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ON NUMERICAL SOLUTION IN THE SPACE OF DIFFERENTIAL FORMS FOR ONE STOCHASTIC SOBOLEV-TYPE EQUATION WITH A RELATIVELY RADIAL OPERATOR

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The paper presents graphs of the trajectories of numerical solutions to the Showalter – Sidorov problem for one stochastic version of the Ginzburg – Landau equation in spaces of differential forms defined on a two-dimensional torus. We use the previously obtained transition from the deterministic version of the theory of Sobolev type equations to stochastic equations using the Nelson – Glicklikh derivative. Since the equations are studied in the space of differential forms, the operators themselves are understood in a special form, in particular, instead of the Laplace operator, we take its generalization, the Laplace – Beltrami operator. The graphs of computational experiments are given for different values of the parameters of the initial equation for the same trajectories of the stochastic process.

Keywords: Sobolev type equation; white noise; Nelson – Gliklikh derivative; Riemannian manifold; differential forms; Laplace – Beltrami operator; numerical solution.

Introduction

The work [1] investigates linear Sobolev type equations, with an irreversible operator at the derivative, of the form

$$L\dot{u} = Mu + f, \tag{1}$$

where $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, the operator $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ belongs to the space of linear and bounded operators, and the operator $M \in Cl(\mathfrak{U};\mathfrak{F})$ belongs to the space of closed densely defined operators in the case of the abstract (L,p)-bounded, (L,p)-sectorial and (L,p)-radial operator M. Based on this research, the work [2] studies the Ginzburg – Landau equation

$$(\lambda - \Delta)u_t = \alpha\Delta u + id\Delta^2 u \tag{2}$$

with the (L,p)-radial operator $M = \alpha\Delta + id\Delta^2$. The Cauchy problem

$$u(0) = u_0$$

for equation (2) is solvable in a subspace called the phase space. The Showalter – Sidorov problem

$$P(u(0) - u_0) = 0$$

for the Ginzburg – Landau equation was also investigated in the work [2]. The paper [3] proposes a transition to the non-deterministic (stochastic) Sobolev type equations

$$L \overset{\circ}{\eta} = M\eta + \omega \tag{3}$$

in spaces of Wiener stochastic processes in the case of the abstract (L,p)-radial operator M. Since Wiener processes are continuous, but nondifferentiable in the usual sense at each point, we use the Nelson-Gliklikh derivative [4]. In this article, we study the numerical solutions to the Showalter – Sidorov problem for the Ginzburg – Landau equation in spaces of differential forms defined on a two-dimensional torus in the form (3) by analogy with our approach used for the solutions to the Barenblatt – Zheltov – Kochina equation obtained in [5] for which a difference analogue of the equation was constructed and a numerical solution to the Showalter – Sidorov problem was found in [6].

1. Structure of Differentiable "Noises" Spaces

Consider the complete probability space $\Omega = (\Omega, \Sigma, P)$ with the probability measure P associated with the sigma-algebra Σ of subsets of the space Ω. If $\mathbb{R}$ is the set of real numbers endowed with a sigma algebra, then the mapping $\xi : \Sigma \mapsto \mathbb{R}$ is called a random variable. The set of random variables ξ, the mathematical expectation of which is equal to zero, i.e. $M\xi = 0$, while variance is finite, i.e. $D\xi < \infty$, form the Hilbert space $\mathfrak{L}_\mathbf{2}$ with the scalar product $(\xi_1, \xi_2) = M\xi_1\xi_2$ and with the norm denoted by $\|\xi\|_{\mathfrak{L}_\mathbf{2}}$. If we take the subalgebra Σ_0 of the sigma-algebra Σ, then we obtain the subspace of random variables $\mathfrak{L}_\mathbf{2}^\mathbf{0} \subset \mathfrak{L}_\mathbf{2}$ measurable with respect to Σ_0.

The measurable mapping $\eta = \eta(t,\omega) : J \times \Sigma \mapsto \mathbb{R}$, where $J = (a,b) \subset \mathbb{R}$, is called a stochastic process, the random variable $\eta(\cdot,\omega), \omega \in \Omega$ is said to be the section of the stochastic process, and the function $\eta(t,\cdot), t \in J$ is said to be the trajectory of the stochastic process. The stochastic process $\eta = \eta(t,\omega)$ is called continuous, if the trajectories $\eta = \eta(t,\omega_0)$ are continuous functions almost sure (i.e. for a.a. (almost all) $\omega_0 \in \Sigma$). The set $\eta = \eta(t,\omega)$ of continuous stochastic processes forms a Banach space $\mathbf{C}\mathfrak{L}_\mathbf{2}$.

