COMPUTATIONAL MATHEMATICS

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NUMERICAL INVESTIGATION OF THE INITAL-FINAL PROBLEM FOR THE BOUSSINESQ – LOVE EQUATION ON A GEOMETRICAL GRAPH

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The article is devoted to the analytical and numerical study of the initial-final problem for the Boussinesq – Love equation describing longitudinal oscillations in a thin elastic rod. The solution of the problem is understood as a function that defines the longitudinal diaplacement in the components of the rod structure. To find it, the Fourier method is used. The first section presents a formal analytical solution to the problem. In the second section, the theorem on the existence of the solution and its uniqueness is proved. The third section describes an algorithm for finding approximate solution to the initial-final problem for the Boussinesq – Love equation defined on graph. In the fourth section, a computational experiment is performed for the graph consisted of two edges.

Keywords: the Boussinesq – Love equation; the initial-final problem; the Sobolev type equation; the Fourier method.

Introduction

Let $G = G(\mathfrak{D}, \mathfrak{E})$ be a finite connected oriented graph , where $\mathfrak{D} = \{V_i\}$ is the set of vertices, and $\mathfrak{E} = \{E_i\}$ is the set of edges. Consider a Boussinesq – Love equation on a geometrical graph

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda')u_t + \beta(\Delta - \lambda'')u, \tag{1}$$

$$u = (u_1(x,t), u_2(x,t), \ldots), \qquad \Delta u = u_{xx},$$

$u_i(x,t)$ is the solution component on the i-th edge of the graph.

At each vertex of the graph G define the conditions

$$\sum_{E_j \in E^{\alpha}(V_i)} u_{jx}(0,t) - \sum_{E_m \in E^{\omega}(V_i)} u_{mx}(l_m,t) = 0, \tag{2}$$

$$u_j(0,t) = u_k(0,t) = u_m(l_m,t) = u_n(l_n,t). \tag{3}$$

Coefficients $\alpha, \beta, \lambda, \lambda', \lambda''$ characterize the properties of the material of the structural elements. Condition (2) indicates the balance of flows through each vertex. Condition (3) indicates that the solution at each vertex must be continuous. Function $u_i(x,t)$ defines the longitudinal displacement at the point x at the moment t for the i-th structural element.

Consider a Hilbert space

$$L_2(G) = \{g = (g_1, g_2, ..., g_j, ...), g_j \in L_2(G)\}$$

with inner product

$$< g, h >= \sum_j d_j \int_0^{l_j} g_j(x)h_j(x)dx$$

and a Banach spaces $\mathfrak{F} = L_2(G)$, and $\mathfrak{U} = \{u = (u_1, u_2, ..., u_j, ...) : u_j \in W_2^1(0, l_j)$ and (2) holds$\}$, with norm

$$||u||^2_{\mathfrak{U}} = \sum_j d_j \int_0^{l_j} (u^2_{jx} + u^2_j)dx.$$

When solving problem (1) – (3), there arises a Sturm – Liouville problem about finding the eigenfunctions of the operator $(-\Delta)$ on a geometric graph:

$$(\mathbb{X})'' + \lambda\mathbb{X} = 0, \ \mathbb{X} = (X_1, ..., X_j, ...), \quad (4)$$

$$\sum_{E_j \in E^\alpha(V_i)} X_j'(0) - \sum_{E_m \in E^\omega(V_i)} X_m'(l_m) = 0, \quad (5)$$

$$X_j(0) = X_k(0) = X_m(l_m) = X_n(l_n), \quad (6)$$

where $E_j, E_k \in E^\alpha(V_i), E_m, E_n \in E^\omega(V_i), t \in \mathbb{R}$.

This problem was investigated, the article [1] contains a generalization of previously achieved results, as well as proofs of the properties of eigenvalues and generalized eigenfunctions.

Set inital-final conditions

$$\begin{aligned} &P_0(u(x,0) - u_0^0(x)) = 0, \\ &P_0(u_t(x,0) - u_1^0(x)) = 0, \\ &P_1(u(x,\tau) - u_0^\tau(x)) = 0, \\ &P_1(u_t(x,\tau) - u_1^\tau(x)) = 0, \end{aligned} \quad (7)$$

where

$$P_0 = \sum_{\lambda_n + \lambda \neq 0, \mu_n^\pm \in \sigma_0^A(\overline{B})} < \cdot, \mathbb{X}_n > \mathbb{X}_n,$$

$$P_1 = \sum_{\lambda_n + \lambda \neq 0, \mu_n^\pm \in \sigma_1^A(\overline{B})} < \cdot, \mathbb{X}_n > \mathbb{X}_n,$$

and A-spectrum $\bar{B}$

$$\sigma^A(\overline{B}) = \{\sigma_0^A(\overline{B}) \cup \sigma_1^A(\overline{B})\}, \quad \sigma_k^A(\overline{B}) \neq \emptyset, k = 0, 1, \quad \sigma_0^A(\overline{B}) \cap \sigma_1^A(\overline{B}) = \emptyset.$$

