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NUMERICAL ANALYSIS OF A ONE-DIMENSIONAL MODEL OF A MELTING-FREEZING SNOWPACK

S. V. Alekseeva[1,2]**,** svetlana.alekseeva.vl@yandex.ru,
S. A. Sazhenkov[1,3]**,** sazhenkovs@yandex.ru
[1]Altai State University, Barnaul, Russian Federation
[2]Novosibirsk State University, Novosibirsk, Russian Federation
[3]Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russian Federation

The article is devoted to a numerical study of a one-dimensional non-stationary problem on thermomechanical processes in a snowpack with account of effects of melting and freezing. Snow is modeled as a continuous medium consisting of water, air and porous ice skeleton. The governing equations of snow are based on the fundamental conservation laws of continuum mechanics. A finite-difference algorithm is constructed and a series of numerical experiments is fulfilled. The results of the computations correspond well to laboratory observations.

Keywords: filtration; phase transition; snow; finite-difference scheme.

Introduction

Adequate mathematical modeling of thermomechanical processes inside and near snowpacks is in a great demand due to the need to build calculations and forecasts of spring flood hydrographs and water quality in receiving reservoirs, to make assessment of the risk of avalanches in the mountains and the risk of collapse of membrane-like constructions caused by the weight of snow, etc. To date, there is a number of works devoted to modeling snow taking into account phase transitions, which use observational data and empirical dependencies. A fairly extensive review on this topic can be found in [1]. The present article is devoted to a study of the one-dimensional mathematical model of air and water filtration in a snowpack in the presence of ‘ice-water’ phase transitions. The snowpack is modeled as a three-phase continuum consisting of water, air (vapor in pores) and immovable porous ice skeleton. The precise formulation of this model is posed further in Sec. 1 along with the necessary explanations on its physical background. Because of nonlinearity and high inter-connection between equations, the rigorous mathematical results on existence and uniqueness of classical or generalized solutions to boundary value problems for the considered model are unavailable, at least, so far. Therefore, in this article we concentrate on a numerical analysis for the initial-boundary value problem for the model. More precisely, in Sec. 2, we construct the finite-difference algorithm and carry out a series of numerical experiments using both the test data and the data of laboratory experiments. The results of the computations appear to correspond well to laboratory observations. Ending this introduction, it is worth to notice that the present study is rather close to a set of recent works devoted to the mathematical modeling of processes in snowpack, see [2, 3, 4], and can be referred to as their extension.

1. The 1D Model of Thermomechanical Processes in a Snowpack

At temperatures close to the freezing point of water, snow can be described as a three-phase medium consisting of water, air (water vapor) and ice. In this case, ice is a solid porous skeleton, and a mixture of water (in liquid form) and air is a two-component continuous medium filtering through the pores. We introduce a mathematical description of the balance of mass, momentum and heat on the basis of the fundamental conservation laws in continuum mechanics, following the presentation from the monographs [1, Ch. 3, Secs. 1-2], [5, Ch. I, Sec. 1.7], [6, Ch. 4, Secs. 4.1-4.2]. In one-dimensional case, with the help of the technical procedure described in detail in [1, Ch. 3, Secs. 2-3], [5, Ch. V, § 1, Sec. 1], the basic three-dimensional model consisting of a system of conservation laws and clarifying and simplifying phenomenological hypotheses is reduced to a nonlinear system of mixed type consisting of five differential equations. Now we formulate the initial-boundary value problem for this system and give the necessary comments on its physical meaning. After that, in Sec. 2, we carry out a numerical analysis of this problem.

We suppose that $\Omega := (0, l)$ is an open bounded interval in the space $\mathbb{R}$ of physical positions x, where the coordinate $x = 0$ corresponds to the bottom surface of a snowpack bordering on a frozen base (ground, ice, rooftop, etc.), and $x = l$ is the upper surface bordering an open air. By t we denote the time variable.

