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NUMERICAL STUDY OF THE UNIQUE SOLVABILITY OF THE SHOWALTER – SIDOROV PROBLEM FOR A MATHEMATICAL MODEL OF THE PROPAGATION OF NERVE IMPULSES IN THE MEMBRANE

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The article is devoted to the study of the existence of one or more solutions of a mathematical model of the propagation of a nerve impulse in a membrane based on a degenerate system of Fitz Hugh – Nagumo equations, given on a certain domain with a smooth boundary or on a connected directed graph with the Showalter – Sidorov initial condition. A nondegenerate mathematical model of the propagation of a nerve impulse in the membrane is widespread and is studied using the theory of singular perturbations. A feature of the process of the described investigated mathematical model is that the rate of change of one of the components of the system can significantly exceed the other, which means that the rate of the derivatives, which is much lower, can be considered equal to zero. Hence, it becomes necessary to study precisely the degenerate system of Fitz Hugh – Nagumo equations. The degenerate system of Fitz Hugh – Nagumo equations belongs to a wide class of semilinear Sobolev type equations. To investigate the existence of solutions of this system of equations, the phase space method will be used, which was developed by G.A. Sviridyuk to study the solvability of semilinear Sobolev-type equations. Conditions for the existence and uniqueness or multiplicity of solutions to the Showalter – Sidorov problem for the model under study are revealed, depending on the parameters of the system. The obtained theoretical results made it possible to develop an algorithm for the numerical solution of the problem based on the modified Galerkin method. The results of computational experiments are presented.

Keywords: Sobolev type equations; Showalter – Sidorov problem; non-uniqueness of solutions.

Introduction

To study the phenomena observed in the cells of striated muscles, intestinal smooth muscles, and the cardiovascular system, it is necessary to understand the dynamic principles of the operation of neurons, axons, dendrites, which are surrounded by a membrane envelope. It has a conductivity that depends on its current state and properties, which in turn are manifested in the dynamics of ionic currents Na_+ and K_+ and changes in the membrane potential and generate a nerve impulse as a response to external influences. The Fitz Hugh – Nagumo system allows one to describe the occurrence of impulses running along a nerve fiber due to the equation characterizing the ion flux in the membrane sheath and belongs to a wide class of reaction-diffusion systems. It was shown that the Fitz Hugh – Nagumo system of equations

$$\begin{cases} \varepsilon_1 v_t = \alpha_1 \Delta v + \beta_1 w - \varkappa_1 v, \\ \varepsilon_2 w_t = \alpha_2 \Delta w + \beta_2 w - \varkappa_2 v - w^3, \end{cases} \tag{1}$$

helps to investigate not only the process of propagation of a nerve impulse, but also blood coagulation, contraction of the heart muscle, and the work of the brain. As in the case with the rest of the reaction-diffusion equations, most researchers considered the system of equations (1) only under the assumption that $\varepsilon_1, \varepsilon_2 \neq 0$. An analytical study of the case when $\varepsilon_2 = 0$ was was carried out in the works [1] and it was shown that the phase space of the Fitz Hugh – Nagumo system of equations contains singularities of the Whitney assembly type and, as a consequence, there can be several different solutions of this system.

Numerical methods for solving direct and inverse problems for the Fitz Hugh – Nagumo model were studied in the works [2–4]. The work uses the Galerkin method, which was also used for numerical studies of Sobole-type equations in the works of G.A. Sviridyuk and his students [5–7]. In the case of degenerate semilinear equations for finding approximate solutions, the Galerkin method was used in the works of M.A. Sagadeeva, S.A. Zagrebina, N.A. Manakova A.A. Zamyshlyaeva, O.N. Tsyplenkova, S.I. Kadchenko, E.A. Soldatova, K.V. Perevozchikova and many others [8–14] In this paper, we will carry out a numerical study of the phenomenon of the existence of several solutions to the mathematical model of the propagation of a nerve impulse in the membrane sheath in the case when $\varepsilon_2 = 0$.

1. Mathematical Model in Case $\varepsilon_2 = 0$ $\varepsilon_2 = 0$

Consider the degenerate system of equations (1) in the case $\varepsilon_2 = 0$, which takes the form

$$\begin{cases} v_t = \alpha_1 \Delta v + \beta_1 w - \varkappa_1 v, \\ 0 = \alpha_2 \Delta w + \beta_2 w - \varkappa_2 v - w^3, \end{cases} \tag{2}$$

in the cylinder $Q = \Omega \times \mathbb{R}_+$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary of class C^∞ with boundary conditions

$$v(s,t) = 0,\ w(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}_+. \tag{3}$$

Reduce the problem (2), (3) to a semilinear Sobolev-type equation

$$L\ \dot{x} = Mx + N(x). \tag{4}$$

Take the spaces $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2 = \overset{\circ}{W}{}^1_2(\Omega) \times \overset{\circ}{W}{}^1_2(\Omega)$, $\mathfrak{U} = \mathfrak{U}_1 \times \mathfrak{U}_2 = L_2(\Omega) \times L_2(\Omega)$. Denote $[x, \zeta] = \langle v, \xi \rangle + \langle w, \eta \rangle$ as the scalar product in the space $\mathfrak{U}$, where $x = (v, w), \zeta = (\xi, \eta)$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(\Omega)$. Let $\mathfrak{F}$ be the conjugate of $\mathfrak{H}$ space with respect to duality $[\cdot, \cdot]$. There are dense and continuous embeddings

$$\mathfrak{H} \hookrightarrow \mathfrak{U} \hookrightarrow \mathfrak{F}. \tag{5}$$

Construct linear operators $L, M : \mathfrak{U} \to \mathfrak{F}$ by the formulas

$$[Lu, \zeta] = \langle w, \xi \rangle,\ u, \zeta \in \mathfrak{U}, \tag{6}$$

$$[Mu, \zeta] = -\alpha_1 \langle v_{s_i}, \xi_{s_i} \rangle - \alpha_2 \langle w_{s_i}, \eta_{s_i} \rangle,\ u, \zeta \in \mathfrak{U},\ \text{where } \operatorname{dom} M = \mathfrak{H}. \tag{7}$$

(Einstein's convention on summation over repeated indices is satisfied everywhere.) By construction, the operator $L \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$.

Denote by $\{\nu_k\}$ the sequence of eigenvalues of the following spectral problem:

$$\begin{aligned} -\Delta\varphi &= \nu\varphi,\ s \in \Omega, \\ \varphi(s) &= 0,\ s \in \partial\Omega, \end{aligned} \tag{8}$$

where the eigenvalues are numbered in non-decreasing order taking into account their multiplicity, $\{\varphi_k\}$ care the corresponding eigenfunctions, orthonormalized in the sense of the scalar product $\langle\cdot,\cdot\rangle$.

