-5}$	$4,30\cdot10^{7}$	$7,08\cdot10^{-6}$
10	$1,00\cdot10^{4}$	$1,90\cdot10^{-3}$	$1,00\cdot10^{6}$	$2,25\cdot10^{-5}$	$1,00\cdot10^{8}$	$3,02\cdot10^{-7}$
11	$1,46\cdot10^{4}$	$1,29\cdot10^{-3}$	$1,77\cdot10^{6}$	$1,26\cdot10^{-5}$	$2,14\cdot10^{8}$	$1,40\cdot10^{-7}$
12	$2,07\cdot10^{4}$	$9,10\cdot10^{-4}$	$2,99\cdot10^{6}$	$7,44\cdot10^{-6}$	$4,30\cdot10^{8}$	$6,95\cdot10^{-8}$
13	$2,86\cdot10^{4}$	$6,59\cdot10^{-4}$	$4,83\cdot10^{6}$	$4,49\cdot10^{-6}$	$8,16\cdot10^{8}$	$3,65\cdot10^{-8}$
14	$3,84\cdot10^{4}$	$4,94\cdot10^{-4}$	$7,53\cdot10^{6}$	$2,96\cdot10^{-6}$	$1,48\cdot10^{9}$	$2,02\cdot10^{-8}$
15	$5,06\cdot10^{4}$	$3,74\cdot10^{-4}$	$1,14\cdot10^{7}$	$1,95\cdot10^{-6}$	$2,56\cdot10^{9}$	$1,16\cdot10^{-8}$
16	$6,55\cdot10^{4}$	$2,09\cdot10^{-4}$	$1,68\cdot10^{7}$	$1,32\cdot10^{-6}$	$4,29\cdot10^{9}$	$6,92\cdot10^{-8}$
17	$8,35\cdot10^{4}$	$2,27\cdot10^{-4}$	$2,41\cdot10^{7}$	$9,21\cdot10^{-7}$	$6,98\cdot10^{9}$	$4,25\cdot10^{-9}$
18	$1,05\cdot10^{5}$	$1,83\cdot10^{-4}$	$3,40\cdot10^{7}$	$6,69\cdot10^{-7}$	$1,10\cdot10^{10}$	$2,75\cdot10^{-9}$
19	$1,30\cdot10^{5}$	$1,53\cdot10^{-4}$	$4,70\cdot10^{7}$	$1,26\cdot10^{-5}$	$1,70\cdot10^{10}$	$1,79\cdot10^{-9}$
20	$1,60\cdot10^{5}$	$2,04\cdot10^{-4}$	$6,40\cdot10^{7}$	$7,18\cdot10^{-7}$	$2,56\cdot10^{10}$	$2,03\cdot10^{-9}$
21	$1,94\cdot10^{5}$	$5,15\cdot10^{-4}$	$8,58\cdot10^{7}$	$7,02\cdot10^{-7}$	$3,78\cdot10^{10}$	$1,45\cdot10^{-9}$

It should be noted that if the number $1,94 \cdot 10^5$ differs from another number by the amount of $5,15 \cdot 10^{-4}$, then they have the same first 8 significant digits.

Table 1 shows the results of calculating eigenvalues for $m = \overline{2,4}$ show that the values of $\widetilde{\mu}_{m_n}$ and $\widehat{\mu}_{m_n}$ starting with the number $n = 4$ are in good agreement. The same trend persists when $m \geq 5$. For example, when $m = 11$ $\widetilde{\mu}_{11_4} = 1,76 \cdot 10^{13}$, $|\widetilde{\mu}_{11_4} - \widehat{\mu}_{11_4}| = 8,49 \cdot 10^{-12}$, and $\widetilde{\mu}_{11_{21}} = 1,23 \cdot 10^{25}$, $|\widetilde{\mu}_{11_{21}} - \widehat{\mu}_{11_{21}}| = 2,37 \cdot 10^{-27}$.

It is interesting to compare the results of calculating the eigenvalues of spectral problems (6), (7), when the sum defining the perturbing operator P_m, consists of a different number of members up to the number $m_v = 2m - 1$.

In the Table 2 the results of such calculations for $m = 5$.

