MSC 35G15, 65N30 **DOI: 10.14529/jcem210404**

NUMERICAL SOLUTION OF THE CAUCHY-WENTZELL PROBLEM FOR THE DZEKZER MODEL IN A BOUNDED DOMAIN

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In terms of the theory of the p-sectorial operator, the Cauchy problem for the Dzekzer equation describing the evolution of the free surface of a filtered liquid with pure Wentzell boundary conditions are investigated. In particular, we consider the relative spectrum in the Dzekzer equation and construct a resolving holomorphic semigroup of the operator in the Cauchy–Wentzell problem. In the article, these problems are solved under the assumption that the initial space in which the Laplace operator operates on the bounded domain is a Lebesgue space $L^2(\Omega)$. The purpose of this work is to show new approach for resolvability of this problem with pure Wentzell boundary conditions. Namely, according to the modified Galerkin method, describe the solution of the Cauchy–Wentzell problem.

Keywords: Wentzell boundary conditions, resolving semigroups, Dzekzer equations.

Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\backslash\{1\}$ be a bounded connected domain with the boundary $\partial\Omega$ of the class C^∞. Let us consider the Cauchy problem with pure Wentzell boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), x \in \Omega, \\ &\Delta u(x,y,z) = 0,\ (x,t) \in \partial\Omega \times \mathbb{R} \\ &u(x,y,z,t) = 0,\ (x,t) \in \partial\Omega \times \mathbb{R} \end{aligned} \tag{1}$$

for the Dzekzer equation

$$(\lambda - \Delta)u_t(x,t) = \alpha\Delta u(x,t) - \beta\Delta^2 u(x,t) + f(x,t),\ (x,t) \in \Omega \times \mathbb{R}, \tag{2}$$

which simulates the evolution of the free surface of the filtered liquid. Here α and $\beta \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$ are real parameters characterizing the medium; the function $f(x,t)$ corresponds to liquid sources.

The purpose of this work is to show new approach for resolvability of problem (1)–(2) with pure Wentzell boundary conditions. Namely, according to the modified Galerkin method, describe the solution of the Cauchy–Wentzell problem.The article contains four sections except introduction, conclusion, and the list of references. Analytical research of the Dzekzer model is given in the first section. The algorithm for the numerical solution of the this model contains in the second section. The ideas of computational implementation describe in the third section. The result of a computational experiment on resolvability of the Cauchy–Wentzell problem in the Dzekzer model are given in the fourth section.

1. Analytical Research of the Dzekzer Model

In this section we recall the main results necessary for further numerical solution of problem (1)–(2). Let us consider the differential operator

$$Au(x) = \Delta u(x), \quad x \in \Omega \tag{3}$$

with Dirichlet boundary condition

$$u(x) = 0. \tag{4}$$

By formulas (3)-(4) we define the linear operator A : dom $A \subset \mathfrak{F} \to \mathfrak{F}$. Here dom $A = \{u \in W_2^2(\Omega)$: conditions (4) is fulfilled$\}$ is the linear manifold as the domain of the operator A, $\mathfrak{F}$ is a Lebesque space $L_2(\Omega)$. Next, in the space $\mathfrak{F}$ let us consider the bilaplacian $A^2u(x) := \Delta^2(x), x \in \Omega$ with the following domain dom $A^2 = \{u \in W_2^4(\Omega) : Au(x) = 0\} \cap \text{dom } A$. It is well known that the embedding dom $A \hookrightarrow \mathfrak{F}$ is densely and compact and the embedding dom $A^2 \hookrightarrow \mathfrak{F}$ is densely and the compact too. Moreover, both spectrum sets are $\sigma(A)$ and $\sigma(A^2)$ are discrete, finite-fold, and have only limit point to ∞.