Construct a nonlinear operator by the formula

$$[N(x),\zeta] = \langle \beta_1 w - \varkappa_1 v, \xi\rangle + \langle \beta_2 w - \varkappa_2 v - w^3, \eta\rangle \tag{9}$$

and put dom $N = \mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 = L_4(\Omega) \times L_4(\Omega)$.

Let's construct an auxiliary space $\mathfrak{U}_\alpha$. Take $\mathfrak{U}_\alpha = \mathfrak{U}_1^0 \oplus \mathfrak{U}_\alpha^1$, where $\mathfrak{U}_1^0 = \{0\} \times W_2^1(\Omega)$, $\mathfrak{U}_\alpha^1 = \mathfrak{U}^\alpha \times \{0\}$, $\mathfrak{U}_\alpha^1 = L_4(\Omega)$. For $n \le 4$ there are dense and continuous embeddings

$$\mathfrak{H} \hookrightarrow \mathfrak{U}_\alpha \hookrightarrow \mathfrak{B} \hookrightarrow \mathfrak{U} \hookrightarrow \mathfrak{B}^* \hookrightarrow \mathfrak{F}, \tag{10}$$

then the operator $N \in C^\infty(\mathfrak{U}_\alpha;\mathfrak{F})$ (see [1]).

Consider the Showalter – Sidorov problem

$$L(x(0) - x_0) = 0, \tag{11}$$

which in this particular case has the following form:

$$v(0) = v_0. \tag{12}$$

Thus, we are interested in the solvability of the problem (2), (3), (12) for any $x_0 = (v_0, w_0) \in \mathfrak{U}_\alpha$.

Definition 1. *Vector-function $x \in C^1((0,\tau);\mathfrak{U}) \cap C((0,\tau);\mathfrak{U}_\alpha$, satisfying the equation (4), is called a solution of the equation. The solution $x = x(t)$ of the equation (4) is called a solution of the problem (4), (11), if* $\lim\limits_{t\to 0+} ||L(x(t)-x_0)||_{\mathfrak{F}}=0$.

Following the phase space method [1,15], we construct $\mathfrak{M}$ of the form

$$\mathfrak{M} = \left\{ u \in \mathfrak{U}_\alpha : -\langle v,\eta\rangle = \left\langle -\frac{\beta_2}{\varkappa_2} w + \frac{1}{\varkappa_2} w^3, \eta \right\rangle + \left\langle \frac{\alpha_2}{\varkappa_2} w_{s_i}, \eta_{s_i} \right\rangle \right\} \tag{13}$$

and note that all solutions of the system of equations (2) satisfying the boundary conditions (3) lie in this set.

In the work [1], conditions were found for the existence of a unique solution to the problem (2), (3), (12).

Theorem 1. [1] *For any $\alpha_2, \varkappa_2 \in \mathbb{R}_+$, $\beta_2 \in (0, \alpha_2\nu_1)$ and $x_0 \in \mathfrak{U}_\alpha$ there is a unique solution to the problem (2), (3), (12).*

Consider the case $\beta_2 = \alpha_2\nu_1$, put

$$\mathfrak{U}^{\alpha\perp} = \{v^\perp \in \mathfrak{U}^\alpha : \langle v^\perp, \varphi\rangle = 0\},\ \mathfrak{H}_2^\perp = \{w^\perp \in \mathfrak{H}_2 : \langle w^\perp, \varphi\rangle = 0\}.$$

If $v \in \mathfrak{U}^{\alpha\perp}$ and $w \in \mathfrak{H}_2$ are represented as $v = v^\perp + r\varphi$ and $w = w^\perp + q\varphi$, where $r, q \in \mathbb{R}$, φ is the eigenfunction of the problem (8), corresponding to the eigenvalue ν_1 and normalized in the sense of $L_2(\Omega)$, then the set $\mathfrak{M}$ takes the following form:

$$\mathfrak{M} = \left\{ u \in \mathfrak{U}_\alpha : \left\{ \begin{array}{l} \int\limits_\Omega -v^\perp \eta^\perp ds = \int\limits_\Omega \left(-\frac{\beta_2}{\varkappa_2} w^\perp \eta^\perp - \frac{\alpha_2}{\varkappa_2} w^\perp_{s_i} \eta^\perp_{s_i} + \right. \\ \qquad\qquad\qquad \left. + \frac{1}{\varkappa_2} (w^\perp + q\varphi)^3 \eta^\perp \right) ds, \\ -\varkappa_2 r = \int\limits_\Omega (w^\perp + q\varphi)^3 \varphi ds \end{array} \right. \right\}. \quad (14)$$

Note that the system of equations defining the set (14), is obtained from the equation defining the set (13), if we put $\delta = \alpha_1\nu_k - 1$ in it and then instead of η substitute first $\eta^\perp$, and then φ_k.

Let us turn to the second equation of the system defining the set (14). Transforming the resulting equation, we get:

$$q^3 ||\varphi||^4_{L_4(\Omega)} + 3q^2 \int\limits_\Omega w^\perp \varphi^3 ds + 3q \int\limits_\Omega (w^\perp)^2 \varphi^2 ds + \int\limits_\Omega \varphi (w^\perp)^3 ds + \varkappa_2 r = 0. \quad (15)$$

Equation (15) is a cubic general equation $aq^3 + bq^2 + cq^2 + d = 0$ with respect to q. According to Cardano's formulas, any cubic general equation by replacing $q = y - \frac{b}{3a}$ can be reduced to the canonical form $y^3 + py + e = 0$ with coefficients

$$\begin{array}{c} a = ||\varphi||^4_{L_4(\Omega)},\ b = 3\int\limits_\Omega w^\perp \varphi^3 ds,\ c = 3\int\limits_\Omega (w^\perp)^2 \varphi^2 ds,\ d = \int\limits_\Omega \varphi (w^\perp)^3 ds - \varkappa_2 r, \\ p = \frac{3ac - b^2}{9a^2},\ e = \frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \right), Q(q, w^\perp) = p^3 + e^2, \\ R(q, w^\perp) = q^2 \|\varphi\|^4_{L_4} + 2q \int\limits_\Omega \varphi^3 w^\perp ds + \int\limits_\Omega \varphi^2 (w^\perp)^2 = 0. \end{array} \quad (16)$$

For convenience of further consideration, introduce the following sets:

$$\mathfrak{H}_2^0 = \{ w \in \mathfrak{H}_2 : R(q, w^\perp) = 0 \},$$
$$\mathfrak{H}_2^+ = \{ w \in \mathfrak{H}_2 : Q(q, w^\perp) > 0 \},$$
$$\mathfrak{H}_2^- = \{ w \in \mathfrak{H}_2 : Q(q, w^\perp) < 0 \}.$$

Theorem 2. [1] *For any $\alpha_2, \varkappa_2 \in \mathbb{R}_+$, $r \in \mathbb{R}$, $\beta_2 = \alpha_2\nu_1$, $n \le 4$, and $x_0 \in \mathfrak{U}^\alpha$ such that*
(i)$v_0 \in \mathfrak{U}^\alpha \bigcap \mathfrak{H}_2^-$ there is exactly one solution to the problem (2), (3), (12);
(ii)$v_0 \in \mathfrak{U}^\alpha \bigcap \mathfrak{H}_2^+$ there are exactly three solutions to the problem (2), (3), (12).