Problem 1D. (The dynamical one-dimensional model of a snowpack.) In the space-time domain $\Omega_T = \Omega \times \{0 < t < T\}$, where $T = \text{const} > 0$ is a given time moment, find the water saturation in pores $s = s(x, t)$, the reduced pressure $p = p(x, t)$, the intensity of the 'ice-water' phase transition $I = I(x, t)$, the temperature of snow $\theta = \theta(x, t)$, and the total velocity of filtration $v = v(x, t)$, satisfying the reduced balance of mass equations

$$\frac{\partial\big[(1-s)\phi(x,\theta)\big]}{\partial t} + \frac{\partial}{\partial x}\left[a(s,\phi(x,\theta))\frac{\partial s}{\partial x} + b(s)v + F(s,\phi(x,\theta))\right] = 0,\ (x,t) \in \Omega_T, \tag{1a}$$

$$\frac{\partial}{\partial x}\Big[K(s,\phi(x,\theta))\frac{\partial p}{\partial x} - f(s,\phi(x,\theta))\Big] = \left(1 - \frac{\rho_3^0}{\rho_1^0}\right)\frac{\partial \phi(x,\theta)}{\partial t}, \quad (x,t) \in \Omega_T, \tag{1b}$$

$$\frac{\partial \phi(x,\theta)}{\partial t} = \frac{I}{\rho_3^0}, \quad (x,t) \in \Omega_T, \tag{1c}$$

the reduced Darcy law

$$v = -K(s,\phi(x,\theta))\frac{\partial p}{\partial x} + f(s,\phi(x,\theta)), \quad (x,t) \in \Omega_T, \tag{1d}$$

the heat balance equation

$$Q(s,\phi(x,\theta))\frac{\partial \theta}{\partial t} + V\left(v, s, \phi(x,\theta), \frac{\partial s}{\partial x}\right)\frac{\partial \theta}{\partial x} - \frac{\partial}{\partial x}\left(\lambda_c(s,\phi(x,\theta))\frac{\partial \theta}{\partial x}\right) = -(c_1 - c_3)\,\theta\, I, \quad (x,t) \in \Omega_T, \tag{1e}$$

and the boundary and initial conditions

$$s|_{x=0} = s_0(0,t), \quad p|_{x=0} = p_0(t), \quad \theta|_{x=0} = \theta_0(0,t), \quad t \in (0,T], \tag{1f}$$

$$s|_{x=l} = s_0(l,t), \quad \frac{\partial p}{\partial x}\Big|_{x=l} = 0, \quad \theta|_{x=l} = \theta_0(l,t), \quad t \in (0,T], \tag{1g}$$

$$s|_{t=0} = s_0(x,0), \quad \theta|_{t=0} = \theta_0(x,0), \quad x \in [0,l]. \tag{1h}$$

In conditions (1f)-(1h), $p_0 = p_0(t)$ is a given function on $[0,T]$ and $s_0 = s_0(x,t)$ and $\theta_0 = \theta_0(x,t)$ are given functions on $\sqcup_T$, where $\sqcup_T = \big(\{0 \le x \le l\} \times \{t=0\}\big) \cup \big(\{x = 0,\, x = l\} \times (0,T]\big)$, i.e., $\sqcup_T$ is the $\sqcup$-shaped part of $\partial\Omega_T$.

In equations (1a)-(1e), the nonlinear (in general) functions ϕ, a, b, f, k, K, F, Q, and V are given. They are uniquely defined by the given physical characteristics of the components of the snow via the following formulas:

$$\phi(x,\theta) = \phi^- \text{ for } \theta < \theta^-, \quad \frac{\partial \phi}{\partial \theta}(x,\theta) \ge 0 \text{ for } \theta \in [\theta^-,\theta^+], \quad \phi(x,\theta) = \phi^+ \text{ for } \theta > \theta^+, \tag{2a}$$

$$a(s,\phi) = -K_0(\phi)\frac{k_{01}(s)k_{02}(1-s)}{\mu_2 k_{01}(s) + \mu_1 k_{02}(1-s)}\, p_c'(s), \quad b(s) = \frac{k_{02}(1-s)}{\mu_2 k(s)}, \tag{2b}$$

$$f(s,\phi) = -K_0(\phi)\left[\left(\frac{k_{01}(s)}{\mu_1}\rho_1^0 + \frac{k_{02}(1-s)}{\mu_2}\rho_2^0\right)\right] g, \quad k(s) = \frac{k_{01}(s)}{\mu_1} + \frac{k_{02}(1-s)}{\mu_2}, \tag{2c}$$

$$K(s,\phi) = k(s)K_0(\phi), \quad F(s,\phi) = -\frac{k_{02}(1-s)}{\mu_2 k(s)} f(s,\phi) - K_0(\phi)\left[\frac{k_{02}(1-s)}{\mu_2}\rho_2^0\, g\right], \tag{2d}$$

$$Q(s,\phi) = c_1 s\phi\rho_1^0 + c_2(1-s)\phi\rho_2^0 + c_3(1-\phi)\rho_3^0, \tag{2e}$$

$$V(v,s,\phi,\zeta) = c_1\rho_1^0\big[(1-b(s))v - a(s,\phi)\zeta - F(s,\phi)\big] + c_2\rho_2^0\big[a(s,\phi)\zeta + b(s)v + F(s,\phi)\big]. \tag{2f}$$

