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COMPARISON OF NUMERICAL MODELING METHODS FOR QUASI-STEADY PROCESS IN CONDUCTING NONDISPERSIVE MEDIUM WITH RELAXATION

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This article deals with different numerical methods of solving the Dirichlet–Cauchy problem for equation modeling the quasi-steady process in conducting nondispersive medium with relaxation. Known proofs of existence and uniqueness of solution to this problem are not constructive. Therefore the necessity of selection the appropriate numerical method arises. Such method should allow us to find a solution of the considered problem in the reasonable time. The comparative analysis of the Galerkin method and the method of straight lines with ε-embedding method and complex Rosenbrock method is performed in the article. The results of numerical experiments for one-dimensional case are shown.

Keywords: Galerkin method, Rosenbrock method, quasi-linear Sobolev type equation, weak generalized solution, numerical modeling.

Introduction

During a research of quasi-steady processes in conducting nondispersive media [1], the Dirichlet–Cauchy problem arises

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0, \tau), \tag{1}$$

$$u(x,0) = u_0(x), x \in \Omega, \tag{2}$$

for the equation modeling the quasi-steady process in conducting nondispersive medium with relaxation

$$(\Delta u - \Phi(u))_t = \Phi(u). \tag{3}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded region with boundary of class C^∞ representing the ideal conductivity region, $\tau \in \mathbb{R}_+$, unknown function u represents the potential of the electric field. Function $\Phi(u) \equiv |u|^{p-2}u, p > 2$ is monotonely increasing and smooth. Problem (1) – (3) was considered earlier in the [2], global solvability in strong generalized meaning was established under some conditions. We consider the equation (3) as a representative of the class of quasi-linear equations of Sobolev type. It enables us to prove a solvability of problem (1) – (3) in a weak generalized meaning by methods developed for this class of equations [3].

Applied nature of the problem causes the necessity of numerical modeling of the process described by the (1) – (3). The Galerkin method and the method of straight lines with ε-embedding method and complex Rosenbrock method [4] are used to solve the (1) – (3) problem numerically.

The article consists of 3 parts. The Galerkin method of solving the (1) – (3) problem is considered in the first part. An illustrative example and the results of computational experiment are present. The numerical modeling of the quasi-steady process in conducting nondispersive medium with relaxation by the method of straight lines with ε-embedding method and complex Rosenbrock method is considered in the second part. The results of computational experiment for this method on the example described in first part are listed. Third part is a comparative analysis of described methods.

1. Galerkin method

We will find the solution of problem (1) – (3) in the form of Galerkin sum

$$u^m(x, t) = \sum_{k=1}^{m} u_k(t)\Phi_k(x), m \in N,$$

where $\Phi_k(x)$ is a set of eigenfunctions of homogeneous Dirichlet problem for Laplace operator in the Ω, orthonormal with scalar product in space $L_2(\Omega)$. Taking the scalar product of (3) with $\Phi_k(x)$, $k = 1, 2, ...m$, we get differential system for unknown functions $u_k(t)$, $k = 1, 2, ...m$.

This method was implemented using the Maple mathematical system. Such selection was caused by the availability of built-in analitycal calculations mechanism and tools for ordinary differential equations solving in this system.

The developed program allows us to find the approximate solution of the Dirichlet–Cauchy problem for the equation modeling the quasi-steady process in conducting medium with relaxation in the segment for given initial conditions and number of Galerkin approximations and shows the graph of approximate solution on the screen.

Example 1. Find the solution of problem (1) – (3) with following conditions: $\Omega = (0, \pi)$, $u_0 = 0, 5 \sin x$, $\Phi(u) = u^3(x, t)$.

Problem (1) – (3) takes the form:

$$\begin{cases} \left(u(x, t) + \dfrac{\partial^2}{\partial x^2}u^3(x, t)\right)'_t = \dfrac{\partial^2}{\partial x^2}u^3(x, t), \\ u(x, 0) = 0.5 \sin x, \\ u(0, t) = u(\pi, t) = 0. \end{cases} \tag{4}$$

We represent the desired function in the form of Galerkin sum:

$$u(x, t) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{m} u_k(t) \sin(kx).$$

Substitute this representation to the equation. Taking the scalar product with eigenfunctions of Laplace operator, we get differential system for the coefficients $u_k(t)$, $k = 1, 2, ..., m$. Solving this system numerically, we get the approximate solution to the problem (4). Graphs of the approximate solution at various time points are shown in Fig. 1.
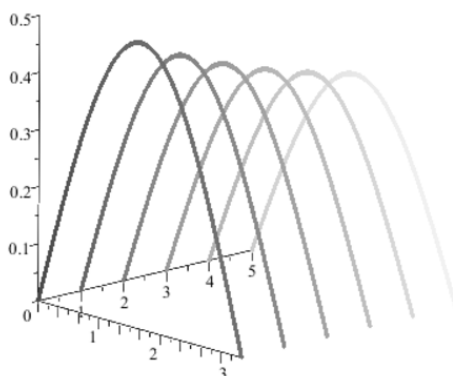
Fig. 1. Graphs of $u(x,t)$ at various time points

