MSC 35R60 DOI: 10.14529/jcem210204

NUMERICAL SOLUTION OF THE HOFF EQUATION WITH ADDITIVE "WHITE NOISE" IN SPACES OF DIFFERENTIAL FORMS ON A TORUS

D. E. Shafranov, South Ural State University, Chelyabinsk, Russian Federation, shafranovde@susu.ru

The paper is devoted to the search for numerical solutions to the Cauchy problem for the linear stochastic equation in space of smooth differential forms on a torus. Based on the previously obtained results on the type of analytical solution to the stochastic version of the Hoff equation in spaces of smooth differential forms on smooth compact Riemannian manifolds without boundary, we choose several terms from the analytical solution in order to construct graphs of the numerical solution for various values of the coefficients and the inhomogeneous term. Since these equations are Sobolev type equations with a degenerate operator at the derivative, we can solve various initial-boundary value problems using the theory of degenerate analytic groups of resolving operators. In the deterministic case, the solution is based on the phase subspace of the original space. In spaces of differential forms, we use the invariant form of the Laplacian, i.e. the Laplace – Beltrami operator. The phase space method is also used in non-deterministic case, but we use the Nelson – Gliklikh derivative due to the non-differentiability of "white noise" in the usual sense. In this paper, a two-dimensional torus plays the role of a smooth compact oriented Riemannian manifold without boundary. Numerical solutions are found using the Galerkin–Petrov method and are presented for several fixed time points as graphs of the coefficients of differential forms obtained in Maple.

Keywords: Sobolev type equation; Nelson – Gliklikh derivative; Laplace – Beltrami operator; differential forms; Riemannian manifold.

Introduction

In its initial statement [1], the Hoff equation

$$(\lambda - \Delta)u_t = \alpha u + f$$

simulates the process of I-beam buckling under a constant load and high temperatures. Different types (including cases of different functional spaces) of initial boundary value problems for this equation were solved by reducing to abstract Sobolev type equation (with noninvertible operator at derivative) [2]

$$L\dot{u} = Mu + f. \tag{1}$$

In turn, the abstract Sobolev type equation with the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{U})$ is reduced to the equivalent system on splitting spaces

$$\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1; \mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1 \tag{2}$$

by the phase space method proposed by G. A. Sviridyuk. Recently, in the Chelyabinsk scientific school of Sobolev type equations, various results were obtained for the Hoff equation, in particular, in spaces of differential forms defined on a Riemannian manifold without boundary [3]. Note that in this case we use the invariant generalization of the Laplace operator in spaces of differential forms, i.e. the Laplace – Beltrami operator [5], and differentiability of, generally speaking, just continuous stochastic processes in the sense of Nelson – Gliklikh [6].

The aim of this paper is to obtain and analyze a numerical solution to the Cauchy problem

$$u(0) = u_0 \tag{3}$$

for equation (1) on a two-dimensional torus, which we use as an example of a Riemannian manifold. To this end, we use a numerical algorithm based on the Galerkin–Petrov approximation method in order to carry out computational experiments for some set of initial parameters of the Hoff equation in Maple. The results are presented in the form of sets of time-fixed values of the coefficients of differential forms.

The paper consists of Introduction, three sections, Conclusion, and References. Section 1 contains preliminary information and describes spaces of differential K - "noises" [4]. Section 2 describes an example for the Hoff equation on a torus. In Section 3, we use the Galerkin–Petrov approximation method in order to construct a numerical solution to the Cauchy problem. References do not pretend to be complete, but only meet the preferences of the author.