Table 2

n	$m_v = 4$ $\widetilde{\mu}_{m_n}$	$\|\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\|$	$m_v = 6$ $\widetilde{\mu}_{m_n}$	$\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\|$	$m_v = 8$ $\widetilde{\mu}_{m_n}$	$\|\widetilde{\mu}_{m_n} - \widehat{\mu}_{m_n}\|$
1	$-4,46 \cdot 10^0$	$3,86 \cdot 10^{-1}$	$-4,66 \cdot 10^0$	$2,08 \cdot 10^{-1}$	$-4,78 \cdot 10^0$	$9,13 \cdot 10^0$
2	$-5,61 \cdot 10^{-1}$	$1,17 \cdot 10^{-1}$	$3,90 \cdot 10^{-1}$	$7,94 \cdot 10^{-1}$	$-1,28 \cdot 10^0$	$1,01 \cdot 10^0$
3	$9,91 \cdot 10^2$	$1,44 \cdot 10^{-1}$	$1,19 \cdot 10^3$	$3,79 \cdot 10^0$	$2,95 \cdot 10^2$	$1,71 \cdot 10^2$
4	$5,90 \cdot 10^4$	$5,50 \cdot 10^{-2}$	$6,11 \cdot 10^4$	$3,80 \cdot 10^1$	$3,71 \cdot 10^4$	$6,96 \cdot 10^3$
5	$1,05 \cdot 10^6$	$1,51 \cdot 10^{-1}$	$1,05 \cdot 10^6$	$1,66 \cdot 10^1$	$8,37 \cdot 10^5$	$8,52 \cdot 10^4$
6	$9,67 \cdot 10^6$	$3,84 \cdot 10^{-2}$	$9,88 \cdot 10^6$	$1,43 \cdot 10^1$	$9,25 \cdot 10^6$	$7,61 \cdot 10^4$
7	$6,05 \cdot 10^7$	$2,19 \cdot 10^{-2}$	$6,07 \cdot 10^7$	$7,24 \cdot 10^1$	$4,78 \cdot 10^7$	$2,13 \cdot 10^5$
8	$2,82 \cdot 10^8$	$1,64 \cdot 10^{-2}$	$2,84 \cdot 10^8$	$1,43 \cdot 10^2$	$2,50 \cdot 10^8$	$5,89 \cdot 10^5$
9	$1,07 \cdot 10^9$	$6,29 \cdot 10^{-3}$	$1,08 \cdot 10^9$	$2,21 \cdot 10^2$	$9,87 \cdot 10^8$	$2,13 \cdot 10^6$
10	$3,49 \cdot 10^9$	$4,40 \cdot 10^{-2}$	$3,49 \cdot 10^9$	$7,71 \cdot 10^2$	$3,27 \cdot 10^9$	$4,21 \cdot 10^6$
11	$1,00 \cdot 10^{10}$	$8,23 \cdot 10^{-3}$	$1,00 \cdot 10^{10}$	$4,07 \cdot 10^2$	$9,50 \cdot 10^9$	$8,67 \cdot 10^6$
12	$2,59 \cdot 10^{10}$	$7,10 \cdot 10^{-3}$	$2,59 \cdot 10^{10}$	$7,66 \cdot 10^2$	$2,49 \cdot 10^{10}$	$1,61 \cdot 10^6$
13	$6,19 \cdot 10^{10}$	$4,40 \cdot 10^{-5}$	$6,19 \cdot 10^{10}$	$6,42 \cdot 10^2$	$5,87 \cdot 10^{10}$	$1,59 \cdot 10^7$
14	$1,38 \cdot 10^{11}$	$2,42 \cdot 10^{-3}$	$1,38 \cdot 10^{11}$	$5,04 \cdot 10^2$	$1,34 \cdot 10^{11}$	$1,48 \cdot 10^7$
15	$2,89 \cdot 10^{11}$	$3,29 \cdot 10^{-3}$	$2,89 \cdot 10^{11}$	$6,01 \cdot 10^2$	$2,82 \cdot 10^{11}$	$4,07 \cdot 10^7$
16	$5,77 \cdot 10^{11}$	$1,21 \cdot 10^{-2}$	$5,77 \cdot 10^{11}$	$5,38 \cdot 10^2$	$5,64 \cdot 10^{11}$	$5,73 \cdot 10^7$
17	$1,10 \cdot 10^{12}$	$1,27 \cdot 10^{-2}$	$1,10 \cdot 10^{12}$	$5,96 \cdot 10^2$	$1,08 \cdot 10^{12}$	$8,56 \cdot 10^7$
18	$2,02 \cdot 10^{12}$	$1,68 \cdot 10^{-2}$	$2,02 \cdot 10^{12}$	$3,37 \cdot 10^2$	$1,98 \cdot 10^{12}$	$1,01 \cdot 10^8$
19	$3,57 \cdot 10^{12}$	$1,70 \cdot 10^{-2}$	$3,57 \cdot 10^{12}$	$3,37 \cdot 10^2$	$1,98 \cdot 10^{12}$	$1,38 \cdot 10^8$
20	$6,13 \cdot 10^{12}$	$3,67 \cdot 10^{-2}$	$2,31 \cdot 10^{12}$	$2,31 \cdot 10^3$	$6,05 \cdot 10^{12}$	$1,38 \cdot 10^8$
21	$1,02 \cdot 10^{13}$	$3,68 \cdot 10^{-2}$	$1,02 \cdot 10^{13}$	$3,01 \cdot 10^3$	$1,01 \cdot 10^{13}$	$3,08 \cdot 10^8$