2. Method of straight lines with ε-embedding method and complex Rosenbrock method

Perform the decomposition of problem (1) – (3) and consider the system

$$
\begin{aligned}
w_t &= |u|^{p-2}u, \\
0 &= w - \Delta u + |u|^{p-2}u, \\
u(x,t) &= 0, (x,t) \in \partial\Omega \times (0,\tau), \\
u(x,0) &= u_0(x), x \in \Omega, \\
w(x,t) &= 0, (x,t) \in \partial\Omega \times (0,\tau), \\
w(x,0) &= \Delta u_0(x) - |u_0|^{p-2}u_0, x \in \Omega.
\end{aligned}
\tag{5}
$$

Apply the method of straight lines to the problem (5). Denote

$$
U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad
W = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix},
$$

$$
|U|^{p-2} = \begin{pmatrix}
|u_1|^{p-2} & 0 & \cdots & \cdots & 0 \\
0 & |u_2|^{p-2} & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots \\
0 & 0 & \cdots & 0 & |u_n|^{p-2}
\end{pmatrix},
$$

M is a grid analog of the Laplas operator. Proceed with differential-algebraic system:

$$
\begin{cases}
\dfrac{d}{dt}W = |U|^{p-2}U, \\
0 = W - MU + |U|^{p-2}U.
\end{cases}
$$

We solve this system by one-stage Rosenbrock method with coefficient $\alpha = \frac{1}{2} + \frac{1}{2}i$. This method and its applying to differential-algebraic systems by the ε-embedding method was considered in detail in [5], selection of the complex coefficient was also explained there. According to the Rosenbrock method at every time step we have

$$\begin{cases} \hat{W} = W + \operatorname{Re} \bar{k}, \\ \hat{U} = U + \operatorname{Re} \bar{l}, \end{cases}$$

and vectors $\bar{k}, \bar{l}$ are found from the linear algebraic system

$$\begin{cases} \bar{k} = \alpha\tau(p-1)|U|^{p-2} + \tau|U|^{p-2}U, \\ -\alpha\bar{k} + \alpha(M - (p-1)|U|^{p-2})\bar{l} = W - MU + |U|^{p-2}U, \end{cases}$$

τ is a step of the time grid.

This method was implemented using the Matlab programming language. Such selection is caused by the availability of effective and easy built-in tools for operating values in vector and matrix forms in this language, including tools for solving large linear algebraic systems.

Fig. 2 shows the solution of problem (4) found by the described method. Operator M in one-dimensional case has the form of

$$M = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix},$$
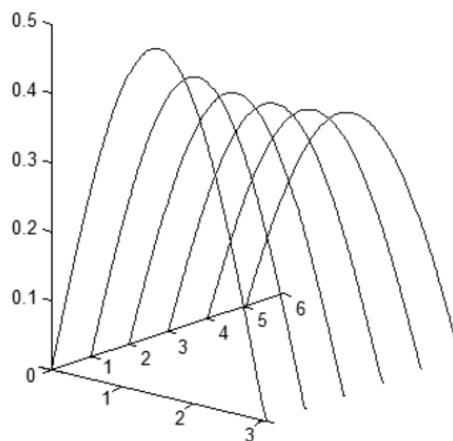
where h is a step of the space grid.

Fig. 2. Graphs of $u(x,t)$ at various time points

3. Comparative analysis of numerical methods efficiency

Perform a comparative analysis of numerical methods efficiency. At first, compare the numerical results. Values found by the Galerkin method ($u^1(x)$) and by the method of straight lines with ε-embedding method and complex Rosenbrock method ($u^2(x)$) are listed with step $\frac{\pi}{10}$ at the time moments $t = 2$ and $t = 5$ in tables 1 and 2.

Table 1

Values of the desired function at the time $t = 2$

x	$u^1(x)$	$u^2(x)$	$|u^1(x) - u^2(x)|$
0.0000	0.0000	0.0000	0.0000
0.3142	0.1322	0.1273	0.0049
0.6283	0.2514	0.2401	0.0113
0.9425	0.3461	0.3271	0.0190
1.2566	0.4068	0.3813	0.0255
1.5708	0.4278	0.3997	0.0281
1.8850	0.4068	0.3813	0.0255
2.1991	0.3461	0.3271	0.0190
2.5133	0.2514	0.2401	0.0113
2.8274	0.1322	0.1273	0.0049
3.1416	0.0000	0.0000	0.0000

Table 2

Values of the desired function at the time $t = 5$

x	$u^1(x)$	$u^2(x)$	$|u^1(x) - u^2(x)|$
0.0000	0.0000	0.0000	0.0000
0.3142	0.1105	0.1048	0.0057
0.6283	0.2102	0.1956	0.0237
0.9425	0.2893	0.2630	0.0263
1.2566	0.3402	0.3034	0.0368
1.5708	0.3577	0.3168	0.0409
1.8850	0.3402	0.3034	0.0368
2.1991	0.2893	0.2630	0.0263
2.5133	0.2102	0.1956	0.0237
2.8274	0.1105	0.1048	0.0057
3.1416	0.0000	0.0000	0.0000

Note that the difference between results calculated by different methods is less than 0.05. So we may say that results are in good agreement.

Compare the amount of computing resources spent by both methods. For the numerical experiment we use IBM PC-compatible computer equipped with Intel Core2 Duo 2.66 GHz CPU, Matlab of version R2006a and Maple of version 15. Elapsed time consumed by the first method is 2.2 seconds, and 0.6 seconds by the second method. The amount of used memory is about 45 megabytes for the first method and about 1 megabyte for the second method.

So the selection of used tools may be considered successful, solution for the problem was calculated with a little resources spent.

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