The Boussinesq – Love equation (1) was first presented in 1935. This equation is a Sobolev type equation with respect to the second-order time derivative. Recently, a large number of studies have been devoted to equations that are not solvable with respect to the higher derivative. For example, in [3], the Boussinesq – Love equation in the domain $\Omega \subset \mathbb{R}^n$ was studied by the phase space method. The approaches of R. E. Showalter and N. A. Sidorov were developed in the article by [7] G. A. Sviridyuk and S. A. Zagrebina. The problem statement for the Boussinesq – Love equation with initial-final conditions appeared relatively recently [4]. When setting such a problem, one part of the data is set at the beginning of the time interval, and the other part is set at the end. Earlier in the studies [8, 9], a solution of the Boussinesq – Love equation for two geometric graphs was obtained using the Fourier method. This paper is devoted to the analytical and numerical study of the initial-final problem for the Boussinesq – Love equation on a geometric graph.

1. Analytical Investigation of the Boussinesq – Love Equation

1.1. Sturm – Liouville Problem on the Geometrical Graph

Problem (4) – (6) is a special case of the Sturm – Liouville problem [1]:

$$a_j(x)u_j - (c_j(x)u_{jx})_x = \lambda u_j, j = 1, 2, ..., \tag{8}$$

$$\sum_{E_j \in E^\alpha(V_i)} d_j c_j(0) u_{jx}(0) - \sum_{E_m \in E^\omega(V_i)} d_m c_m(l_m) u_{mx}(l_m) = 0, \tag{9}$$

$$u_j(0) = u_k(0) = u_m(l_m) = u_n(l_n), \tag{10}$$

where $E_j, E_k \in E^\alpha(V_i), E_m, E_n \in E^\omega(V_i), t \in \mathbb{R}$. Here $E^\alpha(V_i)$ denotes the set of edges starting or ending at the vertex V_i.

Theorem 1. *[1] Eigenvalues $\lambda_1, \lambda_2, ...$ of problem (8) – (10) are real and $\lambda_s \to +\infty$ as $s \to \infty$. Eigenvalues satisfy the inequality*

$$\lambda_s > m = \min_j \left(\min_{x \in (0, l_j)} a_j(x) \right)$$

in all cases, except $a_j(x) = a_i(x) = const$ for all i, j.

If $a_j(x) = a_i(x) = const$ for all i, j, then the eigenvalues satisfy the inequality $\lambda_s \geq m$, $s = 1, 2, ...$, and there is a single eigenvalue equal to m corresponding to eigenfunction

$$\left(\sum_{E_j \in \mathfrak{E}} d_j \int_0^{l_j} c_j(x) dx \right)^{-1} (1, 1, ..., 1, ...).$$

Theorem 2. *[1] Eigenfunctions $u_1(x), u_2(x), ...$ of problem (8) – (10) form an orthonormal basis in $L_2(G)$, i.e. any function $f \in L_2(G)$ is decomposed in a Fourier series*

$$f = \sum_{s=1}^{\infty} f_s u_s, f_s = \langle f, u_s \rangle,$$

converging in $L_2(G)$.

Remark 1. By virtue of theorem 1, nontrivial solutions to the Sturm – Liouville problem (4) – (6) exist only for $\lambda \geq 0$, so further we consider only cases where $\lambda \geq 0$.

Example 1. Let the graph G consist of two edges of lengths l_1, l_2 correspondingly, connecting three vertices (Fig. 1).

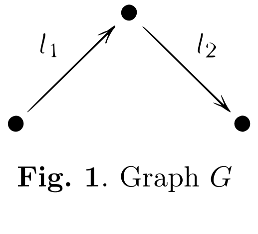

Fig. 1. Graph G

Let's find eigenfunctions and eigenvalues of the Sturm – Liouville problem for this graph. It is needed to find nontrivial solutions $\mathbb{X}(x) = (X^1(x), X^2(x))$ (eigenfunctions) of the problem

$$\begin{cases} (X^1)'' + \lambda X^1 = 0, \\ (X^2)'' + \lambda X^2 = 0, \\ X^1(l_1) = X^2(0), \\ (X^1)'(l_1) - (X^2)'(0) = 0, \\ (X^1)'(0) = 0, \\ (X^2)'(l_2) = 0, \end{cases}$$

and eigenvalues λ.

Solving this problem, we find eigenfunctions and eigenvalues:

$$\begin{cases} X_0^1 = \sqrt{\frac{1}{l_1+l_2}}, \\ X_0^2 = \sqrt{\frac{1}{l_1+l_2}}, \\ \lambda_0 = 0, \end{cases} \quad \begin{cases} X_k^1(x) = \sqrt{\frac{2}{l_1+l_2}} \cos\left(\frac{\pi k}{l_1+l_2} x\right), \ k = 1, 2, ..., \\ X_k^2(x) = \sqrt{\frac{2}{l_1+l_2}} \cos\left(\frac{\pi k}{l_1+l_2}(x + l_1)\right), \ k = 1, 2, ... , \\ \lambda_k = \left(\frac{\pi k}{l_1+l_2}\right)^2, k = 1, 2, ... \end{cases} \tag{11}$$