1. Sobolev Type Equations in the Deterministic Case and Differential K-"Noises" on Differential Form Spaces

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, and the operators L, $M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i.e., the operators are linear and continuous). Consider the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of the operator M. If L-spectrum $\sigma^L(M)$ of the operator M is bounded, then the operator M is called (L,σ)-bounded. If the operator M is (L,σ)-bounded, then there exist the projectors

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)d\mu \in \mathcal{L}(\mathfrak{F}).$$

Here $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are the *right* and the *left L-resolvents* of the operator M, respectively, and the closed contour $\gamma \subset \mathbb{C}$ bounds the domain containing $\sigma^L(M)$. Set $\mathfrak{U}^0(\mathfrak{U}^1) = \ker P(\mathrm{im} P)$, $\mathfrak{F}^0(\mathfrak{F}^1) = \ker Q(\mathrm{im} Q)$ and denote by $L_k(M_k)$ the restriction of the operator $L(M)$ on $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [2] (Sviridyuk's splitting theorem)
Let the operator M be (L,σ)-bounded. Then
(i) the operators $L_k(M_k) \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;
(ii) there exist the operators $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ and $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$.

Construct the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$. Under the conditions of the theorem,

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q$$

for all $\mu \in \rho^L(M)$. The operator M is called (L,p)-*bounded*, $p \in \{0\} \cup \mathbb{N}$, if ∞ is a removable singular point (i.e. $H \equiv \mathbb{O}, p = 0$) or a pole of the order $p \in \mathbb{N}$ (i.e. $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of L-*resolvents* $(\mu L - M)^{-1}$ of the operator M.

A vector function $u \in C^\infty(\mathbb{R};\mathfrak{U})$ is a *solution to equation* (1), if u satisfies the equation on $\mathbb{R}$. A solution u to equation (1) is a *solution to problem* (1), (3), if u satisfies condition (3).

Theorem 2. [2] *For the (L,p)-bounded operator M, $p \in \{0\} \cup \mathbb{N}$ and for any initial $u_0 \in \mathfrak{P} \subset \mathfrak{U}^1$ and $f \in C^\infty(\mathbb{R};\mathfrak{U})$, for problem (1),(3) there exist analytic resolving groups of operators of the form*

$$U^t = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)e^{\mu t}d\mu \in \mathcal{L}(\mathfrak{U}),\ F^t = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)e^{\mu t}d\mu \in \mathcal{L}(\mathfrak{F}), \tag{4}$$

and the solution has the form

$$u(t) = U^t u_0 - \sum_{q=0}^{p} H^p M_0^{-1}\frac{d^q(I-Q)f}{dt^q}(t) + \int\limits_0^t U^{t-s}Qf ds.$$

Let $\Omega \equiv (\Omega, \mathcal{S}, \mathbf{P})$ be a complete probability space with the probability measure $\mathbf{P}$ associated with the σ-algebra $\mathcal{S}$ of subsets of the set Ω. Denote by $\mathbb{R}$ the set of real numbers endowed with the structure of σ-algebra. Then the mapping $\chi : \mathcal{S} \to \mathbb{R}$ is called a *random variable*. The set of random variables $\{\chi\}$ with zero mathematical expectation ($\mathbf{E}\chi = 0$) and finite dispersion ($\mathbf{D}\chi < +\infty$) forms the Hilbert space $\mathbf{L_2}$ with the scalar product $(\chi_1, \chi_2) = \mathbf{E}\chi_1\chi_2$ and the norm $\|\chi\|_{\mathbf{L_2}}$. Let $\mathcal{S}_0$ be a σ-subalgebra of the σ-algebra $\mathcal{S}$. Construct the subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{S}_0$. Denote by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$ the orthoprojector. Consider the random variable $\chi \in \mathbf{L_2}$, then $\Pi\chi$ is called a *conditional expectation* and is denoted by $\mathbf{E}(\chi|\mathcal{S}_0)$.

For some interval $\mathfrak{I} \subset \mathbb{R}$, the measurable mapping $\eta : \mathfrak{I} \times \mathcal{S} \to \mathbb{R}$ is called a *stochastic process*, and the random variable $\eta(\cdot, \omega)$ is called a *section* of the stochastic process, and the function $\eta(t, \cdot)$, $t \in \mathfrak{I}$ is called a *trajectory* of the stochastic process. The stochastic process $\eta = \eta(t, \cdot)$ is called *continuous,* if a.s. (almost sure), i.e. for a.a. (almost all) $\omega \in \mathcal{S}$, the trajectories $\eta(t, \omega)$ are continuous functions. The set $\{\eta = \eta(t, \omega)\}$ of all continuous stochastic processes with values in $\mathbf{L_2}$ forms the Banach space $\mathbf{CL}_2$ with the norm

$$\|\eta\|_{\mathbf{CL}_2} = \sup_{t \in \mathfrak{I}} (\mathbf{D}\eta(t, \omega))^{1/2}.$$

Fix an arbitrary stochastic process $\eta \in \mathbf{CL_2}$.

