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STABILIZATION OF A SYSTEM WITH LINEAR DELAY

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The problem of stabilization of a linear system of differential equations with constant delay containing fast and slow variables is considered. The systems of this kind can be obtained from the systems with linear delay by replacing an argument. An algorithm for stabilizing this system with delay is proposed and implemented using Matlab application software package.

Keywords: asymptotic stability; linear delay; stabilization.

Introduction

The controlled system of differential equations with constant delay

$$\begin{gathered} dx(t)/dt = A_1x(t) + B_1x(t-\tau) + A_2y(t) + B_2y(t-\tau) + C_1u_1(t), \\ dy(t)/dt = \vartheta_0 e^t[A_3x(t) + B_3x(t-\tau) + A_4y(t) + B_4y(t-\tau) + C_2u_2(t)], \\ t \geq 0,\ \ \vartheta_0 = \text{const},\ \ \vartheta_0 > 0,\ \ \tau = \text{const},\ \ \tau > 0 \end{gathered} \tag{1}$$

is considered. The systems of this kind can be obtained from the systems with linear delay $(1-\mu)\vartheta$, $\tau = -ln(\mu)$ by the substitution $t = \ln\frac{\vartheta}{\vartheta_0}$. Systems with linear delay occur in problems of mechanics, physics [1], biology, information exchange. For example, in the study of oscillation process of the current collector of the moving locomotive at interaction with the contact wire (taking into account the impact of elastic supports) [2]. In this case, if a «light» suspension is used during dynamic interaction, then at some (sufficiently high) speeds of movement of the locomotive, a large «separation» of the skid of the pantograph from the contact wire occurs, i.e. instability of movement. The problems of stabilization of other (simpler) systems with linear delay were considered, for example, in [3, 4].

One can see that the considered system for sufficiently large values of the argument is a combination of two subsystems containing fast $y(t)$ and slow $x(t)$ motions. This classification is proposed in [5], where similar complex systems are considered and one of the two subsystems contains a small parameter at the derivative.

Matrices A_j, B_j $(j = 1, 2, 3, 4)$ have dimension $[m \times m]$, vector-functions $x(t), y(t)$ have dimension m. The components $u_j(t)$ of control $u(t) = \{u_1(t), u_2(t)\}^\top$ are r-dimensional vector functions, the matrices C_j have dimensions $[m \times r]$. If for $u(t) \equiv 0$ the solution of the system is unstable, then the problem of stabilizing the system on an infinite time interval [6] arises. The goal of stabilization proposed by the authors is to develop such stabilization algorithms that can be obtained, for example, by solving equations of the Lyapunov–Riccati type. Note that the authors have solved the problem of stabilization of a particular unstable fourth-order system using these algorithms.

1. Exponential Stability Conditions and Stabilization of Weakly Coupled Systems

We consider the linear normalized space $\mathbb{R}^m$ in which the norm of a vector $w = \{w_j\}^\top$ (here w_j $(j = 1, ..., 2m)$ are components of the vector w, $\top$ is transpose icon) is defined, for example, as

$$\|w\| = \sum_{j=1}^{2m} |w_j|.$$

We define the norm of the matrix $D = \{d_{ij}\}$ $(i, j = 1, ..., 2m)$ in accordance with the norm of the vector [7]: $\|D\| = \max_j \sum_i |d_{ij}|$. In favor of this choice, we can say that the norm of a matrix is defined in almost the same way as the norm of a vector.

Consider $u(t) \equiv 0$. Setting $x_{n+1}(t) = x(t + n\tau)$, $y_{n+1}(t) = y(t + n\tau)$: $t \in [0, \tau]$, we pass to the countable differential-difference system on the finite time interval $[0, \tau]$ [1]. We have the relations

$$\begin{gathered} dx_{n+1}(t)/dt = A_1 x_{n+1}(t) + B_1 x_n(t) + A_2 y_{n+1}(t) + B_2 y_n(t), \\ \varepsilon_n dy_{n+1}(t)/d\tau = e^t[A_3 x_{n+1}(t) + B_3 x_n(t) + A_4 y_{n+1}(t) + B_4 y_n(t)], \\ \varepsilon_n = \frac{\mu^n}{\vartheta_0}, \;\; t \in [0, \tau], \;\; x_{n+1}(0) = x_n(\tau), \;\; y_{n+1}(0) = y_n(\tau). \end{gathered} \tag{2}$$