Here,

- $\phi(x,\theta)$ is the snow porosity, i.e., the volumetric part of pores in the specific volume of snow; for fixed x, it is a piece-wise differentiable function of θ and it is postulated following [7], namely, $\phi(x,\theta)$ is a prescribed function on the segment $\{\theta^- \le \theta \le \theta^+\}$, values ϕ^- and ϕ^+ are constant and belong to $(0,1]$, values θ^- and θ^+ are also constant and satisfy the inequality $0 < \theta^- \le \theta^+$, and θ^+ is the temperature of ice melting;

- $K_0(\phi)$ is a nonnegative porous skeleton permeability coefficient such that $K_0(0) = 0$;

- $k_{01}(s) \ge 0$ and $k_{02}(1-s) \ge 0$ are the phase permeability coefficients of water and air (water vapor), respectively, such that $k_{01}(0) = 0$, $k_{02}(0) = 0$;

- μ_1 and μ_2 are the positive constant dynamical viscosity coefficients of water and air, respectively;

- $p_c(s)$ is the capillary pressure, which is the given function that has the properties $p_c(s) > 0$, $p_c(0) = \infty$, $p_c(1) = 0$, and $p_c'(s) < 0$ [5, Ch. 5, Sec. 1.1];

- ρ_1^0, ρ_2^0, and ρ_3^0 are the constant genuine densities of water, air, and ice, respectively;

- $g = \mathrm{const} > 0$ is the acceleration of free fall;

- c_1, c_2, and c_3 are the positive constant coefficients of specific heat capacity at constant volume of water, air, and ice, respectively;

- $\lambda_c(s,\phi)$ is the heat conductivity of snow, which is given according to experimental data and satisfies the positiveness property $\lambda_c(s,\phi) \geq \lambda_- = \text{const} > 0$, $\forall\, s,\phi \in \mathbb{R}$.

The formulation of Problem 1D takes into account the postulate of immobility of the porous ice skeleton, the postulate of equality of the temperature in all three phases in each point of continuum, and the natural observations (see, for example, [6, page 105]) that sublimation is negligible, i.e., the 'ice-air', 'air-ice', 'water-air', and 'air-water' phase transitions are absent. The latter implies that only the intensity I of the 'ice-water' phase transition, or, equivalently, the intensity $-I$ of the 'water-ice' phase transition, takes place in the equations. Also, in (1) and (2) we impose the physically reasonable requirements $\rho_2^0 < \rho_3^0 < \rho_1^0$ on the genuine densities of phases and $c_1 > c_3$ on the specific heat capacities.

We note that, having the five sought functions s, p, I, θ and v found, one can determine separately the velocity of filtration of air v_2 and the velocity of filtration of water v_1 by the formulas

$$v_2 = a(s,\phi(x,\theta))\frac{\partial s}{\partial x} + b(s)v + F(s,\phi(x,\theta)), \quad v_1 = v - v_2, \tag{3}$$

the air pressure p_2 and the hydraulic pressure p_1 in pores by the formulas

$$p_2 = p - \int\limits_s^1 \frac{k_{01}(\xi)p_c'(\xi)}{\mu_1 k(\xi)}d\xi, \quad p_1 = p_2 - p_c(s), \tag{4}$$

and the reduced densities of water (ρ_1), air (ρ_2) and ice (ρ_3) by the formulas

$$\rho_1 = s\phi\rho_1^0, \quad \rho_2 = (1-s)\phi\rho_2^0, \quad \rho_3 = (1-\phi)\rho_3^0. \tag{5}$$

Remark 1. Note that the formulations based on modifications of system (1) have already been considered by various authors. For example, in [2], a numerical study of an initial-boundary value problem for a system of the form (1) was carried out, in which, on the right-hand side of the equation (1e), the sum $-\nu I$ takes place instead of $-(c_1 - c_3)\theta I$, where $\nu = \text{const} > 0$ is the given specific latent heat of the 'ice-water' phase transition and the intensity of the phase transition $I = I(\phi,\theta,s)$ is a given function of a very special form. Note that in the setting considered in the present article, the value I is a sought function, not a given one. The same model (with minor changes) as in [2] was considered earlier in [7], where the unique solvability was proved for the self-similar setting.