Let us move on to the Cauchy–Wentzell (1)–(2) problem for the Dzekzer equation. Note that the boundary conditions are chosen in such a way that the bilaplacian A^2 is also essentially a self-adjoint operator in the space $\mathfrak{F}$ with respect to the norm. In order to solve the problem (1)–(2), we find the L-spectrum of the operator M. Since L is the resolvent of the operator M takes the form

$$(\mu L - M)^{-1} = (\mu(\lambda - A) - \alpha A + \beta A^2)^{-1} = \{\mu + \alpha - \beta A \neq 0\} =$$
$$= (\mu + \alpha - \beta A)^{-1}\Big[\frac{\mu\lambda}{\mu + \alpha - \beta A} - A\Big]^{-1},$$

where $\mu + \alpha - \beta A \neq 0$, then μ lies in the relative spectrum of $\sigma^L(M)$ if

$$\mu = \lambda_k \frac{\beta\lambda_k - \alpha}{\lambda_k - \lambda}.$$

Thus, according the spectrum theorem $\sigma(A)$, where λ_k are eigenvalues of the Laplace operator A, $\mu + \alpha - \beta A \neq 0$, we have a discrete, finite multiplicity L-spectrum of the operator M with the unique limit point at $-\infty$. Let us consider the case of $\mu + \alpha - \beta A \neq 0$. For $\lambda = 0$ we have $\sigma^L(M) = \sigma(A)$. For $\lambda \neq 0$ we have $\sigma^L(M) = \{\varnothing\}$ if $\alpha \neq 0$, and $\sigma^L(M) = \{0\}$ if $\alpha = 0$ and $\beta = 0$. We have described the L-spectrum of the operator M, giving the following lemma.

Lemma 1. *L spectrum of the operator M in the Dzekzer equation with the Wentzell boundary condition is a real, discrete, finite multiplicity with the unique limit point at $-\infty$.*

Hence, the following theorem holds.

Theorem 1. *Suppose that the linear operator A satisfies satisfies the defined above conditions, and $f \in \mathfrak{F}$ is a fixed vector. Then*

(i) if $\lambda \notin \sigma(A)$, then for any $f \in C^1((0,\tau);\mathfrak{F}^0) \cap C^0([0,\tau];\mathfrak{F}^1)$ and $u_0 \in \text{dom } A$ there exists the unique solution $u \in C^1((0,\tau);\text{dom } A) \cap C([0,\tau];\text{dom } A)$ to Cauchy–Wentzell problem (1)–(2), which has the following form

$$u(x,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t \frac{\beta\lambda_k - \lambda}{\lambda - \lambda_k}} < u_0, \varphi_k >_{\mathfrak{F}} \varphi_k(x) + \sum_{k=1}^{\infty} \left(1 - e^{-\lambda_k t \frac{\beta\lambda_k - \lambda}{\lambda - \lambda_k}}\right) \frac{< f, \varphi_k >_{\mathfrak{F}}}{\beta\lambda_k^2 - \alpha\lambda_k} \varphi_k(x);$$

(ii) $\lambda \in \sigma(A)$ and condition

$$\left.\begin{array}{c} \textit{the coefficients } \alpha \in \mathbb{R} \textit{ and } \beta \in \mathbb{R}_+ \textit{ are such that} \\ \textit{no eigenvalue } \lambda_k \in \sigma(A) \textit{ is the root of the equation } \beta\xi^2 - \alpha\xi = 0. \end{array}\right\}$$

be satisfied for any $f \in C^1((0,\tau);\mathfrak{F}^0) \cap C^0([0,\tau];\mathfrak{F}^1)$ *and* $u_0 \in \mathrm{dom}\ A$ *such that*

$$\sum_{\lambda_k=\lambda} < u_0, \varphi_k >_{\mathfrak{F}} \varphi_k = \sum_{\lambda_k=\lambda} \frac{< f(0), \varphi_k >_{\mathfrak{F}} \varphi_k}{\alpha\lambda - \beta\lambda^2},$$

there exists the unique solution $u \in C^1((0,\tau);\mathrm{dom}\ A) \cap C^0([0,\tau];\mathrm{dom}\ A)$ *to Cauchy–Wentzell problem* (1)–(2)*, which has the following form*

$$u(t) = {\sum_{k=1}^{\infty}}' e^{t\mu_k} < u_0, \varphi_k >_{\mathfrak{F}} \varphi_k + \sum_{\lambda_k=\lambda} \frac{< f(t), \varphi_k >_{\mathfrak{F}} \varphi_k}{\alpha\lambda - \beta\lambda^2} +$$
$$+ \frac{1}{2\pi i} {\sum_{k=1}^{\infty}}' (\lambda - \lambda_k)^{-1} \int_0^t ds \int_\Gamma \frac{e^{\mu(t-s)} < f(s), \varphi_k >_{\mathfrak{F}} \varphi_k}{\mu - \mu_k} d\mu\ ,$$

where the dash at the sign of the sum means that there are no summands with numbers k *such that* $\lambda = \lambda_k$*.*