Consider the system of equations on the graph **G**:

$$\left\{ \begin{array}{l} v_{jt} = \alpha_1 v_{jss} + \beta_{12} w_j - \beta_{11} v_j, \\ 0 = \alpha_2 w_{jss} + \beta_{22} w_j - \beta_{21} v_j - w_j^3, \end{array} \right. \quad (17)$$

where each function $v_j = v_j(s,t)$ and $w_j = w_j(s,t)$, $x \in (0, l_j)$ and $t \in \mathbb{R}_+$, satisfies the continuity condition

$$\begin{array}{c} v_j(0,t) = v_k(0,t) = v_m(l_m,t) = v_n(l_n,t), \\ w_j(0,t) = w_k(0,t) = w_m(l_m,t) = w_n(l_n,t), \\ E_j, E_k \in E^\alpha(V_i), \quad E_m, E_n \in E^\omega(V_i), \end{array} \quad (18)$$

flow balance condition

$$\begin{gathered}\sum_{E_j\in E^{\alpha}(V_i)} d_j v_{js}(0,t) - \sum_{E_j\in E^{\omega}(V_i)} d_j v_{js}(l_j,t) = 0,\\ \sum_{E_j\in E^{\alpha}(V_i)} d_j w_{js}(0,t) - \sum_{E_j\in E^{\omega}(V_i)} d_j w_{js}(l_j,t) = 0,\end{gathered} \tag{19}$$

where $E^{\alpha(\omega)}(V_i)$ denotes the set of edges with the beginning or end at the vertex V_i, and the initial condition

$$v_j(0) = v_{j0}. \tag{20}$$

Arguing in the same way as in the derivation of the assertion of Theorem 2, taking into account the peculiarities of the spaces constructed for the graph $\mathbf{G}$, we obtain the following theorem.

Theorem 3. *For any $\alpha_2, \beta_{21} \in \mathbb{R}_+$, $r \in \mathbb{R}$, $\beta_{22} = \alpha_2\nu_1$, and $x_0 \in \mathfrak{X}^\alpha$ such that*
(i)$v_0 \in \mathfrak{U}^\alpha \bigcap \mathfrak{H}_2^-$ there is exactly one solution to the problem (17) – (20);
(ii)$v_0 \in \mathfrak{U}^\alpha \bigcap \mathfrak{H}_2^+$ there are exactly three solutions to the problem (17) – (20).

2. Algorithm for a Numerical Method for Finding a Solution to the Showalter–Sidorov Problem

Let us describe an algorithm for the numerical solution of the problem (2), (3), (12) on a given domain Ω or the problem (17) – (20) on graph $\mathbf{G}$. The algorithm is based on the modified Galerkin – Petrov method and allows one to find approximate solutions on a given domain Ω or a geometric graph $\mathbf{G}$ for given initial values $v_0(s)$ and values of coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \varkappa_1, \varkappa_2$ for the model of the propagation of a nerve impulse in the membrane sheath of the nerve, as well as to obtain graphs of approximate solutions.

Stage 0. Find the eigenfunctions $\{\varphi_k(s)\}$ of the homogeneous Dirichlet problem of the Laplace operator $(-\Delta)$ in the domain Ω or $\{\varphi_k(s)\}$ - the eigenvectors of the operator

$$\langle A\varphi, \psi\rangle = \sum_{E_j\in\mathfrak{E}} d_j \int_0^{l_j} (\varphi_{js}\psi_{js} + \lambda\varphi\psi)ds,$$

where $d_j \in \mathbb{R}_+$, $l_j \in \mathbb{R}_+$ is the length and the cross-sectional area of the edge E_j of a finite connected directed graph $\mathbf{G} = \mathbf{G}(\mathcal{V};\mathcal{E})$, $\mathcal{V} = \{V_g\}_{g=1}^M$ is a set of vertices, $\mathcal{E} = \{E_j\}_{j=1}^K$ is a set of edges.

Stage 1. Following the Galerkin – Petrov method, we seek an approximate solution $\tilde{x} = (\tilde{v}, \tilde{w})$ of the problem (2), (3), (12) for the case $\varepsilon_2 = 0$ as sums

$$\begin{gathered}\tilde{v}_i(s,t) = \sum_{i=1}^{m} v_i(t)\varphi_i(s),\\ \tilde{w}_i(s,t) = \sum_{i=1}^{m} w_i(t)\varphi_i(s).\end{gathered} \tag{21}$$

Stage 2. Taking for $r = v_1(0)$, $q = w_1(0)$, $v^\perp = \sum\limits_{k=2}^{m} v_k(t)\varphi_k$, $w^\perp = \sum\limits_{k=2}^{m} w_k(t)\varphi_k$, and substituting the obtained values into the formulas (16), check uniqueness or multiplicity of the solution to the Showalter – Sidorov problem under the given initial conditions $w_0(s)$

for the case $\varepsilon_1 = 0$ or $v_0(s)$ for the case $\varepsilon_2 = 0$ and the obtained $v_k(0)$ or $w_k(0)$. In the case when $R < 0$, the required problem has three solutions $(v_1(s,t), w_1(s,t))$, $(v_2(s,t), w_2(s,t))$, $(v_3(s,t), w_3(s,t))$, consequently, the system of differential algebraic equations has three solutions and three sets $v_k(t)$ and $w_k(t)$ for each of the solutions, respectively. In this case, all subsequent steps must be done three times for each of the sets $v_k(t)$ and $w_k(t)$.