By the Nelson – Gliklikh derivative of the stochastic process $\eta \in \mathbf{C}\mathfrak{L}_\mathbf{2}$ at the point $t \in J$ we mean the random variable

$$\overset{\circ}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \mapsto 0+} M_t^\eta\left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t \mapsto 0+} M_t^\eta\left(\frac{\eta(t-\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right)\right), \qquad (4)$$

if the limit exists in the sense of a uniform metric on $t \in J$. Here M_t^η is the expectation on a subalgebra of the sigma-algebra Σ that is generated by the random variable $\eta = \eta(t,\omega)$. If there exist the Nelson – Gliklikh derivatives $\overset{\circ}{\eta}(\cdot,\omega)$ of the stochastic process η at almost all points of the interval J, then we say that there exists the Nelson – Gliklikh derivative $\overset{\circ}{\eta}(\cdot,\omega)$ almost sure on J. The set of continuous stochastic processes with continuous Nelson – Gliklikh derivatives $\overset{\circ}{\eta}$ form the Banach space $\mathbf{C}^\mathbf{1}\mathfrak{L}_\mathbf{2}$. Further, by induction, we obtain the Banach spaces $\mathbf{C}^\mathbf{l}\mathfrak{L}_\mathbf{2}$, $\mathbf{l} \in \mathbb{N}$ of the stochastic processes having continuous Nelson – Gliklikh derivatives on J up to the order $\mathbf{l} \in \mathbb{N}$ inclusively with norms of the form $\|\eta\|_{\mathbf{C}^\mathbf{l}\mathfrak{L}_\mathbf{2}} = \sup\limits_{t \in J}(\sum\limits_{k=0}^{\mathbf{l}} D\overset{\circ}{\eta}^{(k)}(t,\omega))^{\frac{1}{2}}$, where $\overset{\circ}{\eta}^{(0)}(t,\omega) = \eta(t,\omega)$.

2. Relatively Radial Operators and Resolving Semigroups

Let $\mathfrak{U}$ and $\mathfrak{F}$ be real separable Hilbert spaces. Denote by $\mathcal{L}(\mathfrak{U};\mathfrak{F})$ the space of linear bounded operators, and by $Cl(\mathfrak{U};\mathfrak{F})$ the space of linear closed and densely defined operators. Let us construct the Hilbert spaces $\mathbf{U}_\mathbf{K}\mathfrak{L}_\mathbf{2}$ and $\mathbf{F}_\mathbf{K}\mathfrak{L}_\mathbf{2}$, where $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}$

is a monotone sequence of numbers such that $\sum_{k=1}^{\infty} \lambda_k^2 < +\infty$. The stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta \tag{5}$$

can be reduced to two equations of the form

$$A \overset{\circ}{\nu} = B\nu.$$

Let us formulate

Lemma 1. *The following statements are true:*
i) the operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ *exactly if* $A \in \mathcal{L}(\mathbf{U_K}\mathfrak{L}_\mathbf{2}; \mathbf{F_K}\mathfrak{L}_\mathbf{2})$*;*
ii) the operator $B \in Cl(\mathfrak{U}; \mathfrak{F})$ *exactly if* $B \in Cl(\mathbf{U_K}\mathfrak{L}_\mathbf{2}; \mathbf{F_K}\mathfrak{L}_\mathbf{2})$*.*

Definition 1. The operator $M \in Cl(\mathbf{U_K}\mathfrak{L}_\mathbf{2}; \mathbf{F_K}\mathfrak{L}_\mathbf{2})$ is said to be p-radial with respect to the operator $L \in \mathcal{L}(\mathbf{U_K}\mathfrak{L}_\mathbf{2}; \mathbf{F_K}\mathfrak{L}_\mathbf{2})$ (for shortness, (L, p)-radial), $p \in \{0\} \cup \mathbb{N}$, if
i) there exists a constant $\alpha \in \mathbb{R}$ such that $[\alpha, +\infty) \subset \rho^L(M)$;
ii) there exists a constant $K_1 > 0$ such that $\forall \mu_q \in [\alpha, +\infty), q = 0, 1, ..., p, \forall n \in \mathbb{N}$ $\max\{\|R^L_{(\mu,p)}(M)\|_{\mathfrak{U}},\ \|L^L_{(\mu,p)}(M)\|_{\mathfrak{F}}\} < \frac{K_1}{\prod_{q=0}^{p} (\mu_q - \alpha)^n}$.
Here $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ is the L -resolvent set, and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ is the L-spectrum of the operator M. For $\mu_q \in \rho^L(M)$, $q = 0, 1, ..., p$, the operator functions $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called the right L-resolvent and the left L-resolvent of the operator M, and $R^L_{(\mu,p)}(M) = \prod_{q=0} p(\mu_q L - M)^{-1}L$ and $L^L_{(\mu,p)}(M) = \prod_{q=0} pL(\mu L - M)^{-1}$ are the (L, p)-right L-resolvent and the left (L, p)-resolvent of the operator.