Consider asymptotic properties of the differential system (2) for small values of ε_n. Obviously, the system contains slow $x_n(t)$ and fast variables $y_n(t)$ [5]. As shown in [3, 9], the problem of obtaining sufficient stability conditions (and hence, in what follows, constructing a stabilization algorithm) is reduced to studying the asymptotic properties of the simpler system

$$\begin{gathered} dx^0_{n+1}(\theta_n)/d\theta_n = \varepsilon_n[(A_1 + B_1)x^0_{n+1}(\theta_n) + A_2 y^0_{n+1}(\theta_n) + B_2 y^0_n(\theta_n)], \\ dy^0_{n+1}(\theta_n)/d\theta_n = e^{\theta_n \varepsilon_n}[(A_3 + B_3)x^0_{n+1}(\theta_n) + A_4 y^0_{n+1}(\theta_n) + B_4 y^0_n(\theta_n)], \\ \theta_n = \frac{t}{\varepsilon_n}. \end{gathered} \tag{3}$$

Since subsystems in system (3) may be weakly connected [6], [3], we assume that for the matrices A_1, B_1 the following terms are valid:
1) the eigenvalues λ of matrix A_1 have negative real part

$$Re(\lambda) < -\beta_1, \;\; \beta_1 = \text{const}, \;\; \beta_1 > 0;$$

2) the eigenvalues $\bar{\lambda}$ of the matrix $A_1 + B_1$ also have a negative real part, i.e.

$$Re(\bar{\lambda}) < -\beta_2, \;\; \beta_2 = \text{const}, \beta_2 > 0; \tag{4}$$

3) for the second subsystem in (3) we suppose that for the eigenvalues $\hat{\lambda}$ of the matrix A_4 the following inequality is valid

$$Re(\hat{\lambda}) < -\beta_3, \;\; \beta_3 = \text{const}, \;\; \beta_3 > 0;$$

4) for the eigenvalues ρ of the matrix $-A_4^{-1}B_4$ the following inequality holds

$$|\rho| < \gamma, \;\; \gamma = \text{const}, \;\; 0 < \gamma < 1. \tag{5}$$

In the work [8] it is shown that conditions 1–4 are often insufficient for the asymptotic stability of the system (2). The asymptotic stability conditions for the original system (1) (using the Laplace transform method) are obtained in [10]. We will obtain somewhat different sufficient conditions (simpler than for stabilizing the system (2)) for asymptotic stability, which essentially takes into account the small parameter ε_n.

First, consider the asymptotic properties of the first subsystem in (3) containing only the variables $x_j^0(\theta_n)$. In general, we assume that at least one element of the matrices A_2, B_2 is nonzero. Obviously, this subsystem does not contain the delay terms. Since the matrix B_1 can be zero, the necessary conditions for asymptotic stability are condition 1. Further, for the «limit» matrix $A_1 + B_1$, conditions 2 are also true since the «limit» system has the form

$$dx_{n+1}^0(\theta_n)/d\theta_n = (A_1 + B_1)x_{n+1}^0(\theta_n), \theta_n = \frac{t}{\varepsilon_n}. \tag{6}$$

This follows in view of relation

$$x_{n+1}(\varepsilon_n\theta_n - \tau) = x_{n+1}\left(\frac{t}{\varepsilon_n}\right) - \tau\varepsilon_n\left[Ax_{n+1}\left(\frac{t}{\varepsilon_n} - \bar{\theta}_n\right) + Bx_{n+1}\left(\frac{t}{\varepsilon_n} - \bar{\theta}_n - \tau\right)\right] =$$

$$= \mathbf{O}(\varepsilon_n)[\sup_{\theta_n}\|x_n(\theta_n)\| + \sup_{\theta_n}\|x_{n-1}(\theta_n)\|], \;\; 0 < \bar{\theta}_n < 1.$$

Finally, from this we obtain that the solution of the system (6) (the solution of the unperturbed system) is exponentially stable.