2. Numerical Study of Problem 1D

2.1. Foreword. Bringing Problem 1D to a Dimensionless Form

Let us carry out a numerical study of Problem 1D, providing it with specific model input data. We develop the finite-difference algorithm similar to the one proposed by A. N. Sibin and A. A. Papin in [2].

We take $K_0(\phi) := B\phi^3$ ($B = \text{const} > 0$), as in [8, Sec. 9.10, formula (9.9)], and $\lambda_c(s,\phi) = a_c + b_c\rho_c^2(s,\phi)$, where $\rho_c(s,\phi) = s\phi\rho_1^0 + (1-s)\phi\rho_2^0 + (1-\phi)\rho_3^0$, $a_c = \text{const} > 0$, and $b_c = \text{const} > 0$, as in [6, Ch. 4, Secs. 4.1-4.2].

We bring Problem 1D to a dimensionless form. The choice of the independent and dependent dimensionless variables (all of them are marked with tildes) is such that equations (1a)-(1e) exactly preserve their forms, whereas the dimensionless quantities in the equations are related with the dimensional ones by the formulas

$$\tilde t = t/t_{sc}, \quad \tilde x = x/x_{sc}, \quad \tilde p = p/p_{sc}, \quad \tilde I = t_{sc} I/\rho_3^0, \quad \tilde\theta = \theta/\theta_{sc}, \quad \tilde v = v/v_{sc},$$
$$\tilde a(s,\phi) = \frac{t_{sc}}{x_{sc}^2}\, a(s,\phi), \quad \tilde b(s) = \frac{t_{sc} v_{sc}}{x_{sc}}\, b(s), \quad \tilde F(s,\phi) = \frac{t_{sc}}{x_{sc}} F(s,\phi),$$
$$\tilde K(s,\phi) = \frac{p_{sc}}{v_{sc} x_{sc}} K(s,\phi), \quad \tilde f(s,\phi) = f(s,\phi)/v_{sc}, \quad \tilde Q(s,\phi) = Q(s,\phi)/(c_3\rho_3^0),$$
$$\tilde\lambda_c(s,\phi) = \frac{a_c t_{sc}}{x_{sc}^2 \rho_3^0 c_3}\Big(1 + \frac{b_c}{a_c}\rho_c^2(s,\phi)\Big), \quad \tilde V\big(\tilde v, s, \phi, \partial s/\partial x\big) = \frac{\theta_{sc}}{\rho_3^0 c_3 x_{sc}} V\big(v, s, \phi, \partial s/\partial x\big),$$

where $x_{sc} = l$, $p_{sc} = \rho_1^0 g l$, $v_{sc} = B\rho_1^0 g/\mu_1$, $\theta_{sc} = \theta^+$, and t_{sc} are the characteristic scales of length, pressure, velocity, temperature, and time, resp. Also, coefficient $-(c_1 - c_3)$ in the right hand side of equation (1e) becomes $-(c_1/c_3 - 1)$ in the dimensionless version of this equation.

2.2. Numerical Algorithm

In this paragraph, for simplicity, we omit tildes over dimensionless terms and refer to the dimensionless equations by the same formula numbers as for dimensional equations, but with the subscript «d-l». Equation $(1a)_{\text{d-l}}$ is approximated using the directional difference for the convective term. Equations $(1b)_{\text{d-l}}$ and $(1e)_{\text{d-l}}$ are approximated by an implicit second order precision scheme. As a result, from $(1a)_{\text{d-l}}$-$(1e)_{\text{d-l}}$ and $(3)_{\text{d-l}}$ we derive the system of difference equations