Theorem 1. *[1] Let the operator* M *be* (L, p)*-radial. Then there exists a* C_0*-semigroup of the operators on the space* $\mathbf{U_K}\mathbf{L}_2$ *(*$\mathbf{F_K}\mathbf{L}_2$*).*

The set $\ker V^* = \{\nu \in \mathbf{U_K}\mathfrak{L}_\mathbf{2}(\mathbf{F_K}\mathfrak{L}_\mathbf{2}) : V^t\nu = 0\}$ is called the kernel, the set $\mathrm{im}V^* = \{\nu \in \mathbf{U_K}\mathfrak{L}_\mathbf{2}(\mathbf{F_K}\mathfrak{L}_\mathbf{2}) \lim_{t \mapsto 0+} V^t\nu = \nu_0\}$ is said to be the image of the analytic semigroup $V^t : t \geq 0$. Denote $\mathfrak{U}^0 = \{\mathbf{U^0_K}\mathfrak{L}_\mathbf{2}\}$ ($\mathfrak{F}^0 = \{\mathbf{F^0_K}\mathbf{L_2}\}$), which form a closure of kernels of semigroups in the norm of the space $\mathfrak{U} = \mathbf{U_K}\mathfrak{L}_\mathbf{2}$ ($\mathfrak{F} = \mathbf{F_K}\mathfrak{L}_\mathbf{2}$). Also, denote $\mathfrak{U}^1 = \{\mathbf{U_K}\mathfrak{L}_\mathbf{2}\}$ ($\mathfrak{F}^1 = \{\mathbf{F^1_K}\mathbf{L_2}\}$), which form a closure $\mathrm{im}R^L_{(\mu,p)}(M)$ ($\mathrm{im}L^L_{(\mu,p)}(M)$) in the norm of the space $\mathfrak{U} = \mathbf{U_K}\mathfrak{L}_\mathbf{2}$ ($\mathfrak{F} = \mathbf{F_K}\mathfrak{L}_\mathbf{2}$). The spaces $\mathbf{U_K}\mathbf{L}_2$ and $\mathbf{F_K}\mathbf{L}_2$ split into the direct sum

$$\mathbf{U_K}\mathbf{L}_2 = \mathbf{U^0_K}\mathbf{L}_2 \oplus \mathbf{U^1_K}\mathbf{L}_2, \mathbf{F_K}\mathbf{L}_2 = \mathbf{F^0_K}\mathbf{L}_2 \oplus \mathbf{F^1_K}\mathbf{L}_2. \tag{6}$$

The following theorem takes place.

Theorem 2. *If the operator* M *is* (L, p)*-radial and there exist splittings (6), then* $\mathrm{im}U^* = \mathbf{U^1_K}\mathfrak{L}_\mathbf{2}$ *and* $\mathrm{im}F^* = \mathbf{F^1_K}\mathfrak{L}_\mathbf{2}$*.*

Previously, the Showalter – Sidorov problem

$$P(\eta(0) - \eta_0) = 0 \tag{7}$$

was investigated [3] in the spaces $\mathfrak{U} = \mathbf{U_K}\mathfrak{L}_\mathbf{2}$ ($\mathfrak{F} = \mathbf{F_K}\mathfrak{L}_\mathbf{2}$), where there exist representations of the form

$$\eta(t,\cdot) = \sum_{k=0}^{+\infty} \lambda_k \xi_k(t,\cdot)\varphi_k. \tag{8}$$

Theorem 3. *Let the operator M be (L,p)-radial and there exist splittings (6), then $\forall \eta_0 \in \mathfrak{U}^1 \subset \mathfrak{U}$ there exists the unique solution to problem (5), (7).*