Definition 1. A random variable

$$\overset{\circ}{\eta}(\cdot, \omega) = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathbf{E}^\eta_t \left(\frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}^\eta_t \left(\frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\Delta t}\right)\right)$$

is called the *Nelson – Gliklikh derivative* of the stochastic process η *at the point* $t \in \mathfrak{I}$, if the limit exists in the sense of a uniform metric on $\mathbb{R}$.

Here $\mathbf{E}^\eta_t = \mathbf{E}(\cdot|\mathcal{N}^\eta_t)$, and $\mathcal{N}^\eta_t \subset \mathcal{S}$ is a σ-algebra generated by the random variable $\eta(t, \omega)$.

If the Nelson – Gliklikh derivatives $\overset{\circ}{\eta}(\cdot,\omega)$ of the stochastic process $\eta(\cdot,\omega)$ there exist for a.a. points of the interval $\mathfrak{I}$, then there exists the Nelson – Gliklikh derivative $\overset{\circ}{\eta}(\cdot,\omega)$ on the interval $\mathfrak{I}$ (a.s. on $\mathfrak{I}$). Denote the set of continuous stochastic processes with continuous Nelson – Glicklikh derivatives $\overset{\circ}{\eta} \in \mathbf{CL}_2(\mathfrak{I})$ by $\mathbf{C}^1\mathbf{L}_2(\mathfrak{I})$. By induction, we can define Banach spaces $\mathbf{C}^l\mathbf{L}_2(\mathfrak{I})$, $l \in \mathbb{N}$, of the stochastic processes having continuous Nelson – Glicklikh derivatives on $\mathfrak{I}$ up to the order $l \in \mathbb{N}$ inclusively.

The norms of these spaces have the form

$$\|\eta\|_{\mathbf{C}^l\mathbf{L}_2} = \sup_{t\in\mathfrak{I}} \left(\sum_{k=0}^{l} \mathbf{D}\, \overset{\circ}{\eta}{}^{l}(t,\omega) \right)^{1/2},$$

where $\overset{\circ}{\eta}{}^{0} \equiv \eta$. Since "white noise" belongs to all the spaces $\mathbf{C}^l\mathbf{L}_2\,(\mathbb{R}_+)$, $l \in \{0\} \cup \mathbb{N}$, then these spaces are called the *spaces of "noises"*.

We also use the spaces of *random* $\mathbf{K}$*-variables*. Let space $\mathfrak{H}$ be a separable Hilbert space with an orthonormal basis $\{\varphi_k\}$, a monotone sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$ $(\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty)$, and a sequence $\{\xi_k\} = \xi_k(\omega) \subset \mathbf{L}_2$ of random variables with norm $\|\xi_k\|_{\mathbf{L}_2} \leq C$ for one $C \in \mathbb{R}_+$ and for all $k \in \mathbb{N}$. We define a $\mathfrak{H}$-valued random $\mathbf{K}$-variable $\xi(\omega) = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k(\omega)\varphi_k$. Complete the linear span of the set $\{\lambda_k\xi_k\varphi_k\}$ with the norm

$$\|\eta\|^2_{\mathbf{H_K}\mathbf{L}_2} = \left(\sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \right)^{1/2}$$