Taking into account the above, consider the problem of stabilization of the system of «slow» motions

$$dx^0(t)/dt = A_1x^0(t) + B_1x^0(t-\tau) + C_1u(t). \tag{7}$$

Let for $u(t) \equiv 0$ the solution of this system be unstable, or stable, but not asymptotically. If condition 1 is not satisfied, we stabilize the system without delay terms, for which we use the method proposed in [6], assuming the control action $u_1^1(t) = -C_1^\top\Gamma_1x^0(t)$, where Γ_1 is a symmetric matrix of dimension $[m \times m]$ satisfying equation

$$\Gamma_1A_1 + (A_1)^\top\Gamma_1 - 2\Gamma_1C_1(C_1)^\top\Gamma_1 = -\alpha_1\Gamma_1 - \delta_1E,$$

E is an identity matrix of dimension $[m \times m]$; δ_1 is a small positive number; α_1 is a positive value that we can set. Since for the eigenvalues λ_j^s of the «corrected» matrix $A_1^s = A_1 - C_1(C_1)^\top\Gamma_1$ the relation $Re(\lambda_j^s) \leq -\frac{\alpha_1}{2}$ is valid, due to the choice of α_1 (making the value of α_1 large enough) it is often possible to achieve that, along with the fulfillment of the inequality (4) it is also true that (5). Otherwise, we perform further stabilization, we set $u_1^2(t) = -C_1^\top\Gamma_2x^0(t-\tau)$ solving the matrix equation

$$\Gamma_2(A_1^s + B_1) + (A_1^s + B_1)^\top\Gamma_2 - 2\Gamma_2C_1(C_1)^\top\Gamma_2 = -\alpha_2\Gamma - \delta_2E, \;\; A_1^s = A_1 - C_1(C_1)^\top\Gamma_1.$$

Now assuming in (7) the control $u_1(t) = -(C_1)^\top\Gamma_1x^0(t) - (C_1)^\top\Gamma_2x^0(t-\tau)$, we obtain the exponential stability of the «limit» system (system of «slow» motions).

Let us now consider the behavior of the second subsystem in (1) in the absence of variables $x(t)$, $x(t-\tau)$. Obviously, for $u_2(t) \equiv 0$ we have system

$$dy^0(t)/dt = \vartheta_0 e^t[A_4 y^0(t) + B_4 y^0(t-\tau)]. \tag{8}$$

To justify the stabilization algorithm for the second subsystem in (1), we prove that inequality 3 is very important in determining the sufficient conditions for the solution of the full system.

Lemma 1. *Let among the eigenvalues* $\hat{\lambda}_j$ *of the matrix* A_4 *there is* $\hat{\lambda}_0$*:* $\max_j Re(\hat{\lambda}_j) = Re(\hat{\lambda}_0) = \bar{\alpha} > 0$*. Then the system* (8) *is unstable for any matrix* B_4*.*

Proof.

Let's make in (8) the substitution

$$y^0(t) = \gamma(t) exp\{\theta_0(\bar{\alpha}+\bar{\varepsilon})e^t\}$$

($\bar{\varepsilon}$ is a sufficiently small positive number, $\gamma(t)$ is a bounded vector function of dimension $m \times 1$). Substituting this expression in (8) and reducing both sides of the resulting relation by $exp\{\theta_0(\bar{\alpha}+\bar{\varepsilon})e^t\}$, we obtain for $\gamma(t)$ equation

$$d\gamma(t)/dt = \theta_0 e^t\left[[A_4 - (\bar{\alpha}+\bar{\varepsilon})E]\gamma(t) + B_4 exp\{\theta_0(\bar{\alpha}+\bar{\varepsilon})(e^{-\tau}-1)e^t\}\gamma(t-\tau)\right]. \tag{9}$$

The corresponding differential-difference system has the form

$$\varepsilon_n d\gamma_{n+1}(t)/dt = e^t[(A_4 - (\bar{\alpha}+\bar{\varepsilon})E)\gamma_{n+1}(t) + B_4 exp\{\theta_0(\bar{\alpha}+\bar{\varepsilon})(e^{-\tau}-1)\mu^{-n}e^t\}e^t\gamma_n(t)].$$

Assuming the terms containing $\gamma_n(t)$ to be inhomogeneous, we write down the solution of the system (9) in integral form [1]

$$\gamma_{n+1}(t) = Y_{n+1}(t,0)\gamma_{n+1}(0) + \int_0^t Y_{n+1}(t,s)B_4 exp\{\theta_0(\bar{\alpha}+\bar{\varepsilon})](e^{-\tau}-1)e^s\}\frac{e^s}{\bar{\varepsilon}}\gamma_n(s)ds. \tag{10}$$