3. Differential Forms and Computational Experiments

Consider a two-dimensional torus obtained by the direct product of two segments $\mathbb{T} = [0,\pi] \times [0, 2\pi]$. The torus is a 2-dimensional smooth compact oriented Riemannian manifold without boundary. Using theory presented in Sections 1 and 2, we construct spaces of smooth differential q-forms with stochastic processes as the coefficient:

$$\omega(t,\omega,x_1,x_2) = \sum_{|i_1,\dots,i_q|=q} \chi_{i_1,\dots,i_q}(t,\omega,x_1,x_2)dx_{i_1} \wedge \dots \wedge dx_{i_2}, \tag{9}$$

where $|i_1,\dots,i_q|$ is a multi-index, and, according to (8), the coefficients have the form

$$\chi_{i_1,i_2,\dots,i_q}(t,\omega,x_1,x_2) = \sum_{k=1}^{\infty} \lambda_k \xi_{k,i_1,\dots,i_q}(t)\varphi_k.$$

As $\mathfrak{U}$, we consider the spaces of differential q-forms defined on a smooth compact oriented Riemannian manifold without boundary and orthogonal to harmonic q-forms. Such spaces take place on the basis of the Hodge – Kodaira theory in the deterministic case for the Cauchy problem for Ginzburg-Landau equation (2). We consider the Showalter – Sidorov problem

$$P(\eta(0) - \eta_0) = 0 \tag{10}$$

for the stochastic version of the Ginzburg – Landau equation

$$(\lambda + \Delta) \overset{\circ}{\eta} = \alpha\Delta\eta + id\Delta^2\eta, \tag{11}$$

and the signs differ from (2) since instead of the Laplace operator we use its generalization (up to a sign) to spaces of differential forms, namely, the Laplace – Beltrami operator. Denote the operators

$$L = (\lambda + \Delta), M = \alpha\Delta, +id\Delta^2$$

and arrive at (5).

For this problem, the work [2] proves (L,p)-radiality of the operator M and constructs the relative spectrum

$$\mu_t = \frac{\alpha\lambda_k + id\lambda_k^2}{\lambda + \lambda_k},$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of the Laplace – Beltrami operator on the torus numbered in increasing order taking into account the multiplicity, and $\{\varphi_k\}$ is the sequence of eigenfunctions, respectively.

Introduce a grid on the torus and construct a difference analogue of the trajectories of the Ginzburg – Landau stochastic equation, and implement the Petrov – Galerkin method in the Maple system.

Here we implement the following algorithm.

Step 1. Enter the parameters of the Ginzburg – Landau equation ($\alpha, d \in \mathbb{R}, \lambda \neq 0$).

Step 2. Construct a grid on the two-dimensional torus $\mathbb{T}$.

Step 3. Calculate eigenvalues and construct eigenfunctions.

Step 4. Represent solutions in the form of expansion in terms of eigenfunctions.

Step 5. Obtain a numerical solution to the problem for a random value that belongs to the probability space Ω.

Step 6. Obtain a graphical representation of the solution and display the solution on the screen.

Fig. 1 shows the only coefficient for solution to the homogeneous Ginzburg – Landau equation for 0-forms (2-forms) at $\alpha = -0,5, d = 0,5, \lambda = 4$.