$$\begin{aligned}\phi_i^n \frac{s_i^{n+1} - s_i^n}{\tau} &= \frac{1}{h^2}\Big(a^n_{i+\frac12}(s^{n+1}_{i+1} - s^{n+1}_i) - a^n_{i-\frac12}(s^{n+1}_i - s^{n+1}_{i-1})\Big) \\ &\quad + \Big(\frac{\partial F}{\partial s}(s_i^n, \phi_i^n) - v_i^n \frac{\partial b}{\partial s}(s_i^n)\Big)\frac{s^{n+1}_{i+1} - s^{n+1}_{i-1}}{2h} \\ &+ \frac{\partial F}{\partial \phi}(s_i^n, \phi_i^n)\frac{\partial \phi}{\partial \theta}(iN^{-1}, \theta_i^n)\frac{\theta^n_{i+1} - \theta^n_{i-1}}{2h} + (1 - s_i^n)\frac{\partial \phi}{\partial \theta}(iN^{-1}, \theta_i^n)\frac{\theta_i^{n+1} - \theta_i^n}{\tau},\end{aligned} \tag{6a}$$

$$\begin{aligned}\frac{1}{h^2}\Big(K_l(p^{n+1}_{i+1} - p^{n+1}_i) - K_\omega(p^{n+1}_i - p^{n+1}_{i-1})\Big) + \frac{f^n_{i+1} - f^n_{i-1}}{2h} &= \\ = \Big(1 - \frac{\rho_3^0}{\rho_1^0}\Big)\frac{\partial \phi}{\partial \theta}(iN^{-1}, \theta_i^n)\frac{\theta_i^{n+1} - \theta_i^n}{\tau},\end{aligned} \tag{6b}$$

$$v_i^n = -K(s_i^n, \phi_i^n)\frac{p^n_{i+1} - p^n_i}{h} + f_i^n, \tag{6c}$$

$$(v_2)_i^n = a(s_i^n, \phi_i^n)\frac{s^n_{i+1} - s^n_i}{h} + b(s_i^n)v_i^n + F(s_i^n, \phi_i^n), \quad (v_1)_i^n = v_i^n - (v_2)_i^n, \tag{6d}$$

$$\begin{aligned}Q_i^n \frac{\theta_i^{n+1} - \theta_i^n}{\tau} &= \frac{1}{h^2}\Big(\lambda^n_{ci+\frac12}(\theta^{n+1}_{i+1} - \theta^{n+1}_i) - \lambda^n_{ci-\frac12}(\theta^{n+1}_i - \theta^{n+1}_{i-1})\Big) \\ &\quad - \frac{(|V_i^n| + V_i^n)\theta^{n+1}_{i+1} - 2|V_i^n|\theta^{n+1}_i + (|V_i^n| - V_i^n)\theta^{n+1}_{i-1}}{2h} \\ &\quad + \left(\frac{c_1}{c_3} - 1\right)\theta_i^n \frac{\partial \phi}{\partial \theta}(iN^{-1}, \theta_i^n)\frac{\theta_i^{n+1} - \theta_i^n}{\tau},\end{aligned} \tag{6e}$$

where $i = 0, \dots, N$, $\quad n = 0, \dots, M-1$, and the following notation is used:

$$\phi_i^n := \phi(iN^{-1}, \theta_i^n), \quad f_i^n := f(s_i^n, \phi_i^n), \quad V_i^n = -(v_1)_i^n \rho_1^0 c_1/(\rho_3^0 c_3) - \rho_2^0 c_2 (v_2)_i^n/(\rho_3^0 c_3),$$

$$a_{i+\frac{1}{2}}^n = \frac{K_0(\phi_{i+1}^n) a(s_{i+1}^n, \phi_{i+1}^n) + K_0(\phi_i^n) a(s_i^n, \phi_i^n)}{2},$$

$$a_{i-\frac{1}{2}}^n = \frac{K_0(\phi_{i-1}^n) a(s_{i-1}^n, \phi_{i-1}^n) + K_0(\phi_i^n) a(s_i^n, \phi_i^n)}{2},$$

$$\lambda_{i-\frac{1}{2}}^n = \frac{2\lambda_c(\phi_{i-1}^n, s_{i-1}^n)\lambda_c(\phi_i^n, s_i^n)}{\lambda_c(\phi_{i-1}^n, s_{i-1}^n) + \lambda_c(\phi_i^n, s_i^n)}, \quad \lambda_{i+\frac{1}{2}}^n = \frac{2\lambda_c(\phi_{i+1}^n, s_{i+1}^n)\lambda_c(\phi_i^n, s_i^n)}{\lambda_c(\phi_{i+1}^n, s_{i+1}^n) + \lambda_c(\phi_i^n, s_i^n)},$$

$$K_l = \frac{2K(\phi_i^n, s_i^n) K(\phi_{i+1}^n, s_{i+1}^n)}{K(\phi_i^n, s_i^n) + K(\phi_{i+1}^n, s_{i+1}^n)}, \quad K_\omega = \frac{2K(\phi_i^n, s_i^n) K(\phi_{i-1}^n, s_{i-1}^n)}{K(\phi_i^n, s_i^n) + K(\phi_{i-1}^n, s_{i-1}^n)}.$$

