

RESEARCH INTO FUNCTIONAL EQUATIONS ARISING FROM MATHEMATICAL MODELING OF CRITICAL STATES OF HETEROGENEOUS SOFT LAYERS

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We obtained the classification of solutions to a functional equation arising from the research into mathematical models of critical states of the plastic layer. The layer is exposed to a tensile stress under conditions of plane deformation. The function of the layer heterogeneity depends presumably on two variables. We demonstrated how the research into the mentioned mathematical models can be reduced to the solution of some nonlinear systems of ordinary differential equations under the conditions of separating the variables for tangent stress and for the heterogeneity function.

Keywords: soft layer, stress state, hypothesis of variables separation, systems of nonlinear differential equations, functional equations.

Introduction

Normally the stresses on the contact surface between the plastic layer and the compressive surfaces are unknown. To define the tensile (compressive) force, it is necessary to solve the inverse boundary problem of defining normal stresses on the contact surface. Various mathematic models of the stress state of a plastically deformed body at compression or tension are based, as a rule, on the differences of technological processes. At compression, what is important is the presence or absence of sliding of contact surfaces as well as the value of the friction factor. At compression and tension of the welded joints, the characteristics of the model are influenced by the deformation of the contact surfaces as well as by involvement of the more solid area in the process of plastic deformation. The suppositions applied quite often take the form of limitations over the function classes in which the solution is being looked for [1, 2]. The hypothesis of the variables separation for tensile stresses has proved to be effective [1 – 4].

$$\tau_{xy} = X(x)Y(y). \quad (1)$$

In practice, not only the whole joint but also its soft layer is heterogeneous. The basis of the mathematical model is the approximate [1 – 5] system of equations of the plastic equilibrium taking the following form for the heterogeneous layer [1, 2, 6, 7]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0; \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0; \quad \sigma_x - \sigma_y = \pm 2(Z + \mu \tau_{xy}^2/Z). \quad (2)$$

Here σ_x , σ_y , τ_{xy} – are normal and tangent stresses, $Z = Z(x, y)$ – is the function of the soft layer heterogeneity. In articles [1, 2, 6, 7] the function of heterogeneity was supposed to be dependent on one variable: $Z_{xy} = Z(y)$. It allowed us to reduce the system of the plastic equilibrium equations (2) under certain simplifying conditions to a combination of

some nonlinear systems of ordinary differential equations. The approach was based on the following statement [1, p. 80, lemma 2; 2, p. 86, lemma 1].

Functional equation

$$f(x) + g(y) + f_1(x)g_1(y) = 0 \quad (3)$$

can be solved if and only if either f and f_1 are constants or g and g_1 are constants.

In articles [8 – 10] the stressed state of the layer with the heterogeneity function of the following type has been studied for the first time:

$$Z(x, y) = U(x)V(y). \quad (4)$$

The research into mathematical models of critical states of the plastic layer was reduced there to the solution to some nonlinear systems of the ordinary differential equations. To do this, the research into the functional equation was carried out

$$f(x) + g(y) + f_1(x)g_1(y) + f_2(x)g_2(y) = 0. \quad (5)$$

If instead of hypothesis (1) we use a more general assumption,

$$\tau_{xy} = X(x)Y(y) + X_1(x)Y_1(y),$$

the method used in articles [8 – 10] will require the research in the properties of solutions to the following equation:

$$f(x) + g(y) + f_1(x)g_1(y) + f_2(x)g_2(y) + \dots + f_5(x)g_5(y) = 0.$$

The purpose of the paper is investigation into the properties of a more general than (5) functional equation

$$f(x) + g(y) + f_1(x)g_1(y) + f_2(x)g_2(y) + \dots + f_n(x)g_n(y) = 0, \quad (6)$$

as the basis of the development of method in papers [8 – 10] used for mathematical modeling of critical states of the plastic layers.

1. Functional equation. Basic theorem

Let functions

$$f(x), f_0(x) \equiv 1, f_1(x), \dots, f_n(x), g(y), g_0(y) \equiv 1, g_1(y), \dots, g_n(y)$$

be defined in set D having, at least, $n + 1$ elements ($n = 1, 2, \dots$) and acquire values in the arbitrary field F with zero characteristic.

Lemma 1. *Let functions f_1, \dots, f_n be linearly independent over field F . Let $\Phi = (f_i(x_j))$ – matrix $n \times n$. Then there exist such $x_1, x_2, \dots, x_n \in D$, that $\det \Phi \neq 0$.*

Proof.