Stage 3. To find the unknowns $v_i(t), w_i(t)$, we substitute the Galerkin sums (21) into the system of equations (2), and then we multiply the resulting system of equations scalarly in $L_2(\Omega)$ or $L_2(\mathbf{G})$ by the eigenfunctions $\varphi_i(s)$, $i = \overline{1,m}$, thus obtaining a system of algebraic-differential equations of the form:

$$\begin{gathered}\alpha_1 \sum_{k=1}^{m} v_k(t)\, \nu_k \langle \nu_k, \varphi_i \rangle + \beta_1 \sum_{k=1}^{m} w_k(t) \langle \varphi_k, \varphi_i \rangle - \\ -\varkappa_1 \sum_{k=1}^{m} v_k(t) \langle \varphi_k, \varphi_i \rangle = 0,\end{gathered} \tag{22}$$

$$\begin{gathered}\sum_{k=1}^{m} \frac{d}{dt} w_k(t) \langle \varphi_k, \varphi \rangle - \alpha_2 \sum_{k=1}^{m} w_k(t)\, \nu_k \langle \varphi_k, \varphi_i \rangle - \\ -\beta_2 \sum_{k=1}^{m} w_k(t) \langle \varphi_k, \varphi_i \rangle + \varkappa_2 \sum_{k=1}^{m} v_k(t) \langle \varphi_k, \varphi_i \rangle + \left(\sum_{k=1}^{m} w_k(t) \langle \varphi_k, \varphi_i \rangle \right)^3 = 0,\end{gathered} \tag{23}$$

with condition

$$\langle w(0) - w_0, \varphi_i \rangle = 0. \tag{24}$$

Stage 4. Find $w_k(0)$ by scalar multiplication in $L_2(\Omega)$ $L_2(\Omega)$ or $L_2(\mathbf{G})$ initial condition (24) to eigenfunctions $\varphi_i(s)$, $i = \overline{1,m}$.

Stage 5. Solving the system of algebraic equations (22) with respect to $w_k(0)$, we obtain the values of $v_k(0)$.

Stage 6. Using the Runge – Kutta method of order 4-5, we find a solution to the system of differential equations (22), (23) with the initial conditions (24).

3. Description of the Operation of Computer Programs

The described algorithm was implemented in the Maple 2017 computer mathematics system for Windows 7, 8.1, 10 as a set of programs. This system of computer mathematics differs from analogs by the presence of a built-in apparatus for analytical calculations of integrals **student**, a package of commands for solving differential equations, including systems, **DEtools**. The program complex «Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the model of the propagation of a nerve impulse in the membrane of the nerve membrane» is intended to find an approximate solution of the Showalter–Sidorov problem for the model of the propagation of a nerve impulse in the membrane of a nerve in the case of uniqueness or multiplicity of solutions. The program implements the modified Galerkin method and the phase space method. The software package consists of the following programs: «Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the model of the propagation of a nerve impulse in the membrane sheath of the nerve axon», «Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the model of the propagation of a nerve impulse in a rectangular membrane», «Numerical study of the non-uniqueness of the Showalter–Sidorov problem for a model of the propagation of a nerve impulse in the membrane sheath of a cubic nerve the shell of the nerve system≫.

Complex of programs ≪Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the model of the propagation of a nerve impulse in the membrane sheath of a nerve≫ can be used to study the propagation of a nerve impulse by the propagation of a nerve impulse along neurons, axons, dendrites, along cells of striated muscles, smooth muscles intestines, cardiovascular system. The described complex of programs is of interest to specialists in the field of biomechanics. The coefficients of the system α_1, α_2, β_1, β_2, $\varkappa_1$, $\varkappa_2$ of the function of initial values $v_0(s)$ for the initial Showalter–Sidorov condition, area parameters Ω or **G**. At the output, each program of the complex produces approximate solutions $(v(s,t), w(s,t))$ and builds their graphs.

The complex of programs is intended only for work on a personal computer with a processor of at least 4 cores, with a frequency of at least 3400 MHz, 16 GB RAM running 64-bit operating systems Windows 7, 8.1, 10 with the Maple 2017 computer mathematics system installed.

Let us describe the logical structure of the operation of each of the programs in the complex in more detail. The program includes the following steps.

Step 1. Introduce system coefficients system coefficients α_1, α_2, β_1, β_2, $\varkappa_1$, $\varkappa_2$ functions of initial values $v_0(s)$ for the case $\varepsilon_2 = 0$ for the initial Showalter – Sidorov condition, the parameters of the domain Ω or **G**, as well as the number of Galerkin approximations m.

Step 2. From a separate file **eigenfunction.mw** using the built-in procedure **read**, the previously found normalized system of functions $\varphi_i(s)$ for the considered domain Ω or graph **G**.

Step 3. The procedure **unapply** allows you to represent the sought approximate solutions as a sum

$$v := unapply(v_1(t)\varphi_1(s) + v_2(t)\varphi_2(s) + \ldots + v_m(t)\varphi_m(s)),$$
$$w := unapply(w_1(t)\varphi_1(s) + w_2(t)\varphi_2(s) + \ldots + w_m(t)\varphi_m(s)).$$

Step 4. Take for

$$\begin{gathered} r := v_1(0), \\ v^{\perp} := v_2(0)\varphi_2(s) + \ldots + v_m(0)\varphi_m(s), \\ w^{\perp} := w_0, \end{gathered} \tag{25}$$

and substitute the obtained values into the formulas

$$\begin{gathered} \Delta(v^{\perp}) := 4\left(int((v^{\perp})^2 w^{\perp}\varphi_k^2, s=\Omega)\right)^2 - 4\cdot int(w^{\perp}\varphi_k^3, s=\Omega)\cdot \\ \cdot int((v^{\perp})^2 w^{\perp}\varphi_k, s=\Omega) + int(\gamma\varphi_k, s=\Omega). \end{gathered}$$

The built-in procedure **if**. . . **else** . . . **fi** is used to check the existence of one or two solutions to the Showalter – Sidorov problem under the given initial conditions.

Step 5. The expressions compiled at step 3 are substituted into the algebraic equation of the system and in the loop **for** i **to 1 do** m **end do** are multiplied by the eigenfunctions φ_i and integrated in the considered domain Ω or on the graph **G** using the procedure **int**. Using the built-in procedures **subs** and **solve**, with the setting **RealDomain**, we solve the resulting system of algebraic equations for the unknowns $v_1(0), \ldots, v_m(0)$, $w_1(0), \ldots, w_m(0)$. In the case when the system of equations has two solutions and, therefore, two sets $v_1(0), \ldots, v_m(0)$, $w_1(0), \ldots, w_m(0)$ for each of the solutions, respectively. In this case, all subsequent steps must be done twice for each of the sets $v_k(t)$ and $w_k(t)$ for the «software package Numerical study of the non-uniqueness of the Showalter–Sidorov problem for the model of nerve impulse propagation in the cell membrane». To realize the possibility of finding two different solutions using the built-in procedure **save**, the initial conditions are saved in the file **usl.mw**, the first set is $v_1(0), \ldots, v_m(0)$, $w_1(0), \ldots, w_m(0)$ is saved in the file **resh1.mw**, the second set is $v_1(0), \ldots, v_m(0)$, $w_1(0), \ldots, w_m(0)$ is saved in the file **resh2.mw**.

