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NUMERICAL INVESTIGATION OF THE INVERSE PROBLEM FOR THE BOUSSINESQ – LOVE MATHEMATICAL MODEL ON A GRAPH

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The article is devoted to a numerical study of the Boussinesq – Love mathematical model considered on a graph. The model under study describes longitudinal vibrations in a construction consisting of thin, elastic rods, taking into account the external load acting on them. To find the solution, the method of successive approximations is used, and the model itself is reduced to an incomplete second-order Sobolev type equation. The first paragraph contains the results on an analytical study of the model. In the second section, the developed algorithm of the numerical method and its scheme are presented. The third section presents the results of computational experiments obtained using the program developed in the Maple environment based on the developed algorithm. All the obtained results can be applied in the field of mathematical modeling, for example, when calculating the longitudinal vibrations in a construction, taking into account the external load acting on it.

Keywords: mathematical model; Boussinesq – Love equation; inverse problem; numerical study; Sobolev type equation; method of successive approximations.

Introduction

Let $G = G(\mathfrak{D}, \mathfrak{E})$ be a finite connected oriented graph, where $\mathfrak{D} = \{V_i\}$ is the set of vertices, and $\mathfrak{E} = \{E_j\}$ is the set of edges. Each edge is characterized by two numbers $l_j, d_j \in \mathbb{R}_+$, denoting the length and cross-sectional area of the edge E_j respectively. On a graph G consider the Boussinesq – Love equations [1]

$$(\alpha - \Delta)v_{tt} = \beta(\Delta - \gamma)v + qf, \tag{1}$$

$$v = (v_1, v_2, ..., v_j, ...), \quad f = (f_1, f_2, ..., f_j, ...),$$

with the conditions at each vertex V_i of the graph

$$\sum_{E_j \in E^\alpha(V_i)} d_j v_{jx}(0, t) - \sum_{E_m \in E^\omega(V_i)} d_m v_{mx}(l_m, t) = 0, \tag{2}$$

$$v_j(0, t) = v_k(0, t) = v_m(l_m, t) = v_n(l_n, t), \tag{3}$$

initial conditions

$$v(x, 0) = \varphi(x), \quad \varphi = (\varphi_1, \varphi_2, ..., \varphi_j, ...), \tag{4}$$

$$v_t(x, 0) = \psi(x), \quad \psi = (\psi_1, \psi_2, ..., \psi_j, ...) \tag{5}$$

and overdetermination condition

$$< v(x, t), K(x) >= \Phi(t), \quad K = (K_1, K_2, ..., K_j, ...), \tag{6}$$

where $f(x,t)$, $\varphi(x)$, $\psi(x)$, $K(x)$ are given vector-functions, $\Phi(t)$ is given function and

$$< v(x,t), K(x) >= \sum_{E_j} \int_0^{lj} v(x,t)K(x)dx$$

is the inner product in space $L^2(G)$. Function $v_j(x,t)$ defines a longitudinal displacement at point x at moment t for the j-th element of construction. The coefficients α, β and γ characterize the properties of the rods material construction. Function $f(x,t)$ sets the known external load and $q(t)$ is its coefficient. Usually, (2) is the «flow balance» condition, and condition (3) means ≪continuity≫ of the solution $v(x,t)$. Condition (4) specifies the initial position, the condition (5) specifies the initial velocity. Condition (6) is necessary to restore the coefficient $q(t)$ in equation (1).

The problem of finding a pair of functions

$$v(x,t) = (v_1(x,t), v_2(x,t), ..., v_j(x,t), ...) \text{ and } q(t)$$

from relations (1)–(6) is called the inverse problem.

Sobolev type equations have already been studied in various aspects [2–9]. In [3], an initial-boundary value problem is considered for the modified Boussinesq equation used to describe the propagation of waves in shallow water under the condition of conservation of mass in the layer and taking into account capillary effects, showing that the solution constructed by the Galerkin method from the system of orthonormal eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator, converges ∗-weakly to the practise solution. Paper [4] presents the results of investigation of the solvability of boundary value problems for some classes of linear Sobolev type equations of the fourth order. In [5], sufficient conditions are presented for the existence of positive solutions to the Showalter – Sidorov and the Cauchy problem for an abstract linear equation of this type. Paper [6] is devoted to the study of the degenerate holomorphic groups of operators in Sobolev spaces generated by linear and continuous operators L and M. Such equations find their application in modeling of various processes and phenomena [7, 8], such as, for example: modeling the vibrations of a rotating viscous fluid using; modeling of gravitational-gyroscopic and internal waves; modeling of sound waves in smectics; modeling of longitudinal vibrations in bars (1)–(6). A numerical study of the direct problem for the Boussinesq – Love equation on graphs has already been carried out in [8]. The numerical study of the mathematical model of the starting regulation of the distribution of the electromagnetic field potential in a crystalline semiconductor, based on the starting control problem and final observation of solutions to a semilinear Sobolev type equation was made in [7].

Works [2, 9–12] are devoted to the study of inverse problems. In [10], the continuity of the solution to the inverse problem for the equation of radiation transfer in multiband regions, in which the scattering coefficient and radiation intensity are located, are studied. Article [11] is devoted to the inverse problem of determining the permittivity tensor and thickness of a thin film deposited on a glass substrate with known optical properties and thickness. Investigation [12] allows to find the function of the electromagnetic field by the known function of the spectrum, which has a finite number of zeros in the frequency interval.