1.2. Inital-Final Problem for the Boussinesq – Love Equation

After determining the eigenvalues and the eigenfunctions, we turn to the search for a solution to the equation. We look for the solution of (1) – (3) in the form of a series

$$u(x, t) = \sum_{n=0}^{\infty} \mathbb{X}_n(x) T_n(t), \tag{12}$$

where $\mathbb{X}_n$ are eigenfunctions of the operator $-\Delta$, and T_n are functions of the variable t. Substituting (12) into (1) and scalarly multiplying the resulting equation by $\mathbb{X}_k(x), k = 0, 1, 2...$ in $L_2(G)$ we get

$$\sum_{n=0}^{\infty} [(\lambda + \lambda_n) T_n''(t) + (\lambda_n + \lambda')\alpha T_n'(t) + (\lambda_n + \lambda'')\beta T_n(t)] < \mathbb{X}_n(x), \mathbb{X}_k(x) >= 0.$$

Since the eigenfunctions $\mathbb{X}_k$ are orthonormal, we get:

$$(\lambda + \lambda_k) T_k''(t) + (\lambda_k + \lambda')\alpha T_k'(t) + (\lambda_k + \lambda'')\beta T_k(t) = 0.$$

Consider the cases:
Case 1: $\lambda + \lambda_k \neq 0$.
We get an ordinary differential equation of the second order with constant coefficients

$$(\lambda + \lambda_k) T_k''(t) + (\lambda_k + \lambda')\alpha T_k'(t) + (\lambda_k + \lambda'')\beta T_k(t) = 0. \tag{13}$$

Define

$$D_k = (\lambda_k + \lambda')^2 \alpha^2 - 4\beta(\lambda_k + \lambda)(\lambda_k + \lambda''). \tag{14}$$

a) If $D_k > 0$, then

$$\mu_k^{\pm} = \frac{-\alpha(\lambda_k + \lambda') \pm \sqrt{D_k}}{2(\lambda_k + \lambda)}.$$

The general solution of (13) has the form:

$$T_k = A_k e^{\mu_k^+ t} + B_k e^{\mu_k^- t}.$$

b) If $D_k = 0$, then

$$\mu_k = \frac{-\alpha(\lambda_k + \lambda')}{2(\lambda_k + \lambda)}.$$

The general solution of (13) has the form:

$$T_k = R_k e^{\mu_k t} + S_k t e^{\mu_k t}.$$

c) If $D_k < 0$, then

$$\mu_k^{\pm} = \frac{-\alpha(\lambda_k + \lambda') \pm i\sqrt{-D_k}}{2(\lambda_k + \lambda)} = \phi_k + i\psi_k.$$

The general solution of (13) has the form:

$$T_k = e^{\phi_k t}(W_k \cos(\psi_k t) + Q_k \sin(\psi_k t)).$$

Case 2: $\begin{cases} \lambda + \lambda_k = 0, \\ \lambda' + \lambda_k \neq 0. \end{cases}$

We obtain an ordinary first-order differential equation with respect to $T_k(t)$ with constant coefficients:

$$(\lambda_k + \lambda')\alpha T_k'(t) + (\lambda_k + \lambda'')\beta T_k(t) = 0.$$

In this case ∞ is an essentially singular point of the resolvent $R_\mu^A(\overline{B})$, and in this case, the solution may be nonunique. Therefore, we exclude it from further considerations.

Case 3: $\begin{cases} \lambda + \lambda_k = 0, \\ \lambda' + \lambda_k = 0, \\ \lambda'' + \lambda_k \neq 0. \end{cases}$

We obtain an algebraic equation with respect to $T_k(t)$:

$$(\lambda_k + \lambda'')\beta T_k(t) = 0.$$

Therefore $T_k(t) \equiv 0$.

So the general solution of (1) – (3) can be written as:

$$u(x,t) = \sum_{\lambda_k + \lambda \neq 0, D_k > 0} [A_k e^{\mu_k^+ t} + B_k e^{\mu_k^- t}]\mathbb{X}_k +$$

$$+ \sum_{\lambda_k + \lambda \neq 0, D_k < 0} e^{\phi_k t}[W_k \cos(\psi_k t) + Q_k \sin(\psi_k t)]\mathbb{X}_k + \sum_{\lambda_k + \lambda \neq 0, D_k = 0} e^{\mu_k^+ t}(R_k + S_k t)\mathbb{X}_k. \quad (15)$$

The coefficients $A_k, B_k, W_k, Q_k, R_k, S_k$ can be found from the initial-final conditions (7).