and call the result by a *space of ($\mathfrak{H}$-valued) random* $\mathbf{K}$*-variables*. Denote the space by $\mathbf{H_K}\mathbf{L}_2$. The obtained space $\mathbf{H_K}\mathbf{L}_2$ is a Hilbert space and contains a random $\mathbf{K}$-variable $\xi = \xi(\omega) \in \mathbf{H_K}\mathbf{L}_2$. Similar, a Banach space of ($\mathfrak{H}$-valued) $\mathbf{K}$-"noises" is defined as a completion of the linear span of the set $\{\lambda_k\xi_k\varphi_k\}$ with the norm

$$\|\eta\|^2_{\mathbf{C}^l\mathbf{H_K}\mathbf{L}_2} = \sup_{t\in\mathfrak{I}} \left(\sum_{k=1}^{\infty} \lambda_k^2 \sum_{m=1}^{l} \mathbf{D}\, \overset{\circ}{\eta}{}^{m}_{k} \right)^{1/2},$$

where a sequence $\{\eta_k\} \subset \mathbf{C}^l\mathbf{L}_2$, $l \in \{0\} \cup \mathbb{N}$. The vector $\eta(t,\omega) = \sum\limits_{k=1}^{\infty} \lambda_k \eta_k(t,\omega)\varphi_k$ belongs to the space $\mathbf{C}^l(\mathfrak{I}; \mathbf{H_K}\mathbf{L}_2)$, if the elements of the sequence of vectors $\{\eta_k\} \subset \mathbf{C}^l\mathbf{L}_2$ and all the Nelson – Gliklikh derivatives of these elements of the sequence up to the order $l \in \{0\} \cup \mathbb{N}$ inclusively are uniformly bounded with respect to the existing norm $\|\cdot\|_{\mathbf{C}^l\mathbf{L}_2}$.

Further, in the spaces of $\mathbf{K}$-"noises", we determine the coefficients of differential forms given on a manifold without boundary. Let M_d be a smooth connected oriented compact Riemannian manifold without boundary of the class C^∞ and of dimension d. On the manifold M_d, consider the vector space $E^q(M_d)$ of the q-forms

$$a(x_1,...,x_n) = \sum_{i_1<i_2<,...,<i_q} a_{i_1,i_2,...,i_q}(x_1,...,x_n) dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_q},$$

where $a_{i_1,i_2,...,i_q}(x_1,...,x_n) \in C^\infty(M_d)$, $q = \{0,1,...,d\}$. On the space E^q, define the scalar product

$$(a,b)_0 = \int\limits_{M_d} a \wedge *b,$$

where $*$ is the Hodge operator that establishes a linear isomorphism of q-forms and $(p-q)$-forms on M_d, but (possible) up to a sign by virtue of $** = (-1)^{q(p-q)}$. In the spaces E^q, the manifold (M_d) is defined by the formula $\Delta = d\delta + \delta d$, where d is the external differentiation operator, $\delta = (-1)^{n(k+1)+1} * d*$ is the Laplace – Beltrami operator. Introduce the following two scalar products:

$$(a,b)_1 = (a,b)_0 + (\Delta a, b)_0, (a,b)_2 = (a,b)_1 + (\Delta a, \Delta b)_0.$$

Denote by $\mathbf{H}^q_k, k = 0,1,2$ a Hilbert space obtained by completion of E^q in the norm $||\cdot||_k$, induced by the scalar product $(\cdot,\cdot)_k$, $k = 0,1,2$, $q \in \{0,1,...,d\}$. The obtained separable Hilbert space H^q_k has a basis of the eigenfunctions of the Laplace – Beltrami operator, be orthonormal with respect to the scalar product $(\cdot,\cdot)_k$, $k = 0,1,2$..

Spaces of random $\mathbf{K}$*-variables* defined on the manifold M_d: $\mathbf{U_K L}_2 = \mathbf{H}^q_{0\mathbf{K}}\mathbf{L}_2$ and $\mathbf{F_K L}_2 = \mathbf{H}^q_{2\mathbf{K}}\mathbf{L}_2$, where $\mathbf{K} = \{\lambda_k\}$ is a monotone sequence of eigenvalues of the Green operator (the eigenvalues are inverted to the eigenvalues of the Laplace – Beltrami operator). The elements of these spaces are the vectors $\alpha = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k \varphi_k$ and $\beta = \sum\limits_{k=1}^{\infty} \lambda_k \xi \psi_k$, respectively, where $\{\varphi_k\}$ and $\{\psi_k\}$ are eigenvectors of the operator orthonormal with respect to $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_2$.