1. Analytical Investigation of the Mathematical Model

Let $\mathcal{U} = \{u \in W_q^{l+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}$, $\mathcal{F} = W_q^l(\Omega)$, $\mathcal{Y} = \mathcal{F}$. Here $W_q^l(\Omega)$ are Soboloev spaces. Similarly to [9] problem (1)–(6) is equivalent to the problem of finding the functions $u \in C^2([0,T];\mathcal{U}^1)$, $w \in C^2([0,T];\mathcal{U}^0)$, $q \in C^1([0,T];\mathcal{Y})$ from the relations

$$u''(t) = Su(t) + (A_1)^{-1}Qq(t)f(t), \tag{7}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{8}$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \tag{9}$$

$$Hw''(t) = w(t) + (B_0)^{-1}Pq(t)f(t), \tag{10}$$

$$w(0) = w_0, \quad w'(0) = w_1, \tag{11}$$

where

$$S = \sum_{\lambda_k \neq \alpha} \frac{\beta(\lambda_k - \gamma)}{\alpha - \lambda_k} < \cdot, \mathbb{X}_k > \mathbb{X}_k, \quad H = \sum_{\lambda_k = \alpha} \frac{\alpha - \lambda_k}{\beta(\lambda_k - \gamma)} < \cdot, \mathbb{X}_k > \mathbb{X}_k,$$

$$Q = \sum_{\lambda_k \neq \alpha} < \cdot, \mathbb{X}_k > \mathbb{X}_k, \quad P = \mathbb{I} - Q = \sum_{\lambda_k = \alpha} < \cdot, \mathbb{X}_k > \mathbb{X}_k,$$

$$(A_1)^{-1} = \sum_{\lambda_k \neq \alpha} \frac{< \cdot, \mathbb{X}_k >}{\alpha - \lambda_k} \mathbb{X}_k, \quad (B_0)^{-1} = \sum_{\lambda_k = \alpha} \frac{< \cdot, \mathbb{X}_k >}{\beta(\lambda_k - \gamma)} \mathbb{X}_k,$$

$$u_0 = \sum_{\lambda_k \neq \alpha} < \varphi, \mathbb{X}_k > \mathbb{X}_k, \quad u_1 = \sum_{\lambda_k \neq \alpha} < \psi, \mathbb{X}_k > \mathbb{X}_k,$$

$$w_0 = \sum_{\lambda_k = \alpha} < \varphi, \mathbb{X}_k > \mathbb{X}_k, \quad w_1 = \sum_{\lambda_k = \alpha} < \psi, \mathbb{X}_k > \mathbb{X}_k,$$

$$C = \sum_{\lambda_k \neq \alpha} < \cdot, K(x) >, \quad \mathcal{U}^0 = \ker P, \quad \mathcal{U}^1 = \operatorname{im} P.$$

Here, according to [9], $\{\lambda_k\} = \sigma(\Delta)$ denotes the set of eigenvalues numbered in non-increasing order with multiplicity, and $\{\mathbb{X}_k\}$ denotes the family of corresponding eigenfunctions orthonormal with respect to the inner product $< \cdot, \cdot >$ in $L^2(G)$.

Theorem 1. *Let $K, u_0, u_1 \in \mathcal{U}^1, f \in C^2([0,T];\mathcal{L}(\mathcal{Y};\mathcal{F})), \Phi \in C^4([0,T];\mathcal{Y})$, one of the conditions $\alpha \notin \sigma(\Delta)$ or $(\alpha \in \sigma(\Delta)) \wedge (\alpha \neq \gamma)$ be fulfilled, the conditions:*

$$\sum_{\lambda_k \neq \alpha} \frac{< f(\cdot, t), K(\cdot) >}{\alpha - \lambda_k} \neq 0,$$

$$\sum_{\lambda_k \neq \alpha} < u_1, K(\cdot) >= \Phi'(0)$$

be satisfied for initial value $u_1 \in \mathcal{U}^1$, and the initial values $w_k = (\mathbb{I} - P)v_k \in \mathcal{U}^0$, $k = 0, 1$ satisfy

$$< w_0 + \frac{q(0)f(\cdot, 0)}{\beta(\lambda_k - \gamma)}, \mathbb{X}_k >= 0 \text{ for } k : \lambda_k = \alpha, \tag{12}$$

$$< w_1 + \frac{q(0)f_t(\cdot,0) + q'(0)f(\cdot,0)}{\beta(\lambda_k - \gamma)}, \mathbb{X}_k >= 0 \; for \; k : \lambda_k = \alpha. \quad (13)$$

Then there exists a unique solution (v, q) *of the inverse problem* (1)–(6)*, where* $q \in C^2([0,T];\mathcal{Y})$*,* $v = u + w$*, whence* $u \in C^2([0,T];\mathcal{U}^1)$ *is a solution of* (7)–(9) *and the function* $w \in C^2([0,T];\mathcal{U}^0)$ *is a solution of* (10)–(11) *given by*

$$w(t) = -\sum_{\lambda_k=\alpha} < \frac{q(t)f(\cdot,t)}{\beta(\lambda_k - \gamma)}, \mathbb{X}_k > \mathbb{X}_k. \quad (14)$$