Example 2. Consider the Boussinesq – Love equation on graph G (Fig. 2)

$$(-4 - \Delta)u_{tt} = 2(\Delta + 4)u_t + 3(\Delta - 1)u, \quad (16)$$

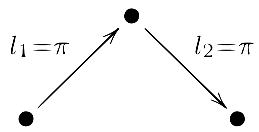

Fig. 2. Graph G

$$u = (u^1(x,t), u^2(x,t))$$

with conditions

$$\begin{aligned} u^1(l_1,t) &= u^2(0,t),\\ u^1_x(l_1,t) &= u^2_x(0,t)\\ u^1_x(0,t) &= 0,\\ u^2_x(l_2,t) &= 0. \end{aligned} \tag{17}$$

Let the relative spectrum of the pencil $\overline{B}$ have the form:

$$\sigma^A(\overline{B}) = \sigma^A_0(\overline{B}) \cup \sigma^A_1(\overline{B}),$$

where

$$\sigma^A_1(\overline{B}) = \{\mu_0, \mu^{\pm}_3, \mu^{\pm}_7\}, \sigma^A_0(\overline{B}) = \{\mu_k\} = \sigma^A(\overline{B}) \setminus \sigma^A_1(\overline{B}).$$

Set inital-final conditions:

$$\begin{aligned} P_0(u(x,0) - u^0_0(x)) &= 0,\\ P_0(u_t(x,0) - u^0_1(x)) &= 0,\\ P_1(u(x,\tau) - u^\tau_0(x)) &= 0,\\ P_1(u_t(x,\tau) - u^\tau_1(x)) &= 0, \end{aligned} \tag{18}$$

where

$u^0_0(x) = \left(3\cos\left(\frac{x}{2}\right) + \cos 2x - 4\cos\left(\frac{5x}{2}\right), 3\cos\left(\frac{x+\pi}{2}\right) + \cos 2(x+\pi) - 4\cos\left(\frac{5(x+\pi)}{2}\right)\right),$

$u^0_1(x) = \left(1 - 9\cos\left(\frac{5x}{2}\right), 1 - 9\cos\left(\frac{5(x+\pi)}{2}\right)\right),$

$u^\tau_0 = (-2 + 6\cos 6x, -2 + 6\cos 6(x+\pi)),$

$u^\tau_1 = (3\cos\left(\frac{x}{2}\right) + \cos 8x, 3\cos\left(\frac{x+\pi}{2}\right) + \cos 8(x+\pi)).$

Search the solution of (16) – (18) in the form:

$$u(x,t) = (u^1(x,t), u^2(x,t)) = \sum_{k=0}^{\infty} T_k(t)\mathbb{X}_k = \sum_{k=0}^{\infty} T_k(t)(X^1_k(x), X^2_k(x)).$$

The eigenvalues and the eigenfunctions of operator $(-\Delta)$ for this graph are given by (11).

Substituting parameters from (16) into equation (13) we get

$$(-4 + \lambda_k)T''_k(t) + 2(-4 + \lambda_k)T'_k(t) + 3(1 + \lambda_k)T_k(t) = 0. \tag{19}$$

Then the points of the relative spectrum are the roots of the equation:

$$(-4 + \lambda_k)\mu^2_k + 2(-4 + \lambda_k)\mu_k + 3(1 + \lambda_k) = 0.$$

Find D_k from (14)

$$\frac{D_k}{4} = (-4+\lambda_k)^2 - 3(-4+\lambda_k)(1+\lambda_k) = -2\lambda_k^2 + \lambda_k + 28.$$

Consider the cases:
Case 1: $\frac{D_k}{4} > 0$, for $\lambda_k \in (0;4)$, therefore, for $k \in \{0,1,2,3\}$

$$\mu_k^{\pm} = \frac{-(-4+\lambda_k) \pm \sqrt{-2\lambda_k^2 + \lambda_k + 28}}{-4+\lambda_k}.$$

Then the function T_k and its derivative have the form:

$$T_k = A_k e^{\mu_k^+ t} + B_k e^{\mu_k^- t},$$

$$T_k' = \mu_k^+ A_k e^{\mu_k^+ t} + \mu_k^- B_k e^{\mu_k^- t}.$$

Case 2: $\frac{D_k}{4} = 0$, for $\lambda_k = 4$, therefore, for $k = 4$. In this case, equation (13) becomes degenerate and the constants in the solution cannot be determined unambiguously, so we exclude it from consideration.
Case 3: $\frac{D_k}{4} < 0$, for $\lambda_k \in (4, +\infty)$, therefore, for $k \in \{5, 6, ...\}$

$$\mu_k^{\pm} = \frac{-(-4+\lambda_k) \pm i\sqrt{2\lambda_k^2 - \lambda_k - 28}}{-4+\lambda_k} = \phi_k + i\psi_k.$$

Then the function T_k and its derivative have the form:

$$T_k = e^{\phi_k t}(W_k \cos(\psi_k t) + Q_k \sin(\psi_k t)),$$

$$T_k' = e^{\phi_k t}\left(\phi_k(W_k \cos(\psi_k t) + Q_k \sin(\psi_k t)) + \psi_k(-W_k \sin(\psi_k t) + Q_k \cos(\psi_k t))\right).$$