Step 6. Using the built-in procedure **read**, the initial conditions and one of the sets $v_1(0), \ldots, v_m(0)$, $w_1(0), \ldots, w_m(0)$ stored in files **resh1.mw** or **resh2.mw**. In the loop (**for** i **to 1 do** m **end do**), obtained in the third step, after substitution (procedure **subs**), the differential equation of the system is multiplied by the eigenfunction $\varphi_i(s)$ and integrates (**int**) in the considered domain Ω or on the graph **G**. As a result of performing steps 5 and 6, we obtain a system of algebraic-differential equations for determining the coefficients of approximation $v_1(t), \ldots, v_m(t)$, $w_1(t), \ldots, w_m(t)$.

Step 7. The system obtained in step 6 is solved with the initial conditions stored in the file **resh1.mw** using the built-in procedure **dsolve**.

Step 8. The solution is compiled and displayed in the form of a graph by the built-in procedures **plot** or **plot3d**.

4. Computational Experiments in the Case $\varepsilon_2 = 0$

Let us consider model examples of the numerical study of the question of the unambiguous solvability of the mathematical model of the propagation of a nerve impulse in the membrane sheath based on the implementation of the algorithm and the complex of programs described in Sections 2, 3.

Example 1. It is required to find a solution to the Showalter – Sidorov problem

$$v(s_1, s_2, 0) = v_0(s_1, s_2), \tag{26}$$

for the system of equations

$$\begin{cases} 0 = w_{s_1s_1} + w_{s_2s_2} - 2w + v + w^3, \\ v_t = v_{s_1s_1} + v_{s_2s_2} - v + w, \end{cases} \tag{27}$$

with the Dirichlet boundary condition

$$v(s_1, s_2, t) = w(s_1, s_2, t) = 0,\ s_1, s_2 \in \partial\Omega, t \in (0, T), \tag{28}$$

if $\Omega = (0, \pi) \times (0, \pi)$, $T = 1$, $v_0(s) = \dfrac{\sin(s_1)\sin(s_2)}{\pi} - \dfrac{2\sin(2s_1)\sin(2s_2)}{\pi}$.

Approximate solutions of the problem (26) – (28) in Ω can be represented as $\tilde{v}(s_1,s_2,t) = v_{1,1}(t)\varphi_{1,1}(s_1,s_2) + v_{2,1}(t)\varphi_{2,1}(s_1,s_2) + v_{1,2}(t)\varphi_{1,2}(s_1,s_2) + v_{2,2}(t)\varphi_{2,2}(s_1,s_2)$, $\tilde{w}(s_1,s_2,t) = w_{1,1}(t)\varphi_{1,1}(s_1,s_2) + w_{2,1}(t)\varphi_{2,1}(s_1,s_2) + w_{1,2}(t)\varphi_{1,2}(s_1,s_2) + w_{2,2}(t)\varphi_{2,2}(s_1,s_2)$, where $\varphi_{k_1,k_2}(s_1, s_2) = \sqrt{\frac{2}{\pi}}\sin(k_1s_1)\sin(k_2s_2)$, $k_1 = 1, 2$, $k_2 = 1, 2$. Using the formulas (16) we get $R = -74.9130174376168$. The problem (26) – (28) has there solutions. The system of differential-algebraic equations:

$$\begin{cases} 0 = \langle \tilde{w}_{s_1 s_1} + \tilde{w}_{s_2 s_2} - 2\tilde{w} + \tilde{v} + \tilde{w}^3, \varphi_{i,j} \rangle, \\ \langle \tilde{v}_t, \varphi_i \rangle = \langle \tilde{v}_{s_1 s_1} + \tilde{v}_{s_2 s_2} - \tilde{v} + \tilde{w}, \varphi_{i,j} \rangle, \end{cases} \tag{29}$$

has three numerical solutions, one is presented in Fig. 1 and in Tables 1, 2, the second – in Fig. 2 and in Tables 3, 4, the third – in Fig. 3 and in Tables 5, 6.

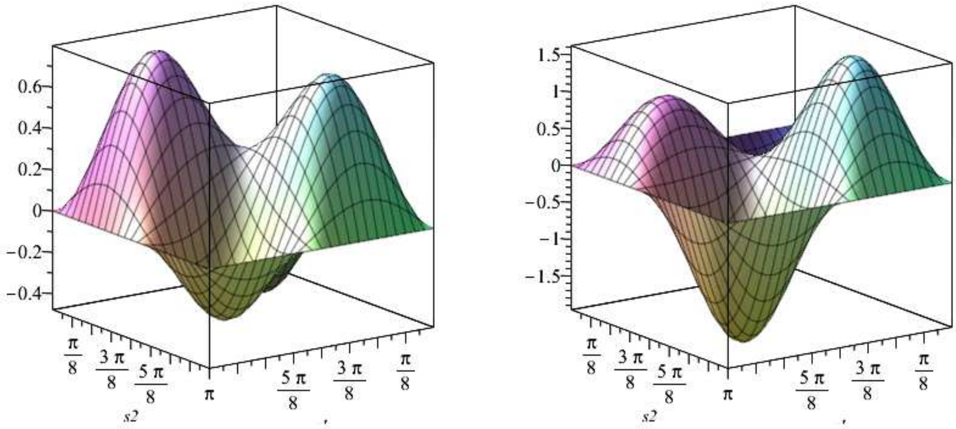

Fig. 1. Numerical solution of the problem (26) – (28) with the first set of initial conditions

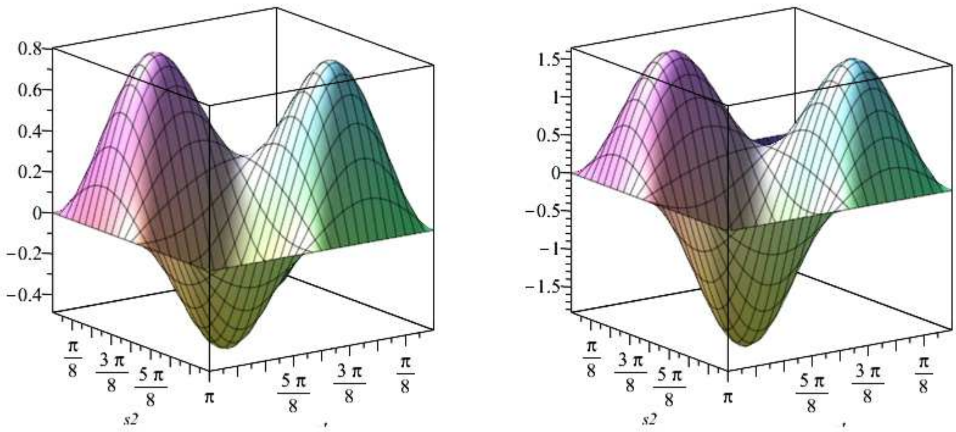

Fig. 2. Numerical solution of the problem (26) – (28) with the second set of initial conditions