Let us suppose that this is not the case. Among all the minors of all numerical matrices obtained by substitution of the arbitrary elements from set D instead of variables x_1, x_2, \dots, x_n in matrix $\Phi = (f_i(x_j))$, let us choose the largest in order minor M , being other

than zero. Its order is presumably smaller n . Let us choose arbitrary minor M_1 , bordering M , and choose in it nonintersecting minor M column $f_i(x_k)$ (parameter i runs a multitude of line numbers of minor M_1). By the construction for every value of $x = x_k$ minor $M_1 = 0$. Let us expand determinant M_1 over the chosen column. We will obtain a linear combination of functions $f_i(x)$, with at least one of the coefficients, namely M , being other than zero. Hence, functions f_0, f_1, \dots, f_n are linearly dependent. This is a contradiction. □

Lemma 2. *Let functions $f_0 \equiv 1, f_1, \dots, f_n$ be linearly independent over field F . Let functions*

$$f(x), f_1(x), \dots, f_n(x), g(y), g_1(y), \dots, g_n(y) \tag{7}$$

satisfy equation (6). Then functions $g(y), g_1(y), \dots, g_n(y)$ are constant.

Proof.

According to lemma 1, there exist such elements x_0, x_1, \dots, x_n of set D that $(n + 1) \times (n + 1)$ determinant $\det \Phi(f_i(x_j)) \neq 0$.

Let us consider a system of linear equations with respect to the unknown g, g_1, \dots, g_n :

$$f(x_i) + f_0(x_i)g(y) + f_1(x_i)g_1(y) + f_2(x_i)g_2(y) + \dots + f_n(x_i)g_n(y) = 0, \quad i = 1, 2, \dots, n.$$

The matrix determinant of this system by condition is other than zero. The only solution to this system does not depend on variable y , therefore it is constant: $g = \text{const}$, $g_1 = \text{const}, \dots, g_n = \text{const}$. □

Theorem 1. *Let functions $f_0 \equiv 1, f_1, \dots, f_k$ be linearly independent over field F and functions f_{k+1}, \dots, f_n linearly depend on them. Let functions (7) satisfy equation (6). Then functions $g(y), g_1(y), \dots, g_k(y)$ linearly depend on functions $g_0(y) \equiv 1, g_{k+1}(y), \dots, g_n(y)$.*

Proof.

By condition,

$$f_s = \sum_{i=1}^k \alpha_{si} f_i(x), \quad s = k + 1, \dots, n.$$

Substituting these expressions in equation (6), we obtain:

$$\sum_{i=1}^k f_i(x)g_i(y) + \sum_{s=k+1}^n \left(\sum_{i=1}^k (\alpha_{si} f_i(x) + \alpha_s) \right) g_s(y) + f + g = 0,$$

$$\sum_{i=1}^k f_i(x) \left(g_i(y) + \sum_{s=k+1}^n \alpha_{si} g_s(y) \right) + \sum_{s=k+1}^n \alpha_s g_s(y) + f + g = 0.$$

Since by condition functions $f_i, i = 0, 1, \dots, k$, are linearly independent, from lemma 2 it is evident that

$$g_i(y) + \sum_{s=k+1}^n \alpha_{si} g_s(y) = c_i; \quad i = 1, \dots, k;$$

$$\sum_{s=k+1}^n \alpha_s g_s(y) + g = c.$$

Hence, functions $g(y), g_1(y), \dots, g_k(y)$ linearly depend on functions $g_0(y) \equiv 1, g_{k+1}(y), \dots, g_n(y)$.

□

We can suggest another statement of theorem 1.

Theorem 2. *Let there be r_f of linearly independent functions over field F among functions $f_0 \equiv 1, f_1, \dots, f_n$, there is r_g of linearly independent functions over field F among functions g, g_0, \dots, g_n and functions*

$$f(x), f_1(x), \dots, f_n(x), g(y), g_1(y), \dots, g_n(y)$$

satisfy equation (6). Then $r_f + r_g \leq n + 2$.

From the viewpoint of applications to heterogeneous plastic media, of particular interest is case $n = 2$, i.e. equation (5).