From inital-final condition (18) we get:

$$\begin{cases} T_0(\tau) = -4\sqrt{\pi}, \\ T_0'(\tau) = 0, \end{cases} \quad \begin{cases} T_1(0) = 3\sqrt{\pi}, \\ T_1'(0) = 0, \end{cases}$$
$$\begin{cases} T_3(\tau) = 0, \\ T_3'(\tau) = \sqrt{\pi}, \end{cases} \quad \begin{cases} T_5(0) = -4\sqrt{\pi}, \\ T_5'(0) = -9\sqrt{\pi}, \end{cases} \tag{20}$$
$$\begin{cases} T_k(0) = 0, \\ T_k'(0) = 0, \\ k \neq 1,4,5, \end{cases} \quad \begin{cases} T_k(\tau) = 0, \\ T_k'(\tau) = 0, \\ k \neq 0,3,7. \end{cases}$$

Thus, from (20), we find the constants in the general solution of (16), (17)

$$\begin{cases} A_0 = \frac{2\sqrt{\pi}e^{(1+\frac{\sqrt{7}}{2})\tau}e^{\sqrt{7}}}{-e^{\sqrt{7}}+1}, \\ B_0 = -\frac{2\sqrt{\pi}e^{(1-\frac{\sqrt{7}}{2})\tau}}{-e^{\sqrt{7}}+1}, \end{cases} \quad \begin{cases} A_1 = \frac{3\sqrt{\pi}(-1+\sqrt{2})}{2\sqrt{2}}, \\ B_1 = \frac{3\sqrt{\pi}(1+\sqrt{2})}{2\sqrt{2}}, \end{cases}$$

$$\begin{cases} A_3 = \frac{7\sqrt{\pi}}{2\sqrt{322}}e^{(1-\frac{\sqrt{322}}{7})\tau}, \\ B_3 = -\frac{7\sqrt{\pi}}{2\sqrt{322}}e^{(1+\frac{\sqrt{322}}{7})\tau}, \end{cases} \quad \begin{cases} W_5 = -4\sqrt{\pi}, \\ Q_5 = \frac{-39\sqrt{\pi}}{\sqrt{78}}. \end{cases}$$

The rest constants are equal to zero. So the solution of (16)–(18) has the form:

$$u(x,t) = \left(\frac{2\sqrt{\pi}e^{(1+\frac{\sqrt{7}}{2})\tau}e^{\sqrt{7}}}{-e^{\sqrt{7}}+1} e^{(-1-\frac{\sqrt{7}}{2})t} - \frac{2\sqrt{\pi}e^{(1-\frac{\sqrt{7}}{2})\tau}}{-e^{\sqrt{7}}+1} e^{(-1+\frac{\sqrt{7}}{2})t} \right) \mathbb{X}_0 +$$

$$+ \left(\frac{3\sqrt{\pi}(-1+\sqrt{2})}{2\sqrt{2}} e^{(-1-\sqrt{2})t} + \frac{3\sqrt{\pi}(1+\sqrt{2})}{2\sqrt{2}} e^{(-1+\sqrt{2})t} \right) \mathbb{X}_1 +$$

$$+ \left(\frac{7\sqrt{\pi}}{2\sqrt{322}} e^{(1-\frac{\sqrt{322}}{7})\tau} e^{(-1-\frac{\sqrt{322}}{7})t} - \frac{7\sqrt{\pi}}{2\sqrt{322}} e^{(1+\frac{\sqrt{322}}{7})\tau} e^{(-1+\frac{\sqrt{322}}{7})t} \right) \mathbb{X}_3 +$$

$$+ \left(-4\sqrt{\pi}\cos\left(\frac{\sqrt{78}t}{3}\right) - \frac{39\sqrt{\pi}}{\sqrt{78}}\sin\left(\frac{\sqrt{78}t}{3}\right) \right) \mathbb{X}_5.$$

2. Justification of the Fourier Method for Inital-Final Problem for the Boussinesq – Love Equation on a Geometric Graph

Let‘s find conditions for the functions $u_0^0, u_1^0, u_0^\tau, u_1^\tau$, under which formula (15) determines the unique solution of problem (1)–(3), (7) on the graph G (Fig. 1). First, we show that (15) determines the solution of equation (1). The first and the last terms in (15) are finite sums, so, due to the uniformity and linearity of equation (1), these sums will be the solutions of equation (1). We show that the infinite summand is also a solution. To do this, it is sufficient to prove uniform convergence of the series

$$\sum_{\lambda_n+\lambda\neq 0, D_n<0} e^{\phi_n t}[W_n\cos(\psi_n t) + Q_n\sin(\psi_n t)]\mathbb{X}_n(x) \tag{21}$$

and ability to differentiate it the desired number of times.
1)After differentiating (21) by t once and twice, the resulting series is majored up to a constant by the series

$$\sum_{n=0}^{\infty} e^{\phi_n t}(|W_n| + |Q_n|). \tag{22}$$

Since $\phi_n t < 0$, for $\lambda_n > max\{-\lambda', \lambda'\}$, then the majorant series (22) converges, hence the initial series converges uniformly.
2) After differentiating (21) by t once or twice, and by x twice, the resulting series, up to a constant, is majored by the series

$$\sum_{n=0}^{\infty} n^2 e^{\phi_n t}(|W_n| + |Q_n|). \tag{23}$$

Here, just like in the previous case, $\phi_n t < 0$, for sufficiently large n. So, $\|e^{\phi_n t}\| < 1$ for such n. Moreover for the convergence of this series $|W_n|$ and $|Q_n|$ should be approximately $n^{-(3+\delta)}$, where $\delta > 0$.