Example 2. It is required to find a solution to the Showalter – Sidorov problem

$$v_1(s, 0) = v_{01}(s),\ v_2(s, 0) = v_{02}(s), \tag{30}$$

for the system of equations

$$\begin{cases} 0 = w_{jss} + 4w_j - v_j - w_j^3, \\ v_{jt} = v_{jss} + w_j - v_j, \end{cases} \quad j = \overline{1, 2}, \tag{31}$$

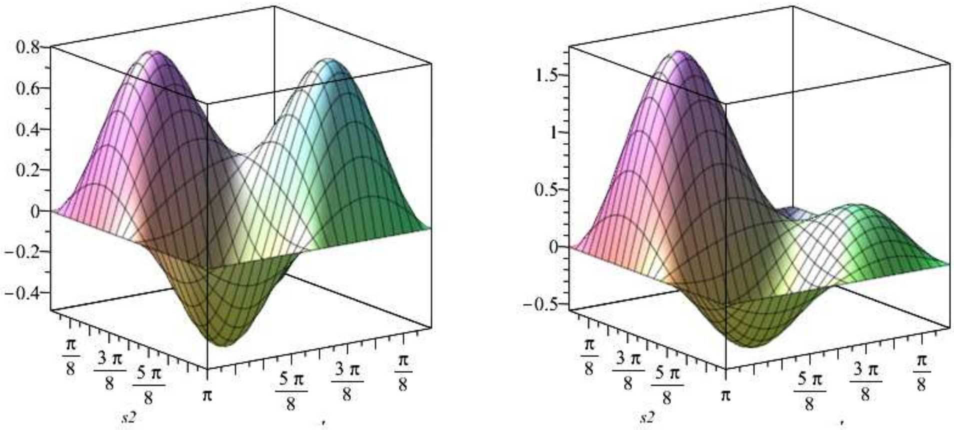

Fig. 3. Numerical solution of the problem (26) – (28) with the third set of initial conditions

Table 1

Numerical solution $v(s,t)$ of the problem (26) – (28) with the first set of initial conditions

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0	0.50000000000000	0	0	−1
0.1	0.50474737464685	−0.00378596339014	−0.01132841745124	−0.99074317029085
0.2	0.50954103298720	−0.00763462825812	−0.02277869603904	−0.98134022817570
0.3	0.51438125054744	−0.01154672318445	−0.03435202378101	−0.97178928680988
0.4	0.51926830164218	−0.01552298765680	−0.04604959619960	−0.96208843443162
0.5	0.52420245910820	−0.01956417144735	−0.05787261474166	−0.95223573603009
0.6	0.52918399297524	−0.02367103538201	−0.06982228451234	−0.94222923275891
0.7	0.53421317038958	−0.02784435257844	−0.08189981575520	−0.93206693850785
0.8	0.53929025561584	−0.03208490845317	−0.09410642386611	−0.92174683988056
0.9	0.54441551003694	−0.03639350072158	−0.10644332939327	−0.91126689619464
1.0	0.54958919215410	−0.04077093939790	−0.11891175803715	−0.90062503948159

Table 2

Numerical solution $w(s,t)$ of the problem (26) – (28) with the first set of initial conditions

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0	0.0275674820000000	0.375485293000000	1.12678818500000	−1.91843959200000
0.1	0.0277003268507354	0.377933394820338	1.12758664566055	−1.92370050341620
0.2	0.0278518193776941	0.380391130307696,	1.12838176241711	−1.92900275951681
0.3	0.0280223833352694	0.382858823660083	1.12917297772719	−1.93434687459740
0.4	0.0282124710580565	0.385336820257002	1.12995969554066	−1.93973339207338
0.5	0.0284225494175228	0.387825477710281	1.13074129859244	−1.94516287085048
0.6	0.0286530973839448	0.390325164106441	1.13151715192417	−1.95063588292930
0.7	0.0289046282421600	0.392836272124868	1.13228657495344	−1.95615303544047
0.8	0.0291776897461898	0.395359219136418	1.13304884127861	−1.96171497079806
0.9	0.0294728641192399	0.397894447203412	1.13380317867878	−1.96732236669955
1.0	0.0297907680536997	0.400442423079641	1.13454876911386	−1.97297593612581

Table 3

Numerical solution $v(s,t)$ of the problem (26) – (28) with the second set of initial conditions

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0	0.500000000000000	0	0	−1
0.1	0.498181166583331	−0.06372573439956	−0.06372574497351	−0.86068505071632
0.2	0.496665288192605	−0.13614269635676	−0.13614271049472	−0.70152839041682
0.3	0.495314693602840	−0.21818527773320	−0.21818529434947	−0.52008875918929
0.4	0.493995360147506	−0.31089424783432	−0.31089427023601	−0.31366388351641
0.5	0.492575542128448	−0.41542632862256	−0.41542635345785	−0.07926300878279
0.6	0.490924690073997	−0.53306483385921	−0.53306486338922	0.18642365550176
0.7	0.488912630697838	−0.66523149145803	−0.66523152433477	0.48705909394149
0.8	0.486409102883660	−0.81349954028520	−0.81349957845022	0.82669793416681
0.9	0.483283897933602	−0.97960793329704	−0.97960797726163	1.20982919656711
1.0	0.479407927169453	−1.16547652430062	−1.16547657386734	1.64142406783334

Table 4

Numerical solution $w(s,t)$ of the problem (26) – (28) with the second set of initial conditions

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0	0.520174093400000	0.596666903400000	0.596666903200000	−2.30147487600000
0.1	0.514621498571012	0.615514766883433	0.615514774422498	−2.34917936453044
0.2	0.510771696656523	0.634511943102271	0.634511952472695	−2.40023189713826
0.3	0.508443293072450	0.653744608178142	0.653744620754558	−2.45467677735339
0.4	0.507472966020829	0.673281592493410	0.673281604812851	−2.51255355502799
0.5	0.507711898258124	0.693176675397257	0.693176687434395	−2.57390248987583
0.6	0.509022293684011	0.713469737844490	0.713469750707949	−2.63876904163250
0.7	0.511273347486573	0.734186819311963	0.734186831694194	−2.70720771405273
0.8	0.514336112718496	0.755338841577821	0.755338856134556	−2.77928586865398
0.9	0.518076466504951	0.776918461902812	0.776918475041570	−2.85508831325540
1.0	0.522344530131669	0.798893759116052	0.798893773913075	−2.93472388889472