Consider the spaces of $\mathbf{K}$-"noises" $\mathbf{C}^l(\mathfrak{I}; \mathbf{H}^q_{0\mathbf{K}}\mathbf{L}_2)$ and $\mathbf{C}^l((\mathfrak{I}; \mathbf{H}^q_{2\mathbf{K}}\mathbf{L}_2)$, $l \in \{0\} \cup \mathbb{N}$, $q \in \{0,1,...,d\}$, $\mathfrak{I} \subset \mathbb{R}$ is an interval.

2. Solution of Stochastic Variant of Hoff Equation on Torus

Consider the homogeneous Hoff equation

$$(\lambda - \Delta)u_t = \alpha u \tag{5}$$

and the Cauchy problem

$$u(0) = u_0. \tag{6}$$

We can [5] to reduce (5), (6) to problem (1), (3). To this end, we define the operators

$$L = (\lambda - \Delta) = (\lambda + d\delta + \delta d), M = \alpha\mathbb{I} \tag{7}$$

and consider the stochastic equation

$$L \overset{\circ}{\eta} = M\eta \tag{8}$$

with the condition

$$\eta(0) - \eta_0 = 0. \tag{9}$$

Theorem 3. [4] *Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for any $\eta_0 \in \mathbf{U_K L}_2$ a.s. there exists the unique solution $\eta \in C^1(\mathfrak{I}; \mathbf{U_K L}_2)$ to Cauchy problem (9) for equation (8) of the form $u(t) = U^t u_0$.*

Let us consider the application of the obtained results to the Hoff equation in the spaces of **K**-"noises", defined on a smooth compact Riemannian oriented manifold without boundary. As such a manifold, we consider the two-dimensional torus $\mathbb{T}^2 = [0, 2\pi] \times [0, \pi]$ and consider the stochastic variant of the Hoff equation

$$(\lambda + \Delta)u_t = \alpha\Delta u. \tag{10}$$

Define the operators L and M by the formulas

$$L = (\lambda + d\delta + \delta d), M = \alpha(d\delta + \delta d). \tag{11}$$

For the two-dimensional torus with coordinates x_1, x_2, taking into account the general representation of the Laplace – Beltrami operator on the manifold M_d with the Riemannian metric g

$$\Delta_{M_d} = \frac{1}{\sqrt{|g|}}(\partial_i g^{ij}\partial_j),$$

we obtain the Laplace – Beltrami operator in the form

$$\Delta_{\mathbb{T}^2} = (8\pi^2\partial^2_{x_1} - 4\pi\partial_{x_1}\partial_{x_2} + \partial^2_{x_2}).$$

The eigenvalues λ_k are

$$\lambda_k = \min_{E\subset\mathbb{Z}^2, |E|=k+1}\ \max_{(c_1,c_2)\subset E} 4\pi^2\left[c_1^2(1+\frac{2\pi^2}{\pi^2}) - 2c_1c_2\frac{2\pi}{\pi^2} + \frac{c_2^2}{\pi^2}\right].$$

Therefore, we have a non-negative, non-decreasing, finite multiple, converging only to ∞ sequence of eigenvalues $\{\lambda_k\}$, and the sequence of corresponding eigenfunctions $\{\varphi_k\}$ forms the necessary orthonormal basis in $\mathfrak{U}^1 = \mathbf{H}^{q\perp}_{0\mathbf{K}\Delta}\mathbf{L}_2$, where $\mathbf{H}^{q\perp}_{0\mathbf{K}\Delta}\mathbf{L}_2$ is a subspace $\mathbf{H}^{q}_{0\mathbf{K}}\mathbf{L}_2$ obtained by the Hodge – Kodaira splitting [5], be orthogonal to harmonic forms for $\lambda \neq \lambda_k$.