Let $\mu_n \in \sigma_0^A(\overline{B})$. Find the coefficients W_n and Q_n by solving the system:

$$\begin{cases} < W_n X_n - u_0^0(x), \mathbb{X}_n >= 0, \\ < [W_n + \psi_n Q_n] X_n - u_1^0(x), \mathbb{X}_n >= 0. \end{cases}$$

Then

$$W_n = \int_0^{l_1} \cos\left(\frac{\pi n}{l_1+l_2}x\right)(u_0^{10}(x)dx + \int_0^{l_2} \cos\left(\frac{\pi n}{l_1+l_2}(x+l_1)\right) u_0^{20}(x)dx =$$

$$= (u_0^{10}(l_1) - u_0^{20}(0))\frac{l_1+l_2}{\pi n}\sin\left(\frac{\pi n l_1}{l_1+l_2}\right) + ((u_0^{10})'(l_1) - (u_0^{20})'(0))\left(\frac{l_1+l_2}{\pi n}\right)^2 \cos\left(\frac{\pi n l_1}{l_1+l_2}\right) -$$

$$-(u_0^{10})'(0)\left(\frac{l_1+l_2}{\pi n}\right)^2 - (u_0^{20})'(l_2)\left(\frac{l_1+l_2}{\pi n}\right)^2 \cos(\pi n) +$$

$$+\int_0^{l_1}\left(\frac{l_1+l_2}{\pi n}\right)^2 \cos\left(\frac{\pi n x}{l_1+l_2}\right)(u_0^{10})'' dx + \int_0^{l_2}\left(\frac{l_1+l_2}{\pi n}\right)^2 \cos\left(\frac{\pi n(x+l_1)}{l_1+l_2}\right)(u_0^{20})'' dx.$$

Since some terms interfere with the fulfillment of the above conditions we require the following:

$$u_0^{10}(l_1) = u_0^{20}(0),$$
$$(u_0^{10})'(l_1) = (u_0^{20})'(0),$$
$$(u_0^{10})'(0) = 0,$$
$$(u_0^{20})'(l_2) = 0.$$

Then coefficient W_n has the form:

$$W_n = \left(\frac{l_1+l_2}{\pi n}\right)^3 \frac{w_n^{(3)}}{n^3},$$

where $w_n^{(3)} = -\int_0^{l_1}\sin(\frac{\pi n x}{l_1+l_2})(u_0^{10})^{(3)}dx - \int_0^{l_2}\sin(\frac{\pi n(x+l_1)}{l_1+l_2})(u_0^{20})^{(3)}dx$ is the Fourier coefficient for the function $(u_0^0)^{(3)}(x)$.

Similarly for Q_n:

$$Q_n = \left(\frac{l_1+l_2}{\pi n}\right)^3 \frac{\frac{q_n^{(3)}}{\psi_n} - w_n^{(3)}}{n^3},$$

where $q_n^{(3)} = -\int_0^{l_1}\sin(\frac{\pi n x}{l_1+l_2})(u_1^{10})^{(3)}dx - \int_0^{l_2}\sin(\frac{\pi n(x+l_1)}{l_1+l_2})(u_1^{20})^{(3)}dx$ is the Fourier coefficient for the function $(u_1^0)^{(3)}(x)$ provided that

$$u_1^{10}(l_1) = u_1^{20}(0),$$
$$(u_1^{10})'(l_1) = (u_1^{20})'(0),$$
$$(u_1^{10})'(0) = 0,$$
$$(u_1^{20})'(l_2) = 0.$$

Having done similar procedures for the coefficients $|W_n|$ and $|Q_n|$ when $\mu_n \in \sigma_1^A(\overline{B})$, we get the following conditions:

$$u_0^{10}(l_1) = u_0^{20}(0), \quad (u_0^{10})'(l_1) = (u_0^{20})'(0),$$
$$(u_0^{10})'(0) = 0, \quad (u_0^{20})'(l_2) = 0,$$
$$u_1^{10}(l_1) = u_1^{20}(0), \quad (u_1^{10})'(l_1) = (u_1^{20})'(0),$$

$$
\begin{gathered}
(u_1^{10})'(0) = 0, \quad (u_1^{20})'(l_2) = 0, \\
u_0^{1\tau}(l_1) = u_0^{2\tau}(0), \quad (u_0^{1\tau})'(l_1) = (u_0^{2\tau})'(0), \\
(u_0^{1\tau})'(0) = 0, \quad (u_0^{2\tau})'(l_2) = 0, \\
u_1^{1\tau}(l_1) = u_1^{2\tau}(0), \quad (u_1^{1\tau})'(l_1) = (u_1^{2\tau})'(0), \\
(u_1^{1\tau})'(0) = 0, \quad (u_1^{2\tau})'(l_2) = 0.
\end{gathered} \tag{24}
$$