Here $Y_{n+1}(t,s) = exp\{\theta_0\mu^{-n}[A_4 - (\bar{\alpha}+\bar{\varepsilon})E][e^t - e^s]\}$, $0 \le s \le t \le \tau$, E is the identity $m \times m$ matrix. Obviously, the eigenvalues $\lambda_{\bar{\varepsilon}}$ of the matrix $A_4 - (\bar{\alpha}+\bar{\varepsilon})E$ satisfy the inequality $Re(\lambda_{\bar{\varepsilon}}) < -\bar{\varepsilon}_1$, $0 < \bar{\varepsilon}_1 < \bar{\varepsilon}$. Then the estimate

$$\|Y_{n+1}(t,s)\| \le M exp\{-\theta_0\mu^{-n}\bar{\varepsilon}_1(e^t - e^s)\}, \quad M = \text{const}, \quad M > 1 \tag{11}$$

is valid. Taking into account this estimate, from (10) we obtain inequality

$$\begin{gathered}\|\gamma(t)_{n+1}\| \le M exp\{-\theta_0\mu^{-n}\bar{\varepsilon}_1(e^t-1)\}\|\gamma_{n+1}(0)\| + \\ + \int_0^t \frac{e^s}{\bar{\varepsilon}} M exp\{-\theta_0\mu^{-n}\bar{\varepsilon}_1(e^t - e^s)\}exp\{e^s\theta_0\mu^{-n}\bar{\varepsilon}_1(\mu-1)\}\|B_4\|\|\gamma_n(s)\|ds.\end{gathered} \tag{12}$$

Consider the integral term on the right-hand side of this inequality. Obviously, the relation

$$\int_0^t \frac{e^s}{\bar{\varepsilon}} M exp\{-\theta_0\mu^{-n}\bar{\varepsilon}_1 e^s)\}exp\{e^s\theta_0\mu^{-n}\bar{\varepsilon}_1(\mu-1)\}\|B_4\|\|\gamma_n(s)\|ds \le$$

$$\leq \frac{M}{\mu}\max_t \|\gamma_n(t)\| exp\{e^t\theta_0\mu^{-n}\bar{\varepsilon}_1\mu\}\|B_4\|$$

is true. Then from (11), (12) we obtain inequality

$$\|\gamma(t)_{n+1}\| \leq Mexp\{-\theta_0\mu^{-n}\bar{\varepsilon}_1(e^t-1)\}\|\gamma_{n+1}(0)\| + \frac{M}{\mu}\max_t \|\gamma_n(t)\| exp\{e^t\theta_0\mu^{-n}\bar{\varepsilon}_1\mu\}\|B_4\|.$$

Hence, for $t=\tau$ we obtain

$$\|\gamma(\tau)_{n+1}\| \leq Mexp\{-\theta_0\mu^{-(n-1)}\bar{\varepsilon}_1(1-\mu)\}\|\gamma_{n+1}(0)\| + \frac{M}{\mu}\max_t \|\gamma_n(t)\| exp\{\theta_0\mu^{-n}\bar{\varepsilon}_1\}\|B_4\| \leq$$

$$\leq \left[Mexp\{-\theta_0\mu^{-(n-1)}\bar{\varepsilon}_1(1-\mu)\} + \frac{M}{\mu}exp\{\theta_0\mu^{-n}\bar{\varepsilon}_1\}\|B_4\|\right]\max_t \|\gamma_n(t)\|.$$

Therefore, for $\max_t \|\gamma_{n+2}(t)\|$ from (12) we have the estimate

$$\max_t \|\gamma_{n+2}(t)\| \leq M\left[Mexp\{-\theta_0\mu^{-(n-1)}\bar{\varepsilon}_1(1-\mu)\} + \frac{M}{\mu}exp\{\theta_0\mu^{-n}\bar{\varepsilon}_1\}\|B_4\|\right]\max_t \|\gamma_n(t)\|+$$

$$+\frac{M}{\mu}\max_t \|\gamma_{n+1}(t)\| exp\{\theta_0\mu^{-(n+1)}\bar{\varepsilon}_1\}\|B_4\|.$$