Table 5

Numerical solution $v(s,t)$ of the problem (26) – (28) with the third set of initial conditions

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0	0.50000000000000	0	0	−1
0.1	0.45720164617699	−0.07052997798714	0.07052997798714	−0.97156402634621
0.2	0.40766281478719	−0.14929685224525	0.14929685224525	−0.93594697610189
0.3	0.35054335018670	−0.23721091501194	0.23721091501194	−0.89212494015368
0.4	0.28491400511567	−0.33527087856784	0.33527087856784	−0.83894967137105
0.5	0.20974713012607	−0.44457159091336	0.44457159091336	−0.77513193721708
0.6	0.12390664054813	−0.56631172670231	0.56631172670231	−0.69922195782794
0.7	0.02613697348867	−0.70180139053204	0.70180139053204	−0.60958567782432
0.8	−0.08494899090398	−0.85246894772638	0.85246894772638	−0.50437505602464
0.9	−0.21088253507104	−1.01986512645267	1.01986512645267	−0.38148942865899
1.0	−0.35335227940796	−1.20566165334988	1.20566165334988	−0.23852126918506

Table 6

Numerical solution $w(s,t)$ of the problem (26) – (28)
with the third set of initial conditions

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0	0.896875988000000	0.666943335600000	-0.666943335600000	-1.25158542500000
0.1	0.917557177184330	0.674507051263782	-0.674507051263782	-1.29021876647179
0.2	0.939482678299100	0.682519434527073	-0.682519434527073	-1.33133732079518
0.3	0.962660021775526	0.690917334609731	-0.690917334609731	-1.37508634912818
0.4	0.987095703978462	0.699624012407593	-0.699624012407593	-1.42163954440554
0.5	1.01279468351171	0.708544342076834	-0.708544342076834	-1.47120939724315
0.6	1.03975937640447	0.717557252823475	-0.717557252823475	-1.52406263784026
0.7	1.06798773707065	0.726503239509183	-0.726503239509182	-1.58054464731150
0.8	1.09746985017752	0.735162625735116	-0.735162625735116	-1.64112122192264
0.9	1.12818187230430	0.743215340433790	-0.743215340433790	-1.70645508358176
1.0	1.16007428787653	0.750158184299418	-0.750158184299418	-1.77756172576667

on the graph **G** (Fig. 4) with the boundary conditions

$$\begin{aligned} &v_{1s}(l_1,t)=v_{2s}(0,t), v_{1s}(0,t)=0, v_{2s}(l_2,t)=0, v_1(l_1,t)=v_2(0,t),\\ &w_{1s}(l_1,t)=w_{2s}(0,t), w_{1s}(0,t)=0, w_{2s}(l_2,t)=0, w_1(l_1,t)=w_2(0,t), \end{aligned} \tag{32}$$

if $l_1=\pi$, $l_2=\pi$, $T=1$, $v_{01}(s)=-\frac{1}{\sqrt{\pi}}+\frac{0.1\cos(0.5s)}{\sqrt{\pi}}$, $v_{02}(s)=-\frac{1}{\sqrt{\pi}}-\frac{0.1\sin(0.5s)}{\sqrt{\pi}}$.

Approximate solutions of the problem (30) – (32) for the j-th edge ($j=1,2$) of the graph have the form $\tilde{v}_j(s,t)=\varphi_{1j}(s)v_1(t)+\varphi_{2j}(s)v_2(t), \tilde{w}_j(s,t)=\varphi_{1j}(s)w_1(t)+$ $+\varphi_{2j}(s)w_2(t)$, where $\varphi_{11}=\frac{1}{\sqrt{\pi}}$, $\varphi_{12}=\frac{1}{\sqrt{\pi}}$, $\varphi_{21}=\frac{1}{\sqrt{\pi}}\cos(\frac{s}{2})$, $\varphi_{22}=-\frac{1}{\sqrt{\pi}}\sin(\frac{s}{2})$. Using the formulas (16) we get $R=-1.97834612093402$. The problem (30) – (32) will have three solutions. System of algebraic-differential equations

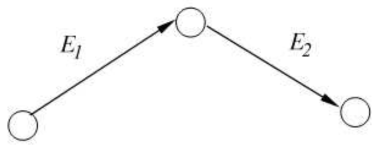

Fig. 4. Graph **G**

$$\begin{cases} 0=\langle w_{1ss}+4w_1-v_1-w_1^3,\varphi_{i1}\rangle, 0=\langle w_{2ss}+4w_2-v_2+w_2^3,\varphi_{i2}\rangle,\\ \langle v_{1t},\varphi_{i1}\rangle=\langle v_{1ss}+w_1-v_1,\varphi_{i1}\rangle, \langle v_{2t},\varphi_{i2}\rangle=\langle v_{2ss}+w_2-v_2,\varphi_{i2}\rangle. \end{cases} \tag{33}$$

has three numerical solutions, one is shown in Fig. 5 and in Table 7, the second – in Fig. 6 and in Table 8, the third – in Fig. 7 and in the Table 9.

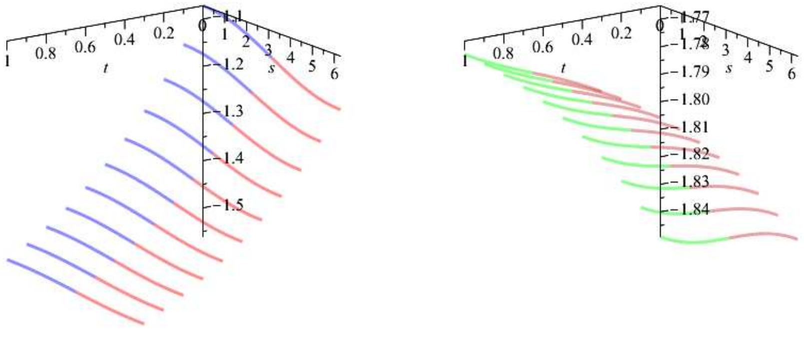

Fig. 5. Numerical solution of the problem (30) – (32) with the first set of initial conditions

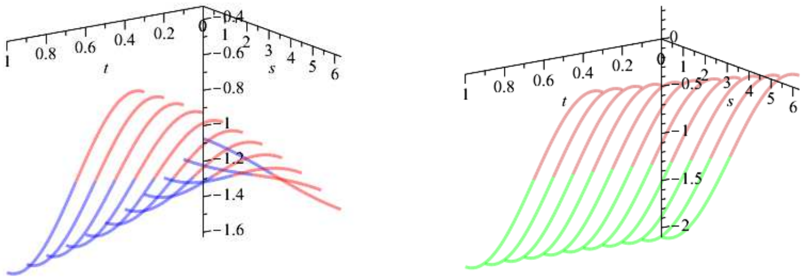

Fig. 6. Numerical solution of the problem (30) – (32)
with the second set of initial conditions