Since the dimension of the manifold is $d = 2$, we have solutions of two types. The first type takes place for 0-forms (and 2-forms isomorphic to them), and the second type takes place for 1-forms.

The relative spectrum of the Hoff equation has the form

$$\sigma^L(M) = \left\{\mu_k = \frac{\alpha}{\lambda - \lambda_k}\right\}, \tag{12}$$

therefore, the operator M is (L, p)-bounded. For the inhomogeneous Hoff equation

$$L \overset{\circ}{\eta} = M\eta + f, \tag{13}$$

it is necessary to find the projections of the inhomogeneity f onto the corresponding subspaces by the existing projectors $Q, (I - Q)$, see Theorem 2.

3. Numerical Solution of Stochastic Hoff Equation on Torus

Numerical solution of the Cauchy problem for the Hoff equation was carried out in Maple using the operators and variables implemented in Maple. The solution algorithm is presented by the following 5 steps.

Step 1. Input the parameters α, λ for the Hoff equation, x, y for the torus, number of steps by time, and the function $f(t)$.

Step 2. Project the Hoff equation to subspace.

Step 3. Construct a numerical analogue for the Hoff equation.

Step 4. Compute the solutions by 4 time steps.

Step 5. Output the solution in the form of a graph.

The numerical solution was obtained by the given steps up to the time $T = 4$. An approximate solution is obtained by approximating the first three basic functions of the Galerkin–Petrov method in the form of product of expansions in cosines of the trigonometric system in the variable x and in sines of the trigonometric system in the variable y. Random values are introduced using the randomize procedure.

The graphs show the solutions at the time instants $t_k, k = 1, ..., 4$, by the corresponding colors: green, blue, red, pink (see Figs. 1 – 3).

Fig. 1 shows the graphs of the solution to the homogeneous Cauchy problem with $\lambda = -3, \alpha = 1$ in the first 4 time cutoffs .

Fig. 2 shows the graphs of the solution to the homogeneous Cauchy problem with $\lambda = 3, \alpha = 0,5$ in the first 4 time cutoffs.

Fig. 3 shows the graphs of the solution to the homogeneous Cauchy problem with $\lambda = 3, \alpha = 1$ and $f(t) = t$ in the first 4 time cutoffs.