Denoting $\max_t \|\gamma_n(t)\| = v_n$, we obtain from the last estimate the inequality

$$v_{n+2} \leq M\left[Mexp\{-\theta_0\mu^{-(n-1)}\bar{\varepsilon}_1(1-\mu)\} + \frac{M}{\mu}exp\{\theta_0\mu^{-n}\bar{\varepsilon}_1\}\|B_4\|\right]v_n+$$

$$+\frac{M}{\mu}\|exp\{\theta_0\mu^{-n+1}\bar{\varepsilon}_1\}\|B_4\|v_{n+1} = \bar{\beta}^1_{n+1}v_{n+1} + \bar{\beta}^2_n v_n.$$

Hence it follows that the value v_{n+2} does not exceed the solution of the second-order system [10]

$$\bar{v}_{n+1} = \bar{A}_n\bar{v}_n,\;\; \bar{v}^1_n = v_n,\;\; \bar{v}^2_n = v_{n+1},\;\; \bar{A}_n = \begin{pmatrix} 0 & 1 \\ \bar{\beta}^1_n & \bar{\beta}^2_{n+1} \end{pmatrix}. \tag{13}$$

Let us note the properties of the matrix $\bar{A}_n$. The "limit" matrix is

$$\bar{A}_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The coefficients of the matrix $\bar{A}_n$ have the following properties: $\bar{\beta}^j_n > 0$, $\sum_n \bar{\beta}^j_n < \infty$ $(j=1,2)$. Since the «limit» matrix has eigenvalues $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$, the difference system (13) is exponentially stable [10]. Therefore, $\gamma_n(t)$ is exponentially stable. Obviously, for the value $\|\gamma_n(t)\|$ we have the estimate $\|\gamma_n(t)\| = \mathbf{O}(exp(-\theta_0\bar{\varepsilon}_1\mu^{-n}))$. Hence it follows $y^0(t) = \mathbf{O}(exp(\theta_0\bar{\alpha}-\varepsilon_1)exp(t))$, which implies the instability of the solution of the system (8) for any matrix B_4. The lemma is proved.

□

Taking into account the lemma, we see that at the first stage it is necessary to stabilize the following system for (8) in the absence of delay terms:

$$d\bar{y}^0(t)/dt = \vartheta_0 e^t[A_4\bar{y}^0(t) + C_2\bar{u}_2(t)].$$

Its stabilization can be carried out, for example, by Furasov's method, which we outlined in the stabilization of the system (7). Having stabilized this system, let us consider its behavior in the presence of delay terms. If the resulting system

$$dy_s^0(t)/dt = \theta_0 e^t[A_{4s}y_s^0(t) + B_4 y_s^0(t-\tau) + C_2u_2(t)]$$

is unstable again with $u_2(t) \equiv 0$, then we will stabilize the "degenerate" difference system [11, 12]

$$\hat{y}^0_{s,n+1} = -A_{4s}^{-1}B_4\hat{y}^0_{s,n} - A_{4s}^{-1}C_2u_{s,n} = \hat{A}\hat{y}^0_{s,n} + \hat{C}_2u_{s,n} \tag{14}$$

with minimization of the functional

$$Q = \sum_{j=0}^{\infty}[(y^0_{s,j})^\top G y^0_{s,j} + u_{s,j}^\top D u_{s,j}],$$

applying the algorithm suggested in [12], namely, setting the control $u_s = [D + (\hat{C}_2)^\top P\hat{C}_2]^{-1}(\hat{C}_2)^\top P\hat{A}\hat{y}_{s,n} = \bar{A}_1\hat{y}_{s,n}$. Here P is a symmetric, positive definite matrix satisfying the equation [12]

$$(\hat{A})^\top P\hat{A} - P + G - [(\hat{C}_2)^\top P\hat{A}]^\top[D + (\hat{C}_2)^\top P\hat{C}_2]^{-1}(\hat{C}_2)^\top P\hat{A} = 0, \tag{15}$$

where G, D are positive definite matrices of dimension, respectively, $m \times m$ and $r \times r$. Note that the nonlinear matrix equation (15) is solvable provided that the system (14) is completely controllable. Thus, setting in the second controlled subsystem of the system (1) the control $u_2(t) = -(C_2)^\top\Gamma_1 y(t) + [D + (\hat{C}_2)^P\hat{C}_2]^{-1}(\hat{C}_2)^P\hat{A}\ y(t-\tau)$, we get that the second subsystem in the absence of terms $x(t), x(t-\tau)$ on the right side is exponentially stable [12].