The algorithm for the numerical solution of the initial-boundary value problem is as follows. We solve (6e) for $\dfrac{\theta_i^{n+1} - \theta_i^n}{\tau}$, by first taking θ_i^0 instead of θ_i^1 in the first two terms in the right hand side. Then we substitute the result into (6b) and, using θ_i^0 and s_i^0, we find p_i^0 $(i = 0, ..., N)$. Further, using p_i^0, s_i^0 and θ_i^0, we determine ϕ_i^0 and v_i^0, $(v_1)_i^0$ and $(v_2)_i^0$ from (6c) and (6d). After this, from (6a) we find s_i^1 and from (6e) we find θ_i^1. Next, we calculate p_i^1 from (6b), and so on. Repeating this algorithm further for $n = 1, 2, \dots, M-1$, we find the values of the sought functions over the entire time interval.

2.3. Numerical Illustrations

For the numerical study of Problem 1D, we use the empirical relations from [9]. Namely, we take $p_c(s) = (s^{-1} - 1)\gamma$, where $\gamma = 0.0007\,Pa$; $x_{sc} = 1\,m$; $t_{sc} = 1\,h$; $g = 9.8\,m/s^2$; $K_0(\phi) = B\phi^3$, where $B = 1\,mkm^2$; $\theta^- = 268.3\,K$; $\theta^+ = 278.15\,K$; $\phi_- = 0.6$;

$$\phi(\theta) = \phi_- \text{ for } \theta < \theta^-, \quad \phi(\theta) = \phi_- + (1 - \phi_-)(\theta - \theta_-)(\theta^+ - \theta^-)^{-1} \text{ for } \theta \in [\theta^-, \theta^+],$$
$$\phi(\theta) = 1 \text{ for } \theta > \theta^+;$$

$$k_{01}(s) = 0 \text{ for } s \le 0, \quad k_{01}(s) = s^3 \text{ for } s \in [0,1], \quad k_{01}(s) = 1 \text{ for } s \ge 1;$$
$$k_{02}(1-s) = 1 \text{ for } s \le 0, \quad k_{02}(1-s) = (1-s)^3 \text{ for } s \in [0,1], \quad k_{02}(s) = 0 \text{ for } s \ge 1.$$

We remark that in this section (Sec. 2.3) the physical quantities are taken in dimensional form for better clarity of illustrations. In the boundary and initial conditions below the unit of time measurement is an hour. These conditions are set up as follows:

$$\theta(x,0) = 268.3\,K \quad s(x,0) = 0.01 + 0.5\,x, \quad x \in (0, x_{sc}) = (0,1),$$
$$p(0,t) = 101325\,Pa, \quad s(0,t) = 0.01, \quad \theta(0,t) = 268.3 + 10\,\sin(t\,\pi/12)\,K, \quad t > 0,$$
$$(\partial p/\partial x)(1,t) = 0, \quad s(1,t) = 0.51 + 0.01\,\cos(t\,\pi/12), \quad \theta(1,t) = 268.3\,K, \quad t > 0.$$

Notice that the initial data for $p(0,t)$ corresponds to the atmospheric pressure. The values of the rest of the coefficients and parameters for numerical simulations are borrowed from [10] as follows: $h = 0.01$, $N = 100$, $\tau = 0.01$, $M = 100$, $\rho_1^0 = 1000\,kg/m^3$, $\rho_2^0 = 1.292\,kg/m^3$, $\rho_3^0 = 916.2\,kg/m^3$, $c_1 = 418\,J/(kg\,K)$, $c_2 = 1005\,J/(kg\,K)$, $c_3 = 206\,J/(kg\,K)$, $\mu_1 = 0.001787\,kg/(m\,s)$, and $\mu_2 = 0.0000171\,kg/(m\,s)$.