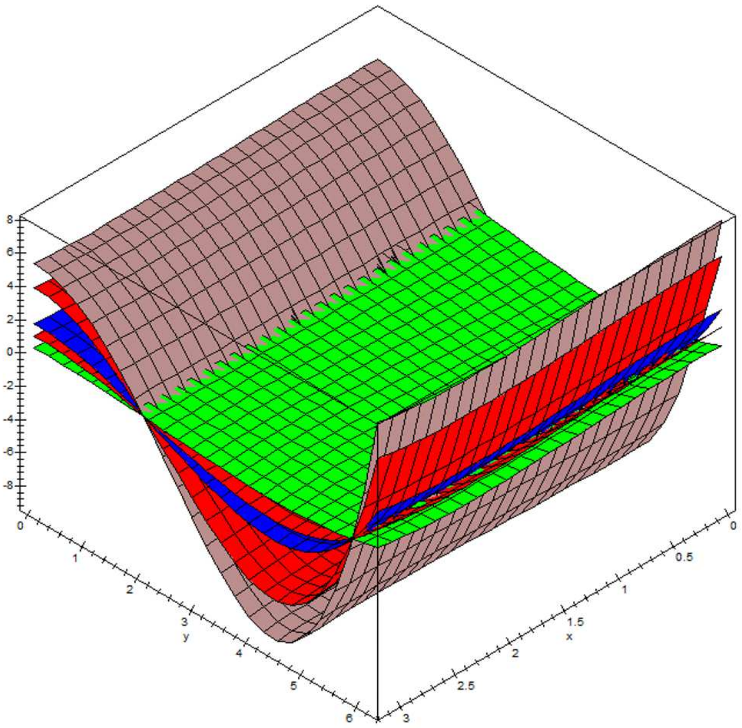

Fig. 1. The graph of the solution with $\lambda = -3, \alpha = 1, f = 0$

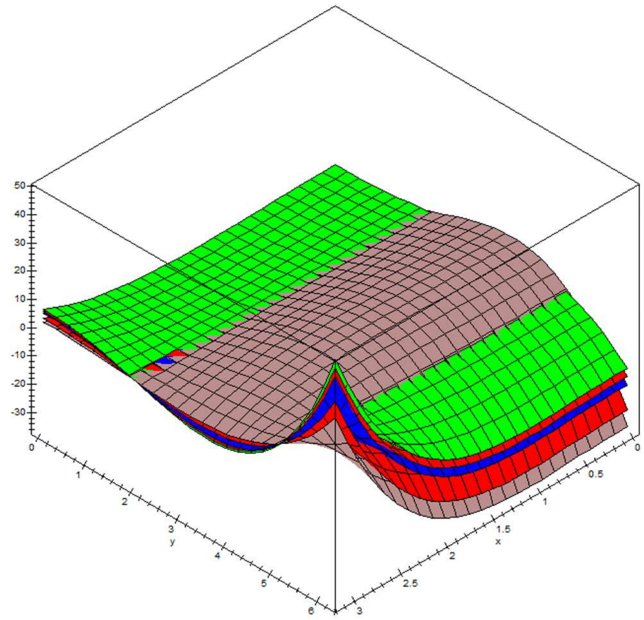

Fig. 2. The graph of the solution with $\lambda = 3, \alpha = 0.5, f = 0$

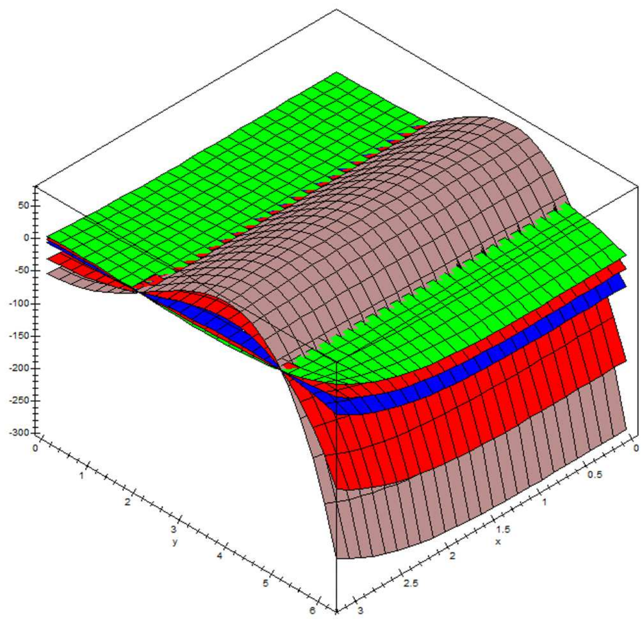

Fig. 3. The graph of the solution with $\lambda = 3, \alpha = 1.5, f(t) = t$

References

1. Sviridyuk G. A., Kazak V. O. The Phase Space of an Initial-Boundary Value Problem for the Hoff Equation. *Mathematical Notes*, 2002, vol. 71, no. 2, pp. 262–266. DOI: 10.1023/A:1013919500605.

2. Sviridyuk G. A. On the General Theory of Operator Semigroups. *Russian Mathematical Surveys*, 1994, vol. 49, no. 4, pp. 45–74. DOI: 10.1070/RM1994v049n04ABEH002390.