2. Stability and Stabilization of the Original System

Obviously, the sufficient conditions for exponential stability for weakly coupled systems obtained in the previous section, are not sufficient conditions for the asymptotic stability of the original system (1). Taking into account the results obtained in the previous section, we first stabilize the first subsystem in (1) (in the absence of the terms $y(t), y(t-\tau)$ on the right-hand side). Methods for stabilizing this system are given earlier in the previous section. Further, considering this subsystem stabilized in this way in the presence of terms containing $y(t), y(t-\tau)$), we obtain from (3) the subsystem

$$dx^0_{n+1}(\theta_n)/d\theta_n = \varepsilon_n\{[(A_1 + B_1)x^0_{n+1}(\theta_n) - {}_1(C_1)^\top\Gamma_1 x^0(t) - {}_1(C_1)^\top\Gamma_2 x^0(t-\tau) +$$

$$+ A_2y^0_{n+1}(\theta_n) + B_2y^0_n(\theta_n)\} \approx \varepsilon_n\{[(A_{1,s} + B_{1,s}])x_{n+1}(\theta_n) + A_2y^0_{n+1}(\theta_n) + B_2y^0_n(\theta_n)\}.$$

Due to the presence of the multiplier ε_n on the right-hand side, the value $x_{n+1}(t)$ satisfies asymptotic equality

$$x_{n+1}(t) = -(A_{1,s}+B_{1,s})^{-1}(A_2^0y^0_{n+1}(t) + B_2^0y^0_n(t)) + \mathbf{O}(\|x_{n+1}(t)\|_\tau + \|y_{n+1}(t))\|_\tau + \|y_n(t))\|_\tau),$$

therefore, the first approximation system is the difference (degenerate) system

$$x^0_{n+1}(\theta_n) = -[A_{1,s} + B_{1,s}]^{-1}[A_2 y^0_{n+1}(\theta_n) + B_2 y^0_n(\theta_n)]\}.$$

whence, taking into account the second subsystem (3) we obtain the controllable first approximation system

$$\begin{aligned} \varepsilon_n dy^0_{n+1}(t)/dt = e^t\{-[A_3 + B_3][A_{1,s} + B_{1,s}]^{-1}[A_2 y^0_{n+1}(t) + B_2 y^0_n(t)] + \\ +A_4 y^0_{n+1}(t) + B_4 y^0_n(t)\} = e^t\{-A^0 y^0_{n+1}(t) + B^0 y_n(t) + C_2 u_2(t)\}. \end{aligned} \quad (16)$$

This system (in the case of instability or stability, but not asymptotic) is quite simply stabilized by the algorithms presented in the previous section. In the papers [8, 13] it was proved that in the case of exponential stability of the stabilized subsystem (16) and exponential stability of the stabilized system (7) the original system is exponentially stable.

Due to the fact that the matrices A_j, B_j $j = 1, 2, 3, 4$ are constant, when stabilizing the system (1) the Lyapunov–Riccati equations with constant coefficients are solved by the methods proposed in the Matlab package. To implement the methods outlined by the authors, a software based on the Matlab package has been created, which makes it possible effectively stabilize the systems under consideration. The example given below and the numerical calculation of the solution illustrate the applicability of the proposed algorithms for stabilizing the considered system.

3. Example

Consider a fourth-order system

$$dx_1(t)/dt = x_1(t) + 0.1x_2(t) + x_1(t-1) + y_1(t) + y_1(t-1) + u_1(t),$$

$$dx_2(t)/dt = 0.1x_1 + x_2(t) + y_2(t) + y_2(t-1) + u_1(t),$$
$$dy_1(t)/dt = e^t[x_1(t) + x_1(t-1) + y_1(t) + 2y_1(t-1) + u_2(t)],$$
$$dy_2(t)/dt = e^t[x_2(t) + x_2(t-1) + 0.2y_1(t) + y_2(t) + y_2(t-1) + u_2(t)].$$

The original system is unstable, which can be seen from the graph shown in Figure 1. The system of slow variables is also unstable, which can also be seen from the corresponding graph shown in Figure 2. Let's stabilize the system of slow variables. This system becomes exponentially stable, as shown in Figure 3. Let us now consider the system of «fast» movements. This system is unstable because matrix A_4 has the eigenvalues $\lambda^1_4 = \lambda^2_4 = 1$.