Proof. The conditions of [9, Lemma 2] are satisfied. Since $K \in \mathcal{U}^1$, then $\mathcal{U}^0 \subset ker\ C$. For $y \in \mathcal{Y}$ due to the orthonormality of the system of eigenfunctions in $L_2(G)$

$$C(A^1)^{-1}Qy = \left(\sum_{\lambda \neq \lambda_k} \frac{< f(\cdot,t), \varphi_k >< \varphi_k, K >}{\lambda - \lambda_k} \right) y = \left(\sum_{\lambda \neq \lambda_k} \frac{< f(\cdot,t), K >}{\lambda - \lambda_k} \right) y.$$

This operator is reversible in $\mathcal{Y}$ when

$$\sum_{\lambda \neq \lambda_k} \frac{< f(\cdot,t), K >}{\lambda - \lambda_k} \neq 0,$$

and the inverse operator is continuously differentiable by t.

Thus, all the conditions of [2, Theorem 4] are satisfied, then there exists a unique solution (v, q) of inverse problem (1)–(6), where $q \in C^2([0,T];\mathcal{Y})$, $v = u + w$, whence $u \in C^2([0,T];\mathcal{U}^1)$ is the solution of (7)–(9) and the function $w \in C^2([0,T];\mathcal{U}^0)$ is a solution of (10), (11) given by (14).

□

2. Algorithm of Numerical Method

Let us describe the algorithm developed for the numerical solution of the inverse problem for the Boussinesq – Love equation in steps corresponding to the block diagram presented in Figures 1.

Start of the program.

Step 1. Input the incidence matrix of the graph and parameters: the coefficients of the Boussinesq – Love equation α, β, γ; the lengths of graph edges l_j; the time limit T; the permissible error ε for the desired function $q(t)$; the minimum number N of terms of the Galerkin sum. Input functions: the known part of the external load $f(x,t)$; the initial position of the rods $\varphi(x)$; the initial velocity $\psi(x)$; the kernel $K(x)$; the righthand side $\Phi(t)$ of the overdetermination condition.

Step 2. Solution of the Sturm – Liouville problem on a graph and obtaining the number n of terms of the Galerkin sum.

Step 2.1. Find λ_k from the system of equations.

Step 2.2. Find the normalized eigenfunctions $\mathbb{X}_k(x)$ on the graph, using the eigenvalues λ_k.

Step 2.3. Specifying the number of terms of the Galerkin sum, taking into account the given N and the possible coincidence of λ_k with the parameter α for some k.