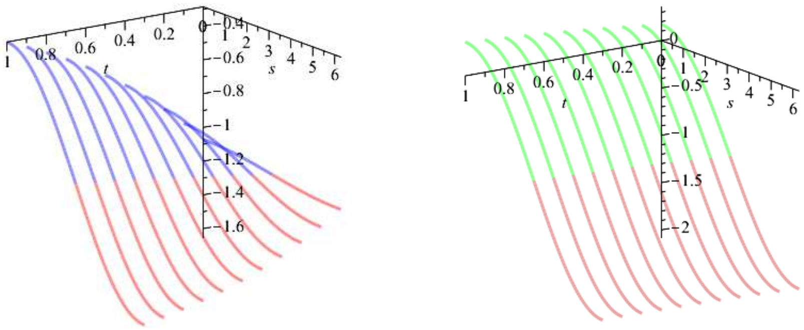

Fig. 7. Numerical solution $v(s,t)$ of the problem (30) – (32)
with the third set of initial conditions

Table 7

Numerical solution of the problem (30) – (32) with the first set of initial conditions

t	$v_1(t)$	$v_2(t)$	$w_1(t)$	$w_2(t)$
0	−2	0.10000000000000	−3.26186277900000	−0.0155999674700000
0.1	−2.11912321080735	0.08687045867908	−3.24234664666796	−0.0138129921390193
0.2	−2.22514173576231	0.07544283146120	−3.22464146698823	−0.0122081852938851
0.3	−2.31946847952800	0.06550066779187	−3.20860931693606	−0.0107710211819444
0.4	−2.40336874288482	0.05685435887580	−3.19411783341095	−0.00948745328897828
0.5	−2.47797511278246	0.04933786674653	−3.18104078450014	−0.00834400852713155
0.6	−2.54430094304637	0.04280584883696	−3.16925850534588	−0.00732786647687465
0.7	−2.60325181696900	0.03713127103391	−3.15865820955832	−0.00642692164068414
0.8	−2.65563685704343	0.03220310314814	−3.14913414982495	−0.00562982580336433
0.9	−2.70217858573811	0.02792436963864	−3.14058767996199	−0.00492601581335811
1.0	−2.74352145710651	0.02421049539553	−3.13292722382306	−0.00430572520358850

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Table 8

Numerical solution of the problem (30) – (32) with the second set of initial conditions

t	$v_1(t)$	$v_2(t)$	$w_1(t)$	$w_2(t)$
0	−2.	0.10000000000000	−1.72889837400000	−1.99048868700000
0.1	−1.9724672826105	−0.09963991899961	−1.69365461874690	−2.00651223474666
0.2	−1.944371897897	−0.27727034558707	−1.66188522635914	−2.02137060046327
0.3	−1.916080089710	−0.43537648838320	−1.63322699785695	−2.03523392749820
0.4	−1.887890777513	−0.57616540364265	−1.60735894782894	−2.04823079447017
0.5	−1.860045661714	−0.70159534680157	−1.58399581060419	−2.06045888061259
0.6	−1.832737843105	−0.81340258652561	−1.56288311024144	−2.07199295770342
0.7	−1.806119138499	−0.91312547251681	−1.54379331194129	−2.08289089381452
0.8	−1.780306411616	−1.00212631127421	−1.52652267461806	−2.09319820418380
0.9	−1.755387118062	−1.08161138445297	−1.51088857273152	−2.10295150309030
1.0	−1.731423915532	−1.15264826107150	−1.49672727285616	−2.11218104841890

Table 9

Numerical solution of the problem (30) – (32) with the third set of initial conditions

t	$v_1(t)$	$v_2(t)$	$w_1(t)$	$w_2(t)$
0	−2	0.100000000000000	−1.69619066600000	1.99750109900000
0.1	−1.96951985348201	0.276776135539865	−1.66429844386661	2.01308075952797
0.2	−1.93905895100012	0.434193370747943	−1.63552572682035	2.02760829357643
0.3	−1.90889638201649	0.574434207291382	−1.60954947182180	2.04121686142368
0.4	−1.87925574678267	0.699435652921811	−1.58608299530547	2.05400817751996
0.5	−1.85031363698218	0.810915513012296	−1.56487087922689	2.06606065282851
0.6	−1.82220685520647	0.910396154177116	−1.54568498559748	2.07743550278310
0.7	−1.79503866311456	0.999226187366611	−1.52832119906028	2.08818136392446
0.8	−1.76888418148664	1.07860009998420	−1.51259668833203	2.09833777124319
0.9	−1.74379483328738	1.14957507319829	−1.49834766134174	2.10793770214928
1.0	−1.71980244366911	1.21308710081427	−1.48542727871022	2.11700950512056

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ОДНОЗНАЧНОЙ РАЗРЕШИМОСТИ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ РАСПРОСТРАНЕНИЯ НЕРВНЫХ ИМПУЛЬСОВ В МЕМБРАНОЙ ОБОЛОЧКЕ

О. В. Гаврилова

Статья посвящена исследованию существования одного или нескольких решений математической модели распространения нервного импульса в мембране на основе вырожденной системы уравнений Фитц Хью – Нагумо, заданной на некоторой области с гладкой границей или на связном ориентированном графе с начальным условием Шоуолтера – Сидорова. Невырожденная математическая модель распространения нервного импульса в мембранной оболочке является распространенной и исследуется с помощью теории сингулярных возмущений. Особенностью процесса описываемого исследуемой математической модели является то, что скорость изменения одной из компонент системы может значительно превосходить другую, а значит ту из производных скорость которой значительно ниже, можно считать равной нулю. Отсюда и возникает необходимость в исследовании именно вырожденной системы уравнений Фитц Хью – Нагумо. Вырожденная система уравнений Фитц Хью – Нагумо относится к широкому классу полулинейных уравнений соболевского типа. Для исследования существования решений данной системы уравнений будет использован метод фазового пространства, который был разработан Г.А. Свиридюком для исследования разрешимости полулинейных уравнений соболевского типа. Выявлены условия существования и единственности или множественности решений задачи Шоуолтера – Сидорова для исследуемой модели, в зависимости от параметров системы. Полученные теоретические результаты позволили разработать алгоритм численного решения задачи, основанный на модифицированном методе Галеркина. Приведены результаты вычислительных экспериментов.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; неединственность решений.

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