Since for any numbers $a > 0$ and $k > 0$, the inequality

$$\frac{a}{k} \leq \frac{1}{2}(\frac{1}{k^2} + a^2)$$

holds, requiring piecewise continuity of $(u_0^0)^{(3)}(x), (u_1^0)^{(3)}(x), (u_0^\tau)^{(3)}(x), (u_1^\tau)^{(3)}(x)$ on $[0, l_1]$ and $[0, l_2]$ and using the closure equation

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{2}{l_1 + l_2}\left(\int_0^{l_1}(u_1)^2(x)dx + \int_0^{l_2}(u_2)^2(x)dx\right)$$

for these functions, we make conclusion that the series $\sum_{n=0}^{\infty}|a_n^{(3)}|^2, \sum_{n=0}^{\infty}|b_n^{(3)}|^2, \sum_{n=0}^{\infty}|w_n^{(3)}|^2, \sum_{n=0}^{\infty}|q_n^{(3)}|^2$ converge, and hence the convergence of the series (23) takes place.

So we proved

Theorem 3. *Let the derivatives of functions $u_0^0(x), u_1^0(x), u_0^\tau(x), u_1^\tau(x)$ up to the second order be continuous, the third order derivatives of these functions be piecewise continuous and condition (24) hold. Then there exists a unique solution of problem (1) – (3), (7) on two-edged graph G. Moreover this solution can be represented as (15).*

3. Algorithm of Numerical Method

An approximate solution of problem (1)–(3), (7) can be found by the Galerkin method. The general scheme for finding an approximate solution to (1)–(3), (7) is shown in Fig. 3, Fig. 4.

4. Computational Experiment

Consider the initial-final problem for the Boussinesq – Love equation on a two-edged graph G (Fig. 2). The existence of analytical solution to this problem was already proved earlier in section 2. Now obtain solution and the solution as a result of the algorithm.

Example 3. Consider equation (1) when

$$\lambda = -4, \lambda' = -4, \lambda'' = 1, \alpha = 2, \beta = 3.$$

Let

$$l_1 = \pi, l_2 = \pi,$$

and the relative spectrum have the form:

$$\sigma^A(\overline{B}) = \sigma_0^A(\overline{B}) \cup \sigma_1^A(\overline{B}), \quad \sigma_1^A(\overline{B}) = \{\mu_0, \mu_3^\pm, \mu_7^\pm\}.$$

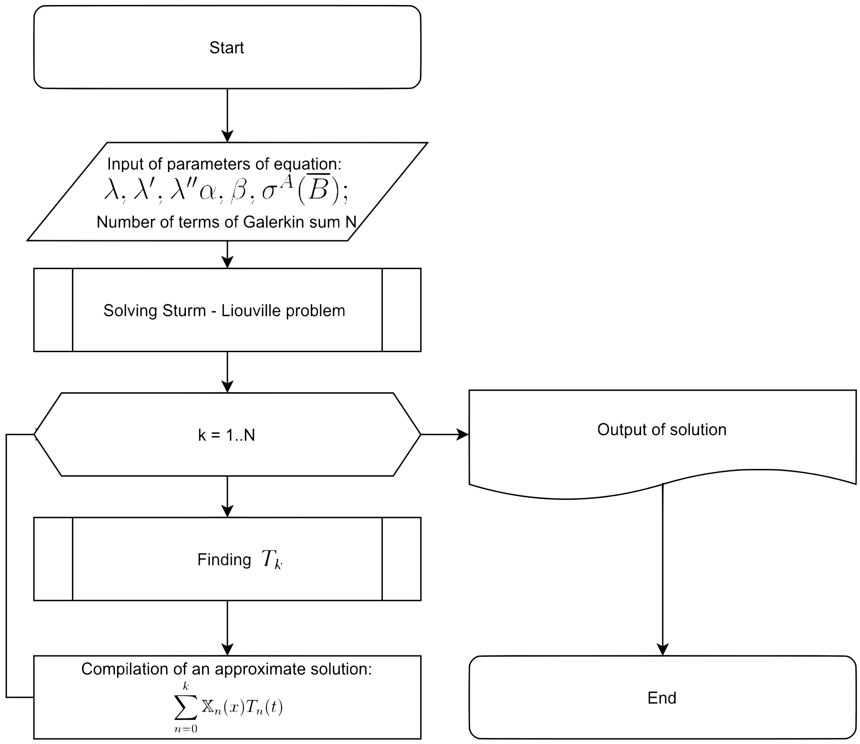

Fig. 3. Algorithm of numerical method for solving inital-final problem for the Boussinesq – Love equation on a geometric graph