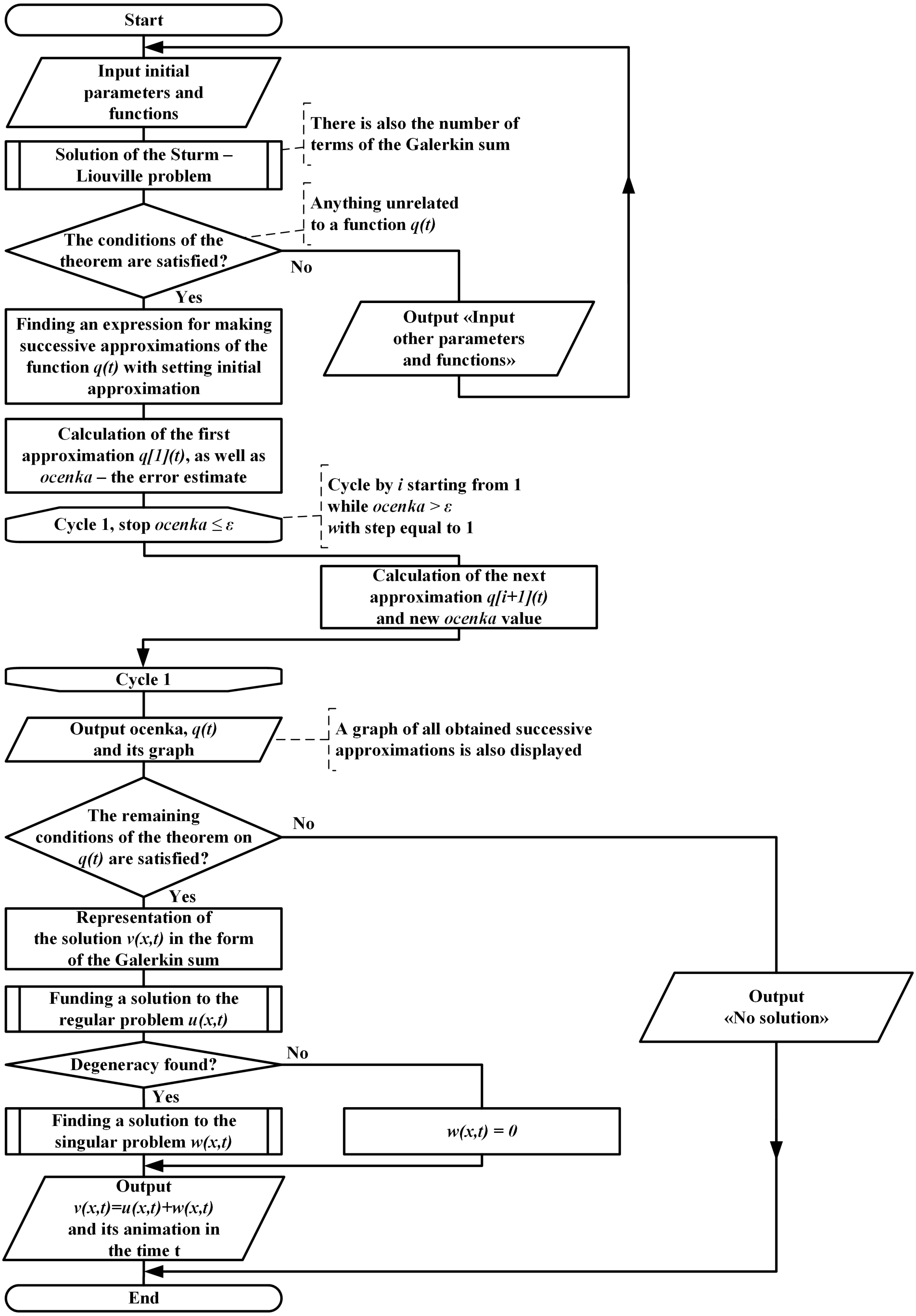

Fig. 1. Diagram of the algorithm

Step 3. Check the conditions of Theorem 1, which do not contain the function $q(t)$. If all conditions are met, then go to step 5, otherwise go to step 4.

Step 4. Print ≪Input another parameters and functions≫ Go to step 1.

Step 5. Find an expression for successive approximations of the function $q(t)$ using the formula from work [2]:

$$
\begin{aligned}
q[i+1](t) = -\,R(t)\Bigg(&\Psi^{(n)}(t) - CSV_{1,1}(t)u_0 - CSV_{1,2}(t)u_1 - \ldots - \\
&- CSV_{1,n}(t)u_{n-1} - CS\int_0^t V_{1,n}(t-s)h(s)ds - Ch(t)\Bigg) + \\
&+ R(t)CS\int_0^t V_{1,n}(t-s)\Phi(s)q[i](s)ds.
\end{aligned} \tag{15}
$$

Step 6. Calculate the first approximation $q[1](t)$ from the given initial approximation $q[0](t) = 0$, using formula (15). Calculate the estimation error *ocenka*, which is equal to the norm (in space $L^2(G)$) of the difference between the 1st approximation and the initial one.

Step 7. Cycle over i starting from 1 while $ocenka > \varepsilon$. If the loop condition is satisfied go to step 8, otherwise go to step 10.

Step 8. Calculate the next approximation $q[i+1](t)$ from the previous approximation $q[i](t)$ by formula (15). Calculate a new estimation error *ocenka*, which is equal to the norm of the differences between the $(i+1)$-th approximation and the i-th.

Step 9. Increse index i by one, go to step 7.

Step 10. Print the found approximate solution $q(t)$, as well as the resulting estimation error *ocenka* for the found function. Plot the found function $q(t)$ and the functions of all obtained successive approximations.

Step 11. Check the remaining conditions of Theorem 1, which are related to the found function $q(t)$. If all conditions are met, then go to step 13, otherwise go to step 12.

Step 12. Print ≪No solution≫. Stop the program.

Step 13. Represent the solution $v(x,t)$ as a Galerkin sum.

Step 14. Obtain an approximate solution of the Boussinesq – Love equation for the calculated function $q(t)$. The cycle in i, starting from 1, until $i \leq N$ with the step equal to 1. Multiplication of the Boussinesq – Love equation, as well as the initial conditions, by the eigenfunction $\mathbb{X}_i$. Solution of second order ordinary differential equations with initial conditions. Go to the next iteration. Bottom line: getting a solution to the regular problem $u(x,t)$ from (7)–(9).

Step 15. Check the degeneracy. If no degeneracy is found, then go to step 16, otherwise go to step 17.

Step 16. The solution $w(x,t)$ to the singular problem (10)–(11) is equal to zero. Go to step 18.

Step 17. Find the solution $w(x,t)$ to the singular problem (10)–(11) given by (14).

Step 18. Calculate the required function $v(x,t)$ as the sum of two previously obtained functions $u(x,t)$ and $w(x,t)$. Print the resulting function $v(x,t)$. Plot an animated graph of the function $v(x,t)$ by variable t.

End of program.

3. Computational Experiments

Present the results of computational experiments carried out using the developed algorithm, which was implemented in the Maple software package.

Example 1. Let the graph G_1 (Figure 2) consist of two edges with lengths $l_1 = l_2 = \pi$, connecting three vertices.