The results of calculations of eigenvalues for $m = 5$ shown in table 2 show that the values $\widetilde{\mu}_{m_n}$ and $\widehat{\mu}_{m_n}$ starting from the number $n = 4$ agree well.

Conclusion

Using the example of spectral problems generated by Sturm-Liouville operators of arbitrary even order, the paper investigates the possibility of using linear formulas (5) to find eigenvalues with large ordinal numbers. Comparison of the formulas (14), which calculate the eigenvalues of problems found by linear formulas (5) with the known asymptotic formulas (13), in the considered problems (12), (7) showed that they differ from each other only in the order of errors.

The result suggests that the formulas (5) can be used to calculate eigenvalues with large ordinal numbers of any discrete semi-bounded operators. At the same time, they are similar to asymptotic formulas and differ from them only in the order of errors.

Comparison of the results of calculating eigenvalues with the first ordinal numbers using the formulas (5) and calculated using the Galerkin method showed that they agree well. Therefore, the formulas (5) can be used to calculate the eigenvalues of the spectral problems (6), (7) over the entire range of changes in their ordinal numbers.

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ВЫЧИСЛЕНИЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ ЗАДАЧ, ПОРОЖДЕННЫХ ОПЕРАТОРАМИ ШТУРМА – ЛИУВИЛЛЯ ПРОИЗВОЛЬНОГО ЧЕТНОГО ПОРЯДКА

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Проблема вычисления собственных значений дискретных полуограниченных дифференциальных операторов является одной из важных задач численного анализа. Несмотря на простату постановки, для решения многих задач, которые встречаются на практике, нельзя предложить единого алгоритма вычисления собственных значений. Известные методы вычисления собственных значений линейных дифференциальных операторов основываются на сведении задач к дискретным моделям, при помощи в основном методов сеток или проекционных методов, сводящие задачи к нахождению спектральных характеристик систем линейных алгебраических уравнений. Ввиду плохой разделенности собственных значений матриц, полученных из соответствующих систем уравнений, применение традиционных методов решения требует весьма значительного объема вычислений. Это часто приводит к необходимости решать некорректно поставленные задачи. Как правило выбор алгоритмов приближенного нахождения собственных значений матриц обусловлен в основном их видом. Это сильно сужает возможности использования вычислительных методов для нахождения их собственных значений матриц. При это необходимо отметить, что задача нахождения всех точек спектра для матриц высокого порядка еще не имеет удовлетворительного численного решения.

Используя численные метод регуляризованных следов и метод Галеркина, ранее были получены линейные формулы для вычисления приближенных собственных значений дискретных полуограниченных операторов. Они позволяют находить приближенные собственные значения с любым порядковым номером. При этом не возникают вычислительных трудностей, которые имеют место в других методах. Сравнение результатов вычислительных экспериментов показали, что собственные значения, найденные по линейным формулам, методом Галеркина, а также известные собственные значения спектральных задач, хорошо согласуются.

В статье исследована возможность использования линейных формул, полученных в статьях авторов для нахождения собственных значений операторов Штурма–Лиувилля произвольного четного порядка. На рассмотренных примерах показано, что собственные значения, найденные по линейным формулам и известным асимптотическим формулам, вычислительно совпадают.

Ключевые слова: собственные значения и собственные функции; дискретные, самосопряженные и полуограниченные операторы; метод Галеркина; асимптотические формулы.

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