The numerical algorithm allows us to determine all thermomechanical characteristics of snow. In this article, we focus on defining porosity ϕ, the water saturation in pores s and temperature θ. Figs. 1-6 show the results of the performed simulations. Fig. 1 shows the porosity profile at $t = 1.74\,h$. Fig. 2 shows the distribution of porosity; one can see that, over time, porosity increases with increasing temperature. Figs. 3 and 4 show the temperature distribution. Figs. 5 and 6 show the saturation distribution; it can be seen that over time the saturation also increases with increasing temperature, like the porosity does.

From Figs. 1-6 we can notice that the lifespan of the numerical solution for the given input data is not short, about three hours. This allows to conclude that the numerical solution of Problem 1D is rather good for description of fairly regular weather regimes, despite the fact that the global existence theorems for Problem 1D cannot be achieved.

Thus, numerical illustrations indicate good consistency of Problem 1D with real physics of the melting/freezing processes. Worth noticing that the analogous experiments also were carried out in [2] and [11] and gave somewhat similar results.

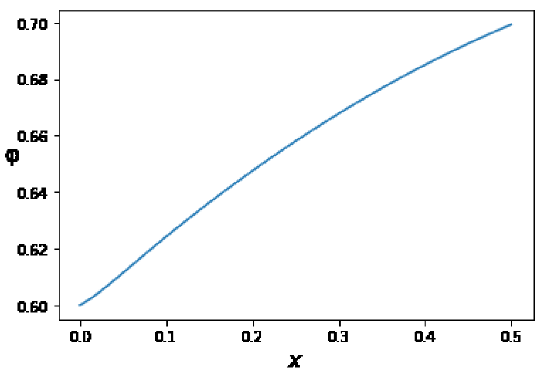

Fig. 1. Profile of porosity at $t = 1.74\,h$. The porosity ϕ is plotted as the function of the hight x of the snow mass

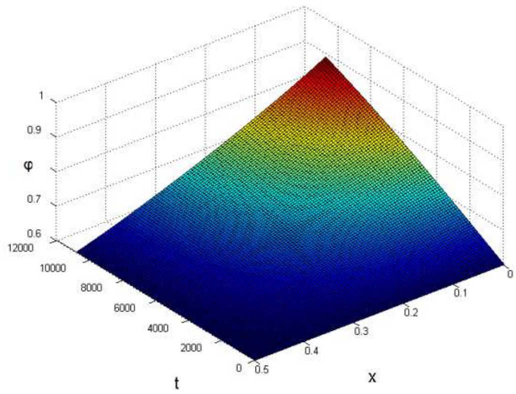

Fig. 2. Porosity distribution in the snowpack plotted as the function of the hight $x\,(m)$ and time $t\,(s)$

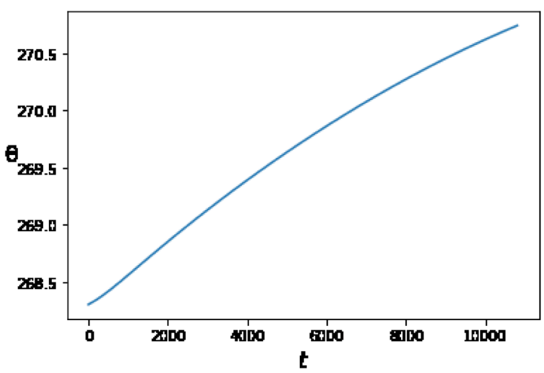

Fig. 3. Temperature distribution on the snow surface plotted as the function of $t\,(s)$

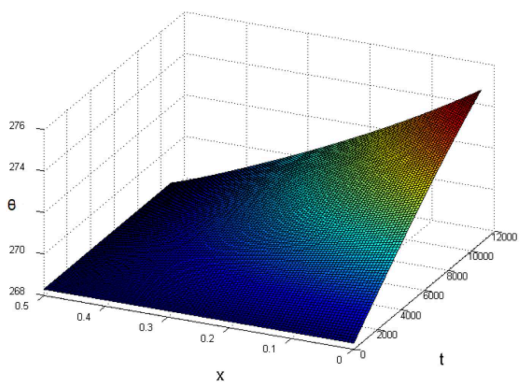

Fig. 4. Temperature distribution in the snowpack as the function of $x\,(m)$ and $t\,(s)$