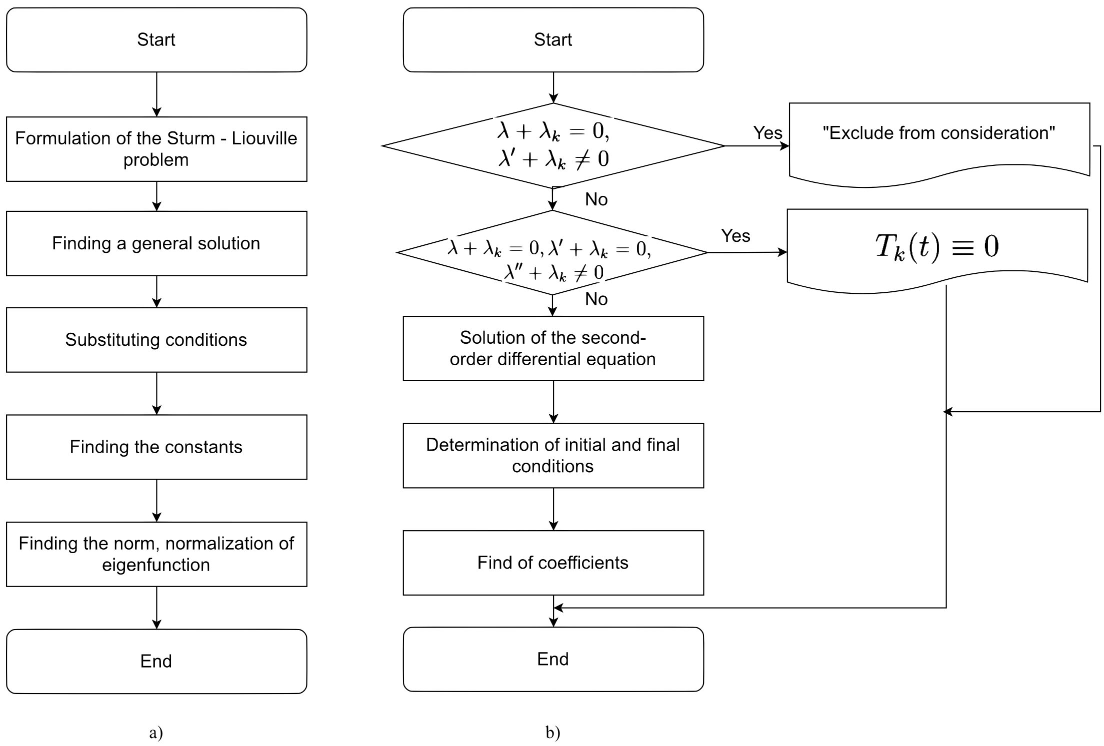

Fig. 4. a) Algorithm of solving the Sturm – Liouville problem, b) Algorithm of finding T_k

Projectors a given by (18). Let the inital-final functions $u_0^0(x), u_1^0(x), u_0^\tau(x), u_1^\tau(x)$ be the same as in example 2. Note that for these functions all the conditions of theorem 3 are satisfied.

Using the developed algorithm, an approximate solution of the problem was found when the number of Galerkin summands is equal to $N = 20$.

$$
\begin{aligned}
u_1(x,t) = \big(&-0.6118e^{2.3229\tau}e^{-2.3229t} - 4.4015e^{-0.3229\tau}e^{0.3229t}\big)\,0.3989\\
&+ \big(4.3341e^{0.4142t} + 0.7436e^{-2.4142t}\big)\,0.5642\cos(0.5x)\\
&+ \big(-0.3039e^{3.5635\tau}e^{-3.5635t} + 0.3039e^{-1.5635\tau}e^{1.5635t}\big)\,0.5642\cos(1.5x)\\
&+ \big(-7.4742\sin(2.9440t) - 6.7703\cos(2.9439t)e^{-t}\big)\,0.5642\cos(2.5x),\\
u_2(x,t) = \big(&-0.6118e^{2.3229\tau}e^{-2.3229t} - 4.4015e^{-0.3229\tau}e^{0.3229t}\big)\,0.3989\\
&- \big(4.3341e^{0.4142t} + 0.7436e^{-2.4142t}\big)\,0.5642\sin(0.5x)\\
&+ \big(-0.3039e^{3.5635\tau}e^{-3.5635t} + 0.3039e^{-1.5635\tau}e^{1.5635t}\big)\,0.5642\sin(1.5x)\\
&- \big(-7.4742\sin(2.9440t) - 6.7703\cos(2.9439t)e^{-t}\big)\,0.5642\sin(2.5x).
\end{aligned}
$$

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ НАЧАЛЬНО-КОНЕЧНОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ БУССИНЕСКА – ЛЯВА НА ГЕОМЕТРИЧЕСКОМ ГРАФЕ

А. Д. Бородина, А. А. Замышляева

Статья посвящена численному исследованию начально-конечной задачи для уравнения Буссинеска – Лява, описывающего продольные колебания в тонком упругом стержне. Под решением задачи понимается однозначное определение функции, которая задает продольное смещение в компонентах конструкции из стержней. Для ее нахождения использовался метод Фурье. В первом параграфе представлено формальное аналитическое решение задачи. Во втором разделе доказана теорема о существовании решения и его единственности. В третьем разделе описывается алгоритм нахождения решения начально-конечной задачи для уравнения Буссинеска – Лява. В четвертом параграфе проводится вычислительный эксперимент для ранее рассмотренного графа.
Ключевые слова: уравнение Буссинеска – Лява; начально-конечная задача; уравнения соболевского типа; метод Фурье.

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