2. The Algorithm for the Numerical Solution of the Dzekzer Model

We would like to find an approximate solution using the modify Galerkin method, since the Dzekzer model may be degenerate. Let us construct Galerkin approximations solution of the Cauchy–Wentzell problem in the following form

$$\widetilde{u}(x,t) = u^N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x), \tag{5}$$

where $\{\varphi_k : k \in \mathbb{N}\}$ are eigenfunctions of the Laplace operator A and correspond to its eigenvalues, orthonormal by the norm $< \cdot, \cdot >_{\mathfrak{F}}$,which are numbered in non-increasing order taking into account the multiplicity.

Substitute approximate solution (5) in the equation (2) and multiply by a scalar it to eigenfunctions $\varphi_k(x)$ by the norm $< \cdot, \cdot >_{\mathfrak{F}}$. We obtain the following system

$$\begin{cases} (\lambda - \lambda_1)u_1'(t) = (\alpha\lambda_1 - \beta\lambda_1^2)u_1(t) + f_1(t), \\ (\lambda - \lambda_2)u_2'(t) = (\alpha\lambda_2 - \beta\lambda_2^2)u_2(t) + f_2(t), \\ \dots \\ (\lambda - \lambda_N)u_N'(t) = (\alpha\lambda_N - \beta\lambda_N^2)u_N(t) + f_N(t). \end{cases} \tag{6}$$

Depending on the parameters λ , we have algebraic or first-order differential equations in the system (6). Let us consider these conditions in more details.

(i) $\lambda \notin \sigma(A)$. Due to this fact, the mathematical model is non-degenerate, and all the equations in the resulting system are ordinary differential equations of the first order. For the solvability this system with respect to $u_k(t)$, we multiply by a scalar the initial conditions (1) to eigenfunctions $\varphi_k(x)$ by the norm $< \cdot, \cdot >_{\mathfrak{F}}$. Then, we solve the system (6) with appropriate initial conditions and find the coefficients $u_k(t)$ in the approximate solution $\widetilde{u}(x,t)$.

(ii) $\lambda \in \sigma(A)$. Let without loss of generality $\lambda = \lambda_{m_1} = \cdots = \lambda_{m_r}$, where r is the multiplicity of the root. Then, the part of equations will be algebraic, the other part will be ordinary differential equations of the first order. Let us consider separately systems composed of algebraic equations and differential equations of the first order. Note that the solution of the original problem exists, according to the Theorem 1, if the initial function $v_0(x)$ belongs to the phase space

$$\mathfrak{B}_f = \left\{ u \in \text{dom } A : u = u^0 + u^1, \ < u^1, \varphi_k >_{\mathfrak{F}} = 0, u^0 = \sum_{\lambda_k = \lambda} \frac{< f, \varphi_k >_{\mathfrak{F}} \varphi_k(x)}{\alpha\lambda - \beta\lambda^2} \right\}$$

3. The Computational Implementation for the Numerical Solution of the Dzekzer Model

Since the Galerkin method is not of most interest, we describe the main ideas, in the author' view, associated with implementing a numerical solution.The full algorithm is shown in the Figure 1.