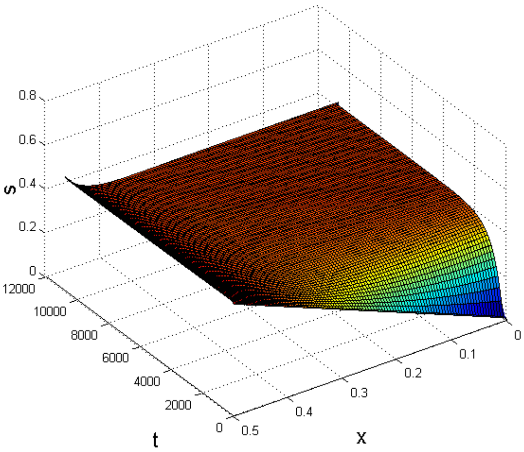

Fig. 5. The water saturation distribution in the snowpack plotted as the function of the hight $x\,(m)$ and time $t\,(s)$

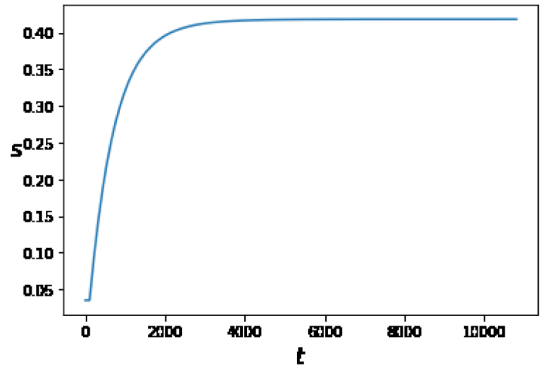

Fig. 6. Profile of the water saturation in pores at the hight $h = 0.02\,m$, i.e., near the frozen bottom. The water distribution is plotted as the function of time $t\,(s)$

Conclusion

For a highly nonlinear one-dimensional problem of description of thermomechanical processes in a snowpack, the finite-difference algorithm is constructed and the series of numerical experiments is fulfilled. The results of these experiments correspond well to laboratory data.

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Svetlana V. Alekseeva, MS (Math), Junior Research Scientist at the Laboratory for Mathematical and Computer Modeling in Natural and Industrial Systems, Altai State University (Barnaul, Russian Federation), and Postgraduate Student at the Department of Mechanics and Mathematics, Novosibirsk State University (Novosibirsk, Russian Federation), svetlana.alekseeva.vl@yandex.ru.

Sergey A. Sazhenkov, PhD (Math), Docent, Principle Research Scientist at the Laboratory for Mathematical and Computer Modeling in Natural and Industrial Systems, Altai State University (Barnaul, Russian Federation), and Senior Research Scientist at the Theoretical Department, Lavrentyev Institute of Hydrodynamics (Novosibirsk, Russian Federation), sazhenkovs@yandex.ru.

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ЧИСЛЕННЫЙ АНАЛИЗ ОДНОМЕРНОЙ МОДЕЛИ ТАЮЩЕГО ИЛИ ЗАМЕРЗАЮЩЕГО СНЕЖНОГО ПОКРОВА

С. В. Алексеева, С. А. Саженков

Статья посвящена численному исследованию нестационарной одномерной задачи описания термомеханических процессов в снегу с учетом эффектов таяния и промерзания. Снег моделируется как сплошная среда, состоящая из воды, воздуха и пористого ледяного скелета. Базовые уравнения, описывающие состояние снега, основаны на фундаментальных законах сохранения механики сплошных сред. Проводится построение конечно-разностного алгоритма и выполняется серия численных экспериментов. Результаты расчетов хорошо соответствуют лабораторным наблюдениям.

Ключевые слова: фильтрация; фазовый переход; снег; конечно-разностная схема.

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Алексеева Светлана Владимировна, магистр математики и компьютерных наук, младший научный сотрудник, лаборатория математического и компьютерного моделирования в природных и индустриальных системах, Алтайский государственный университет (г. Барнаул, Российская Федерация); аспирант, кафедра теоретической механики, механико-математический факультет, Новосибирский национальный исследовательский государственный университет (г. Новосибирск, Российская Федерация), svetlana.alekseeva.vl@yandex.ru.

Саженков Сергей Александрович, доктор физико-математических наук, доцент, главный научный сотрудник, лаборатория математического и компьютерного моделирования в природных и индустриальных системах, Алтайский государственный университет (г. Барнаул, Российская Федерация); старший научный сотрудник, лаборатория краевых задач механики сплошных сред, Институт гидродинамики им. М.А. Лаврентьева СО РАН (г. Новосибирск, Российская Федерация), sazhenkovs@yandex.ru.

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