Theorem 3. *With $n = 2$ the following cases are realized.*

Case 1: $r_f + r_g = 2$. *Then all the functions*

$$f(x), f_1(x), f_2(x), g(y), g_1(y), g_2(y)$$

are constant. These constants have to satisfy equation (5):

$$f + g + f_1 g_1 + f_2 g_2 = 0.$$

Case 2: $r_f + r_g = 3$. *Let $r_f = 1, r_g = 2$. Then functions*

$$f(x), f_1(x), f_2(x)$$

are constant. Functions

$$g(y), g_1(y), g_2(y)$$

are not constant. For some elements of field F $\alpha, \alpha_1, \alpha_2, \alpha^2 + \alpha_1^2 + \alpha_2^2 \neq 0$ the following equations holds:

$$\alpha g(y) + \alpha_1 g_1(y) + \alpha_2 g_2(y) = 0,$$

as well as equation (5)

$$f + g(y) + f_1 g_1(y) + f_2 g_2(y) = 0.$$

Functions $g_1(y)$ u $g_2(y)$ depend linearly on $g(y)$. If $r_f = 2, r_g = 1$, then functions

$$g(x), g_1(x), g_2(x)$$

are constant. This case is symmetrical to the previous one.

Case 3: $r_f + r_g = 4$. *If $r_f = 1, r_g = 3$, then functions $f(x), f_1(x), f_2(x)$ are constant. This variant was considered in case 1. Let $r_f = 2, r_g = 2$. Then two subcases are possible.*

Case 3.1. *There exist $\alpha, \beta \neq 0, \gamma, \delta \in R$ such that*

$$f_2(x) = \alpha + \beta f_1(x),$$

$$\begin{aligned} f(x) + \delta f_1(x) + \gamma &= 0, \\ g(y) = \alpha g_2(y) &= \gamma, \\ g_1(y) + \beta g_2(y) &= \delta. \end{aligned}$$

Case 3.2. There exist $\alpha \neq 0, \beta, \gamma \in R$ such that

$$\begin{aligned} f_1(x) &\equiv \alpha, \\ f(x) + \beta f_2(x) + \gamma &= 0, \\ g_2(y) &= \beta, \\ g(y) + \alpha g_1(y) &= \gamma. \end{aligned}$$

2. Example of reducing system (2) to a system of ordinary differential equations

The result of system (2) under conditions (1) and (3) is equation [8, 9]:

$$(2U'/X)(2V'/Y) - (1/X)(X^2/U)'(1/Y)(Y^2/V)' + X''/X - Y''/Y = 0. \quad (8)$$

Suppose, using designations from (1),(4) and (5),

$$\begin{aligned} f(x) &= U'/X, \quad g(y) = 2V'/Y, \quad f_1(x) = -(1/X)(X^2/U)', \\ g_1(y) &= (1/Y)(Y^2/V)', \quad f_2(x) = X''/X, \quad g_2(y) = -Y''/Y. \end{aligned}$$

Then each of the three mentioned cases of theorem 3 results in the system of ordinary differential equations. As an example, let us consider the case when all the 6 values of g, g_1, g_2, f, f_1, f_2 are constant. Then the following system of equations with an independent variable y occurs:

$$V'/Y = g, \quad (1/Y)(Y^2/V)' = g_1, \quad Y''/Y = g_2, \quad (9)$$

as well as the similar system with variable x . Solution to system (9) with initial conditions (7) yields three cases depending on the sign of constant g_2 . Suppose $g_2 = -a^2, a \neq 0$. Then from (9) follows equation

$$Y'' + a^2Y = 0.$$

Solution to this equation under condition (7) is function

$$Y = C_1 \sin(ay),$$

where C_1 is an arbitrary constant. Using the other equations of system (9), we can obtain

$$a = \sqrt{-gg_1}, \quad V = 0.5(1 + \cos \sqrt{-gg_1}y), \quad Y = 0.5\sqrt{-g_1/g} \sin \sqrt{gg_1}y.$$

If $g_2 = a^2, a \neq 0$, we can obtain similarly:

$$V = 0.5(1 + \operatorname{ch}(\sqrt{gg_1}y)), \quad Y = 0.5\sqrt{g_1/g} \operatorname{sh}(\sqrt{gg_1}y).$$

Thus we get similar expressions of the second system of equations for the functions depending on x . Depending on the increase or decrease of functions U, V , four variants of solution to equation (6) are obtained. For example, if functions $U(x), V(y)$ are decreasing, then

$$Z(x, y) = U(x)V(y) = 0.25(1 + \cos \sqrt{-gg_1}y)(1 + \cos \sqrt{-ff_1}x).$$

Tangent stresses in the layer are calculated according to formula (1). Using equations of equilibrium and the condition of plasticity (2), by integration we can get analytical expressions for calculating normal stresses and critical load.

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