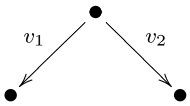

Fig. 2. Graph G_1

Set the parameters

$$\alpha = -0.25,\ \beta = -3,\ \gamma = 1,\ \varepsilon = 1,\ n = 3,\ T = 1$$

and functions

$$\varphi(x) = \Big(\cos(2x) - 1, \cos(2(x-\pi)) - 1\Big),$$

$$\psi(x) = \left(\frac{2(\cos(2x) - 1)}{\pi}, \frac{2(\cos(2(x-\pi)) - 1)}{\pi}\right),$$

$$f(x) = \Big(\sin(6x), \sin(6x)\Big),\ K(x) = \Big(\cos(x), \cos(x)\Big),\ F(t) = -\cos(t) + 1$$

be given. Consequently, the Boussinesq – Love equation (1) takes the form

$$(-0.25 - \Delta)v_{tt} = -3(\Delta - 1)v_t + q(t)\sin(6x), \quad v(x,t) = (v_1(x,t), v_2(x,t)),$$

condition of ≪balance of flows≫ (2) and ≪continuity of the solution≫ (3):

$$v_{1t}(\pi, t) = 0, \quad v_{1t}(0,t) + v_{2t}(0,t) = 0, \quad v_{2t}(\pi, t) = 0, \quad v_1(0,t) = v_2(0,t),$$

the initial condition for the position of (4) and the velocity of the edges of the graph (5):

$$v(x, 0) = \cos(2x) - 1, \quad v_t(x, 0) = \frac{2(\cos(2x) - 1)}{\pi},$$

and the overdetermination condition (6):

$$\int\limits_0^\pi v(x,t)\cos(x)dx = -\cos(t) + 1$$

are set. For such parameters and initial functions, all the conditions of Theorem 1 are satisfied. Using the developed algorithm an approximate solution to the problem was found:

$$q(t) = \frac{35\cos(t)}{44},$$

reaching admissible error $0.678389994 < \varepsilon$ at the 1-th step of approximation. Figure 3 shows the graph of the function $q(t)$.

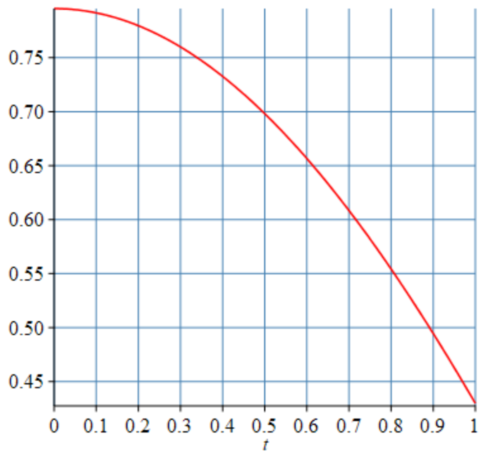

Fig. 3. Function $q(t)$ graph

Further in the program, the required functions

$$v_1(x,t) = u_1(x,t) + w_1(x,t), \quad v_2(x,t) = u_2(x,t) + w_2(x,t),$$

where

$$u_1(x,t) = \frac{32}{21\pi^{2.5}}\Bigg(\Big(\cos(1.5x)-1\Big)\Big(\frac{-4\sqrt{2}}{\sqrt{39}}+\pi\Big)\Big(\cos(1.5x)+1\Big)e^{-\frac{\sqrt{2}\sqrt{39}t}{4}}+$$

$$+\Big(\cos(1.5x)-1\Big)\Big(\frac{4\sqrt{2}}{\sqrt{39}}+\pi\Big)\Big(\cos(1.5x)+1\Big)e^{\frac{\sqrt{2}\sqrt{39}t}{4}}+$$

$$+\frac{7\cos(x)\pi^{1.5}\Big(e^{-2\sqrt{2}t}+e^{2\sqrt{2}t}-2\cos(t)\Big)}{264}\Bigg),$$

$$u_2(x,t) = \frac{-32}{21\pi^{2.5}}\Bigg(\Big(\cos(1.5x)-1\Big)\Big(\frac{-4\sqrt{2}}{\sqrt{39}}+\pi\Big)\Big(\cos(1.5x)+1\Big)e^{-\frac{\sqrt{2}\sqrt{39}t}{4}}+$$

$$+\Big(\cos(1.5x)-1\Big)\Big(\frac{4\sqrt{2}}{\sqrt{39}}+\pi\Big)\Big(\cos(1.5x)+1\Big)e^{\frac{\sqrt{2}\sqrt{39}t}{4}}+$$

$$-\frac{7\cos(x)\pi^{1.5}\Big(e^{-2\sqrt{2}t}+e^{2\sqrt{2}t}-2\cos(t)\Big)}{264}\Bigg),$$

$$w_1(x,t) = 0, \quad w_2(x,t) = 0$$

representing longitudinal vibrations in the rods were found. The last step of the program was in construction of a time-animated graph of the found vector-function $v(x,t)$. Figures 4 and 5 show the graphs of the vector-function $v(x,t)$ at different time t.