3. Shafranov D. E., Shvedchikova A. I. The Hoff Equation as a Model of Elastic Shell. *Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming and Computer Software*, 2012, no. 18 (277), issue 12, pp. 77–81. (in Russian)
4. Shafranov D. E., Kitaeva O. G. The Barenblatt – Zheltov – Kochina Model with the Showalter – Sidorov Condition and Additive "White Noise" in Spaces of Differential Forms on Riemannian Manifolds without Boundary. *Global and Stochastic Analysis*, 2018, vol. 5, no. 2, pp. 145–159.
5. Warner F. *Foundations of Differentiable Manifolds and Lie Groups.* N. Y., Springer-Verlag., 1983.
6. Gliklikh Yu. E. *Global and Stochastic Analysis with Applications to Mathematical Physics*. London, Springer-Verlag, 2011. DOI: 10.1007/978-0-85729-163-9.

Dmitry E. Shafranov, PhD (Math), Docent, Associate Professor at the Department of Equations of Mathematical Physics, South Ural State University (Chelyabinsk, Russian Federation), shafranovde@susu.ru.

Received February 25, 2021.

УДК 517.9 **DOI: 10.14529/jcem210204**

ЧИСЛЕННОЕ РЕШЕНИЕ УРАВНЕНИЯ ХОФФА С АДДИТИВНЫМ «БЕЛЫМ ШУМОМ» В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ НА ТОРЕ

Д. Е. Шафранов

Работа посвящена поиску численных решений задачи Коши для линейного стохастического уравнения Хоффа в пространстве гладких дифференциальных форм на торе. Исходя из ранее полученных результатов по виду аналитического решения стохастическог варианта уравнения Хоффа в пространствах гладких дифференциальных форм на гладких компактных римановых многоообразиях без края и выбирая из аналитического решения несколько слагаемых, строятся графики численного решения для различных значений коэффициентов и неоднородного члена. Это уравнения относится к уравнениям соболевского типа с вырожденным оператором при производной, что и позволило решить различные начально-краевые задачи с помощью теории вырожденных аналитических групп и полугрупп разрешающих операторов. В детерминированном случае решение строится на фазовом подпространстве исходного пространства. В пространствах дифференциальных форм используется инвариантная форма лапласиана – оператор Лапласа – Бельтрами. Метод фазового пространства также используется в недетерминированный случае, но, в силу недифференцируемости «белого шума» в обычном понимании, мы используем производную Нельсона – Гликлиха. Двумерный тор в нашей статье играет роль гладкого компактного ориентированного риманового многообразия без края. Численные решения находятся при помощи метода Галеркина – Петрова и представлены для нескольких фиксированных моментов времени, как графики коэффициентов дифференциальных форм, полученных в системе Maple.

Ключевые слова: уравнения соболевского типа; производная Нельсона – Гликлиха; оператор Лапласа – Бельтрами; дифференциальные формы; риманово многообразие.

Литература

1. Свиридюк, Г. А. Фазовое пространство начально-краевой задачи для уравнения Хоффа / Г. А. Свиридюк, В. О. Казак // Мат. заметки. – 2002. – Т. 71, № 2. – С. 292–297.
2. Свиридюк, Г. А. К общей теории полугрупп операторов / Г. А. Свиридюк // Успехи математических наук. – 1994. – Т. 49, № 4 (298). – С. 47–74.
3. Шафранов, Д. Е. Уравнение Хоффа как модель упругой оболочки / Д. Е. Шафранов, А. И. Шведчикова / Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2012. – № 2. – С. 77–81.
4. Shafranov, D. E. The Barenblatt – Zheltov – Kochina Model with the Showalter – Sidorov Condition and Additive «White Noise» in Spaces of Differential Forms on Riemannian Manifolds without Boundary / D. E. Shafranov, O. G. Kitaeva // Global and Stochastic Analysis. – 2018. – V. 5, № 2. – P. 145–159.
5. Уорнер, Ф. Основы теории гладких многообразий и групп Ли / Ф. Уорнер. – М.: Мир, 1987.
6. Gliklikh, Yu. E. Global and Stochastic Analysis with Applications to Mathematical Physics / Yu. E. Gliklikh. – London: Springer-Verlag, 2011.

Шафранов Дмитрий Евгеньевич, кандидат физико-математических наук, доцент, доцент кафедры уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), shafranovde@susu.ru.

Поступила в редакцию 25 февраля 2021 г.