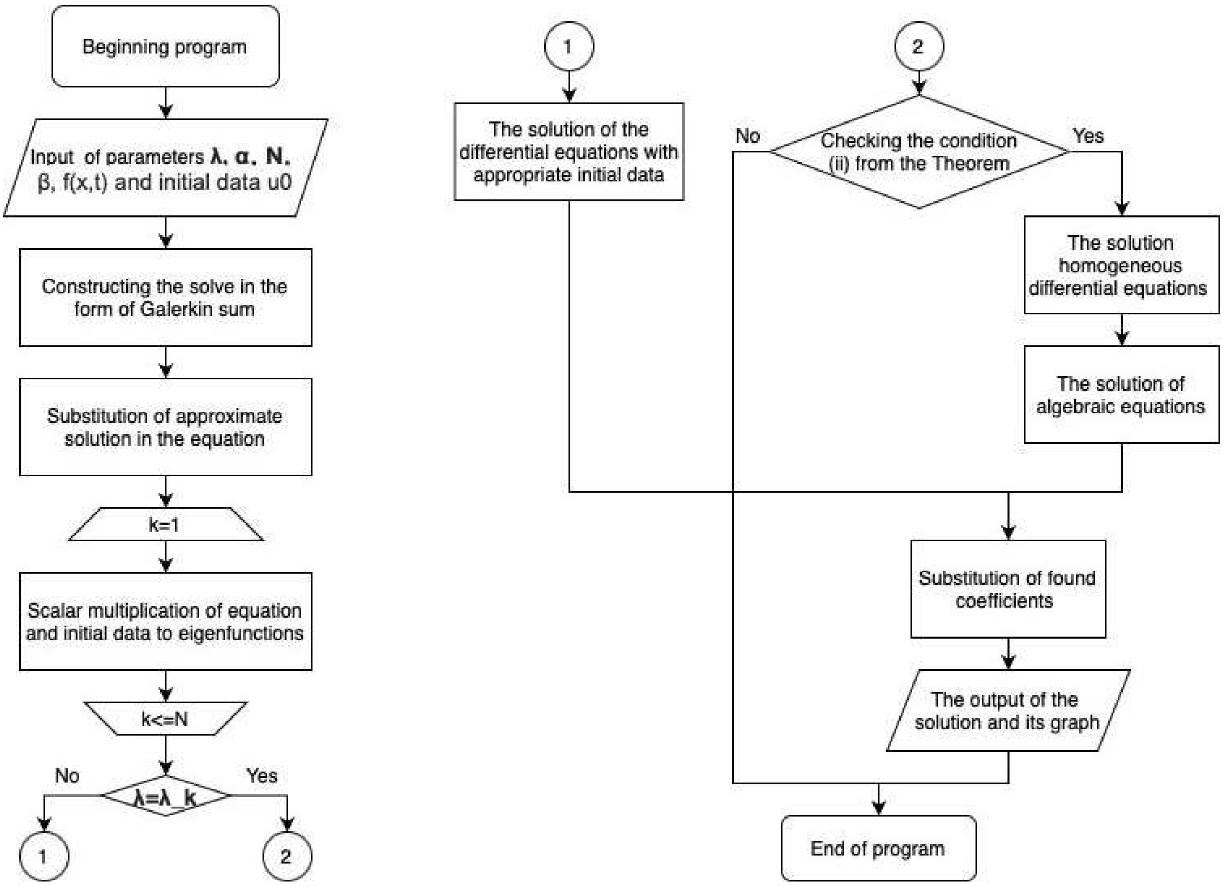

Fig. 1. The block diagram of algorithm

Remark 1. Let's consider an approach that allows us to find the eigenvalues of the Laplace operator in a cube $[0, \pi] \times [0, \pi] \times [0, \pi]$ a. The implementation is present below.

```
import math
from math import cos, sin
from mpmath import *
from sympy import *

N=n
for k in range(N):
    for m in range(N:
        for n in range(N:
            print(math.sin(k*1)*math.sin(m*1)*math.sin(n*1))
            print(k,m,n)
            print('lambda', k**2+m**2+n**2)

print (v)
```

Remark 2. Let's consider an approach that allows us to find the eigenfunctions of the Laplace operator in a cube $[0, \pi] \times [0, \pi] \times [0, \pi]$ a. The implementation is present below.

```
from scipy import integrate
import numpy as np

f = lambda x, y, z : 2*math.sin(x)*math.sin(3*y)*math.sin(z)*math.exp(math.sin(x))
g = lambda x : 0
h = lambda x : math.pi
q = lambda x, y : 0
r = lambda x, y : math.pi
v, err = integrate.tplquad(f, 0, math.pi, g, h, q, r)
```