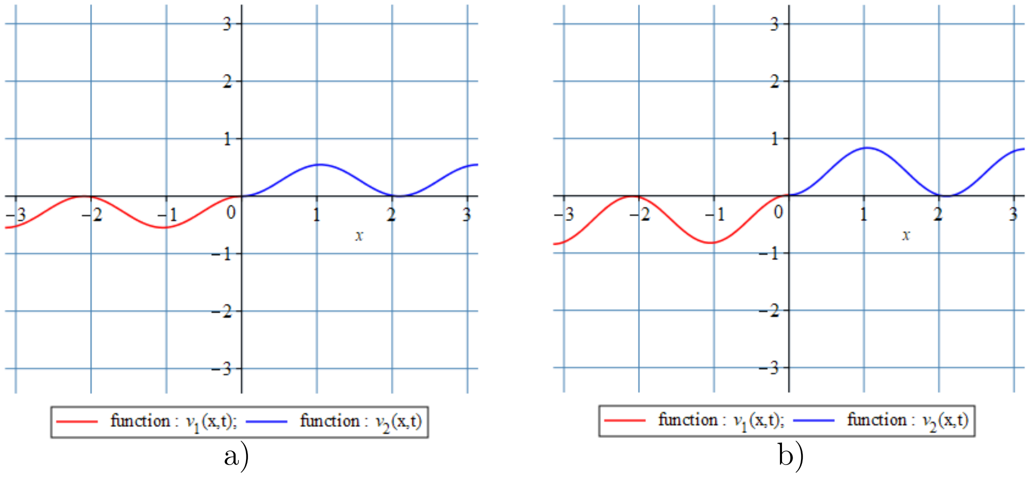

Fig. 4. Function $v(x,t)$ graph at: a) $t = 0$; b) $t = 0.33$

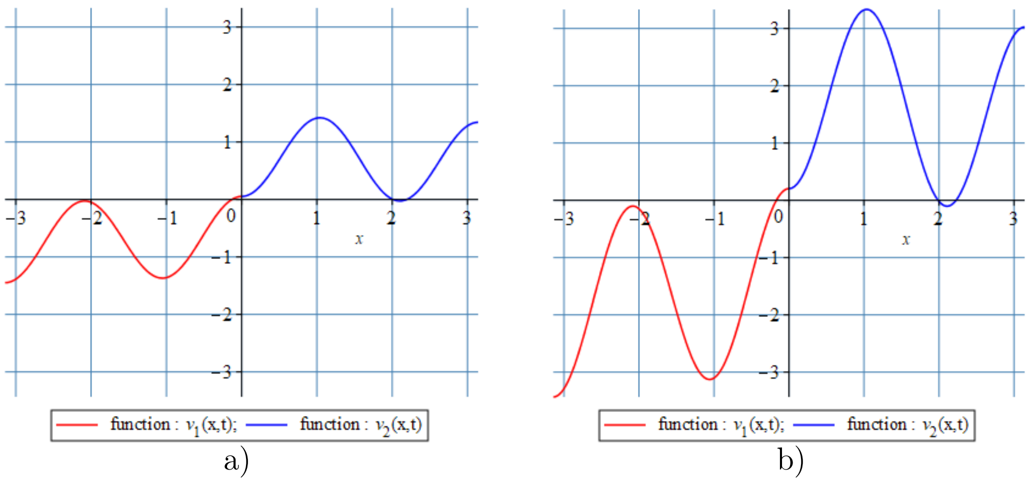

Fig. 5. Function $v(x,t)$ graph at: a) $t = 0.6$; b) $t = 1$

Example 2. Let the graph G_2 (Figure 6) consist of two edges with lengths $l_1 = l_2 = \pi$, connecting three vertices.

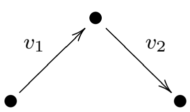

Fig. 6. Graph G_2

Set the parameters

$$\alpha = 4,\ \beta = 1,\ \gamma = 1,\ \varepsilon = 0.8,\ n = 3,\ T = 10,\ l = \Big(\pi, \pi\Big)$$

and functions

$$\varphi(x) = \Big(\cos(x), \cos(x+\pi)\Big),\quad \psi(x) = \Big(\cos(5x), \cos(5(x+\pi)\Big),$$

$$f(x) = \Big(\cos(x), \cos(x)\Big), \; K(x) = \Big(\cos(x), \cos(x)\Big), \; F(t) = \frac{4\cos(t)}{3}.$$

Consequently, the Boussinesq – Love equation (1) takes the form

$$(4-\Delta)v_{tt} = (\Delta - 1)v_t + q(t)\cos(x), \quad v(x,t) = (v_1(x,t), v_2(x,t)),$$

condition of ≪balance of flow≫ (2) and ≪continuity of the solution≫ (3):

$$v_{1t}(0,t) = 0, \quad v_{1t}(\pi,t) = v_{2t}(0,t), \quad v_{2t}(\pi,t) = 0, \quad v_1(\pi,t) = v_2(0,t),$$

the initial condition for the position of (4) and the velocity of the edges of the graph (5):

$$v_1(x,0) = \cos(x), \quad v_2(x,0) = -\cos(x), \quad v_{1t}(x,0) = \cos(5x), \quad v_{2t}(x,0) = -\cos(5x),$$

and the overdetermination condition (6):

$$\int_0^\pi v(x,t)\cos(x)dx = \frac{4\cos(t)}{3}$$

are set. For such parameters and initial functions, all the conditions of Theorem 1 are satisfied. Using the developed algorithm an approximate solution to the problem was found:

$$q(t) = \frac{4\cos(t)(1762152484\cos^2(t) - 8037989418\cos(t) + 17837462559)}{20647703175},$$

reaching a admissible error $0.6551933817 < \varepsilon$ at the 3-rd step of approximation. Figure 7 shows the graphs of the function $q(t)$ and successive approximations.