We will stabilize the system in two stages. The corresponding subsystem becomes exponentially stable, which can be seen from the graph shown in Figure 4. However, the original system stabilized in this way is unstable, since the matrix A^0 has eigenvalues $\lambda^0_1 = 1.135 + 2.3i$, $\lambda^0_2 = 1.135 - 2.3i$. Finally, having stabilized in two stages the system

$$dy^0(t) = \vartheta_0 e^t[A^0 y^0(t) + B^0 y^0(t-1)],$$

from the graph shown in Figure 5 we see that the original system also becomes asymptotically stable. Note that this graph shows the solution in variable ϑ.

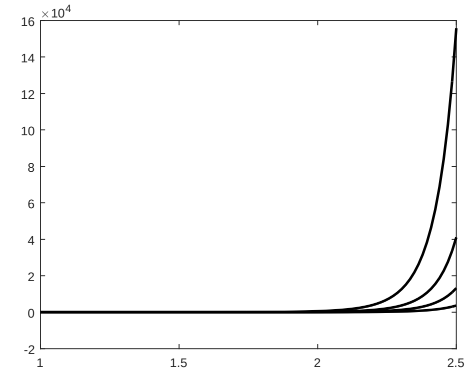

Fig. 1

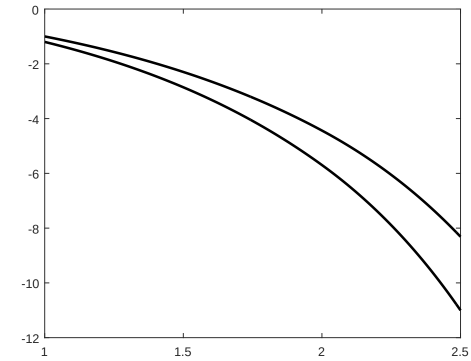

Fig. 2

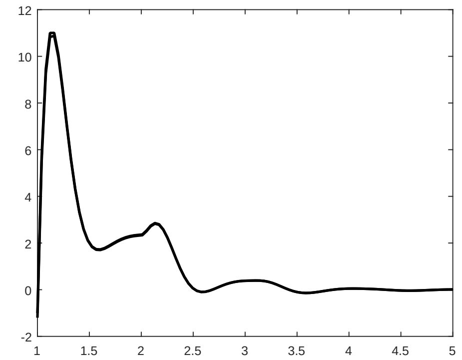

Fig. 3

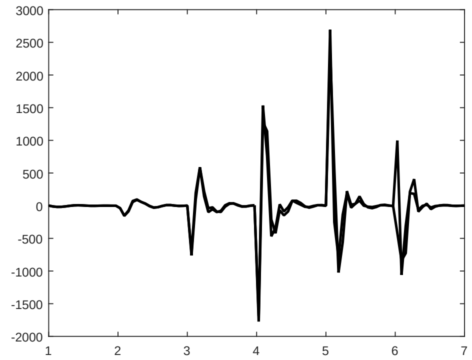

Fig. 4

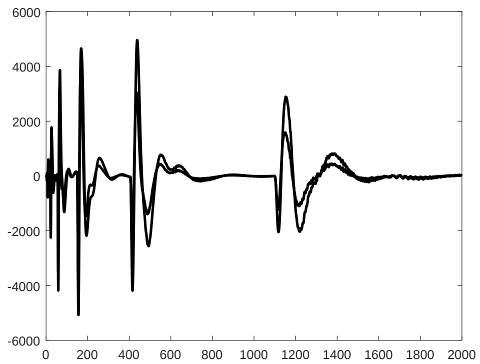

Fig. 5

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СТАБИЛИЗАЦИЯ ОДНОЙ СИСТЕМЫ С ЛИНЕЙНЫМ ЗАПАЗДЫВАНИЕМ

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Рассматривается задача стабилизации линейной системы дифференциальных уравнений с постоянным запаздыванием, содержащей быстрые и медленные переменные. К системам подобного вида заменой времени (аргумента) приводятся системы с линейным запаздыванием. Предложен алгоритм стабилизации этой системы запаздыванием, который реализован с помощью пакета Matlab.

Ключевые слова: асимптотическая устойчивость; линейное запаздывание; стабилизация.

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