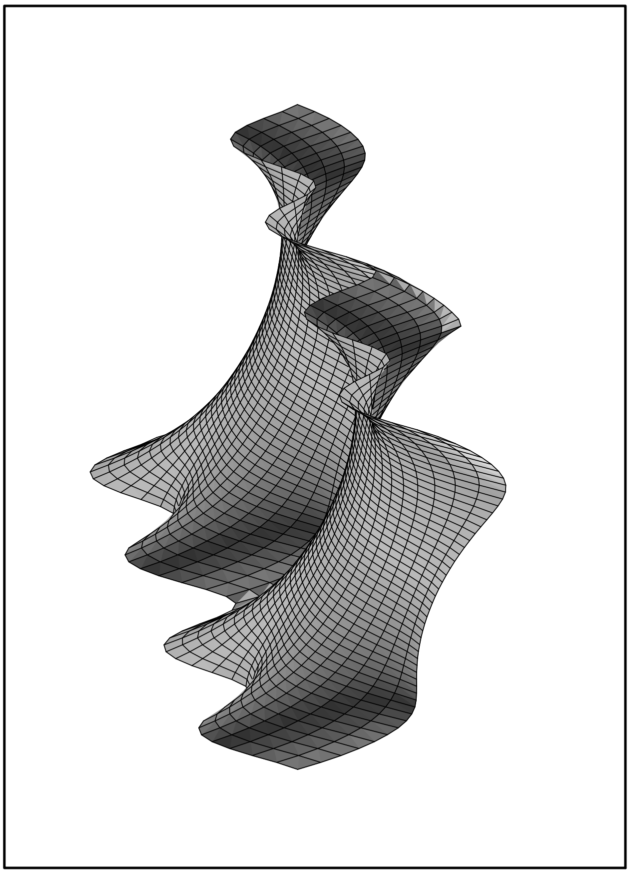

Fig. 1. Solution to (10), (11) for $\alpha = -0,5, d = 0,5, \lambda = 4$

Figs. 2 and 3 show the coefficients at dx and dy, respectively, for the solution to the homogeneous Ginzburg – Landau equation for $\alpha = 1, d = 2, \lambda = 0$.

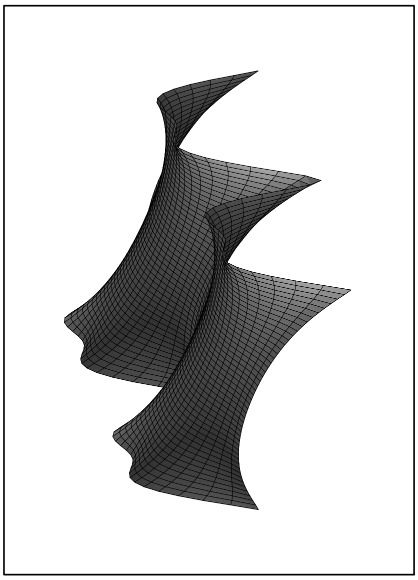

Fig. 2. Coefficient at dx of the solution to (10), (11) for $\alpha = 2, d = 1, \lambda = 0$

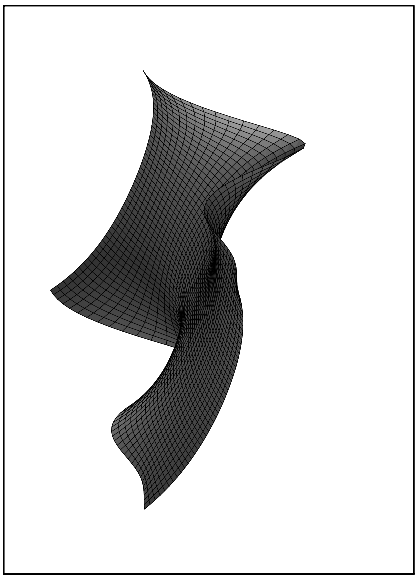

Fig. 3. Coefficient at dy of the solution to (10), (11) for $\alpha = 2, d = 1, \lambda = 0$

Conclusion

The Ginzburg – Landau equation belongs to the Sobolev type equations with a relatively radial operator. As a result of studying the numerical solutions to the homogeneous version of the Ginzburg – Landau equation, we obtain the graphs of the solution for two model cases on the torus.

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О ЧИСЛЕННОМ РЕШЕНИИ В ПРОСТРАНСТВЕ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ ДЛЯ ОДНОГО СТОХАСТИЧЕСКОГО УРАВНЕНИЯ СОБОЛЕВСКОГО ТИПА С ОТНОСИТЕЛЬНО РАДИАЛЬНЫМ ОПЕРАТОРОМ

Д. Е. Шафранов

В работе представлены графики траекторий численных решений задачи Шоуолтера – Сидорова для одного стохастического варианта уравнения Гинзбурга – Ландау в пространствах дифференциальных форм, определенных на двумерном торе. Используется ранее полученные переход от детерминированного варианта теории уравнений соболевского типа к стохастическим уравнениям с помощью производной Нельсона – Гликлиха. Так как уравнения исследуются в пространстве дифференциальных форм, то и сами операторы понимаются в специальном виде, в частности, вместо оператора Лапласа берется его обобщение оператор Лапласа – Бельтрами. Графики вычислительных экспериментов приведены для разных значений параметров исходного уравнения для одних и тех же траекторий стохастического процесса.

Ключевые слова: уравнение соболевского типа; белый шум; производная Нельсона – Гликлиха; риманово многообразие; дифференциальные формы; оператор Лапласа – Бельтрами; численное решение.

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