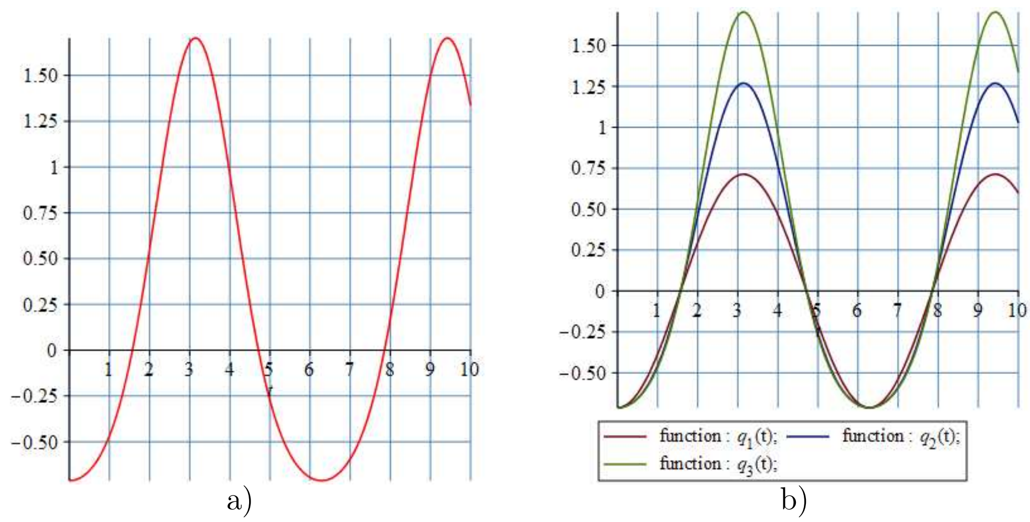

Fig. 7. Graph: a) of function $q(t)$; b) of functions of all approximations

Further in the program, the required functions

$$v_1(x,t) = \frac{1}{625092946330178576 25\pi^2}\Bigg(416728630886785717 50\Big(\sqrt{\pi}\cos\Big(\frac{x}{2}\Big) +$$

$$+\sqrt{\pi}\sin\left(\frac{x}{2}\right)-\frac{17109150420728}{4718607460875}\Big)\cos\left(\frac{x}{2}\right)\cos\left(\frac{\sqrt{85}t}{17}\right)+7501115355962142915\times$$

$$\times\Big(\sqrt{\pi}\cos\left(\frac{3x}{2}\right)-\sqrt{\pi}\sin\left(\frac{3x}{2}\right)-\frac{15870782349800}{5500823428529}\Big)\cos\left(\frac{3x}{2}\right)\cos\left(\frac{\sqrt{13}t}{5}\right)+$$

$$+252562806598051950\cos\left(\frac{x}{2}\right)\sqrt{85\pi}\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)\sin\left(\frac{\sqrt{85}t}{17}\right)-$$

$$-1585189213009751250\cos\left(\frac{3x}{2}\right)\sqrt{13\pi}\left(\cos\left(\frac{3x}{2}\right)-\sin\left(\frac{3x}{2}\right)\right)\sin\left(\frac{\sqrt{13}t}{5}\right)+$$

$$+\Big(103115726103963911520\cos(t)-41201066141796745 60\cos(2t)+$$

$$+192244245554109840\cos(3t)+519133433386638994 56\Big)\cos\left(\frac{x}{2}\right)+$$

$$+18560830698713504073\cos\left(\frac{3x}{2}\right)\left(\cos(t)-\frac{2679329806\cos(2t)}{92602205123}+\right.$$

$$\left.+\frac{440538121\cos(3t)}{338477025622}+\frac{8037989418}{41511333331}\right)\Bigg),$$

$$v_2(x,t)=\frac{1}{6250929463301785762 5\pi^2}\Bigg(\Big(\Big(-41672863088678571750\sqrt{\pi}\cos\left(\frac{x}{2}\right)+$$

$$+151101207074002246256\Big)\sin\left(\frac{x}{2}\right)+41672863088678571750\left(\cos\left(\frac{x}{2}\right)-1\right)\times$$

$$\times\sqrt{\pi}\left(\cos\left(\frac{x}{2}\right)+1\right)\Big)\cos\left(\frac{\sqrt{85}t}{17}\right)+\cos\left(\frac{\sqrt{13}t}{5}\right)\Big(\Big(75011153559621429150\times$$

$$\times\sqrt{\pi}\cos\left(\frac{3x}{2}\right)-216419542895695230000\Big)\sin\left(\frac{3x}{2}\right)+75011153559621429150\times$$

$$\times\sqrt{\pi}\left(\cos\left(\frac{3x}{2}\right)-1\right)\left(\cos\left(\frac{3x}{2}\right)+1\right)\Big)+25256280659805195\sqrt{85\pi}\times$$

$$\times\left(\cos^2\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\sin\left(\frac{x}{2}\right)\right)-1\right)\sin\left(\frac{\sqrt{85}t}{17}\right)-1585189213009751250\sqrt{13\pi}\times$$

$$\times\left(\cos^2\left(\frac{3x}{2}\right)+\cos\left(\frac{3x}{2}\sin\left(\frac{3x}{2}\right)\right)-1\right)\sin\left(\frac{\sqrt{13}t}{5}\right)+$$

$$+(-103115726103963911520\cos(t)+412010661417967456 0\cos(2t)-$$

$$-192244245554109840\cos(3t)-519133433386638994 56)\sin\left(\frac{x}{2}\right)+$$

$$+185608306987135040736\sin\left(\frac{3x}{2}\right)\left(\cos(t)-\frac{2679329806\cos(2t)}{92602205123}+\right.$$

$$+\frac{440538121\cos(3t)}{338477025622}+\frac{8037989418}{415113333331}\Big)\Big)$$

representing longitudinal vibrations in the rods were found. The last step of the program was in construction of a time-animated graph of the found vector-function $v(x,t)$. Figures 8 and 9 show the graphs of the vector-function $v(x,t)$ at different time t.

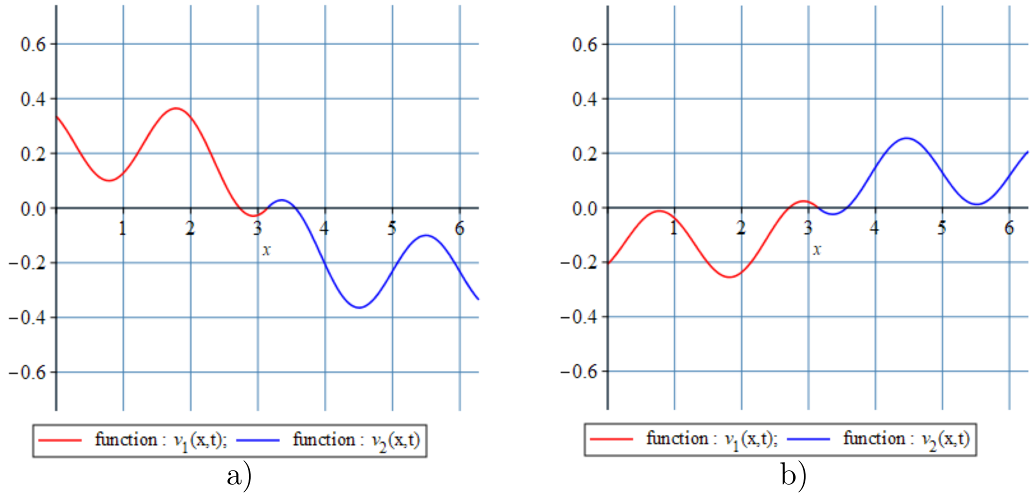

Fig. 8. Function $v(x,t)$ graph at: a) $t=0$; b) $t=3.33$

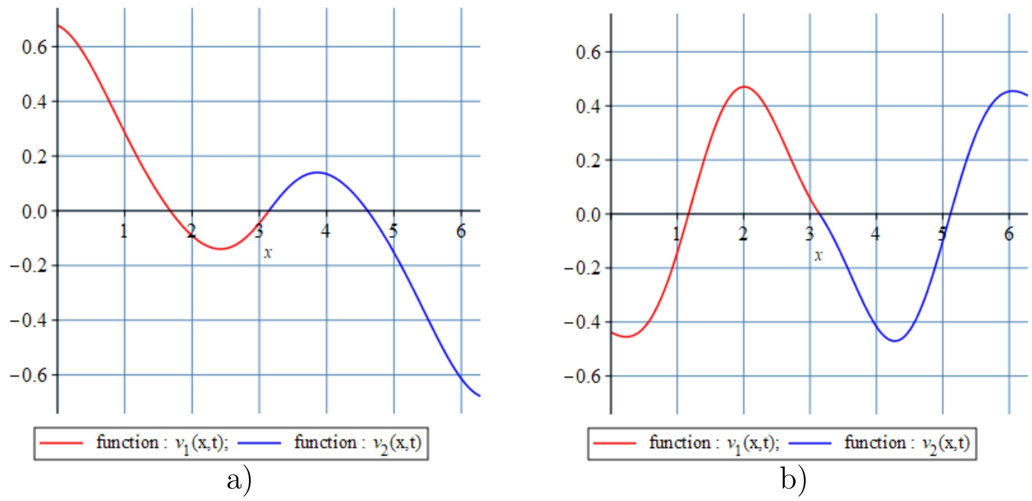

Fig. 9. Function $v(x,t)$ graph at: a) $t=6.66$; b) $t=10$

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ОБРАТНОЙ ЗАДАЧИ ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ БУССИНЕСКА – ЛЯВА НА ГРАФЕ

А. В. Лут, А. А. Замышляева

Статья посвящена проведению численного исследования математической модели Буссинеска – Лява рассматриваемой на геометрическом графе. Исследуемая модель описывает продольные колебания в конструкции состоящей из тонких, упругих стержней при учете внешней создаваемой нагрузкой на них. Для нахождения решения используется метод последовательных приближений, а сама модель сводится у неполному уравнению соболевского типа второго порядка. В первом параграфе приведены результаты аналитического исследования данной модели. Во втором параграфе приведен разработанный алгоритм численного метод и его схема. В третьем приведены результаты вычислительных экспериментов программы разработанной в среде Maple на основе полученного алгоритма. Все полученные результаты могут быть применены в области математического моделирования, например, при расчете продольных колебаний конструкции при учете создаваемой на нее внешней нагрузки.

Ключевые слова: математическая модель Буссинеска – Лява; обратная задача; численное исследование; уравнение соболевского типа; метод последовательных приближений.

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