4. The Result of a Computational Experiment of the Cauchy–Wentzell Problem in the Dzekzer Model

Example. Let us consider the Cauchy–Wentzell problem

$$\begin{aligned} &u(x,0) = 2e^{sin(x)},\\ &\Delta u(x,y,z) = 0,\ (x,y,z,t) \in \partial\Omega \times \mathbb{R}\\ &u(x,y,z,t) = 0,\ (x,y,z,t) \in \partial\Omega \times \mathbb{R}. \end{aligned}$$

for the equation

$$(\lambda - \Delta)u_t(x,y,z,t) = \alpha\Delta u(x,y,z,t) - \beta\Delta^2 u(x,y,z,t) + f(x,y,z,t), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{7}$$

where $\lambda = 1$, $\alpha = 1$, $\beta = 1$, $\Omega = \{(x,y,z) : [0,\pi] \times [0,\pi] \times [0,\pi]\}$, $f(x,y,z,t) = sin(x) + cos(x)$

Let $N = 7$, then the approximate solution have the following form

$$\widetilde{u}(x,t) = u^7(x,t) = \sum_{k=1}^{7} u_k(t)\varphi_k(x). \tag{8}$$

The eigenfunctions φ_{kmn} of the homogeneous Dirichlet problem for the Laplace operator in the cube $[0,\pi] \times [0,\pi] \times [0,\pi]$ are $\sin(kx)\sin(my)\sin(nz)$. Find and write down the eigenvalues and the eigenfunctions of the Laplace operator.

We have eigenvalues $\lambda_{131} = -11$, $\lambda_{111} = -3$, $\lambda_{311} = -11$, $\lambda_{113} = -11$, $\lambda_{513} = -35$, $\lambda_{153} = -35$, $\lambda_{531} = -35$.

Substitute approximate solution (8) in the equation (7) and multiply by a scalar it to eigenfunctions $\varphi_k(x)$ by the norm $< \cdot, \cdot >_{\mathfrak{F}}$. We obtain the following system

$$\begin{cases} 12u_1'(t) + 132u_1(t) + 2.094395 = 0, \\ 4u_2'(t) + 12u_2(t) + 39.478417 = 0, \\ 12u_3'(t) + 132u_3(t) = 0, \\ 12u_4'(t) + 132u_4(t) + 2.09439 = 0, \\ 36u_5'(t) + 1260u_5(t) = 0, \\ 36u_6'(t) + 1260u_6(t) + 0.418879 = 0, \\ 36u_7'(t) + 1260u_6(t) = 0. \end{cases} \tag{9}$$

Due to the fact that $\lambda \notin \sigma(A)$, the mathematical model is non-degenerate, and, according to the algorithm, all the equations in the resulting system are ordinary differential equations of the first order. Let us solve the system (9) with initial conditions

$$\begin{aligned} u_1(0) &= 3.4594415, \\ u_2(0) &= 5.9919285, \\ u_3(0) &= 1.884511856, \\ u_4(0) &= 3.4594415, \\ u_5(0) &= 1.01085648, \\ u_6(0) &= 1.54710, \\ u_7(0) &= 1.0120856. \end{aligned}$$

and find Galerkin coefficients

$$\begin{aligned} u_1(t) &= 3.44357 \cdot e^{-11t} + 0.0158666, \\ u_2(t) &= 2.702060 \cdot e^{-3t} + 3.299868, \\ u_3(t) &= 1.884511856 \cdot e^{-11t}, \\ u_4(t) &= 3.4436 \cdot e^{-11t} + 0.015871, \\ u_5(t) &= 1.0120856 \cdot e^{-35t}, \\ u_6(t) &= 1.5468 \cdot e^{-35t} + 0.00033244, \\ u_7(t) &= 1.0120856 \cdot e^{-35t}. \end{aligned}$$

Substituting them in the representation we obtain an approximate solution to the original problem. The graph of the solution is shown in the Figure 2.

Conclusion

We constructed an algorithm and implementation for the numerical solution of the Cauchy–Wentzell problem in the cube $[0, \pi] \times [0, \pi] \times [0, \pi]$ for Dzekzer model. To this end, we used the numerical methods theory. Further, we plan to continue the results of the paper by applying the Wentzell boundary conditions in directions related to [5].

Acknowledgments: The research was partially funded by RFBR and Chelyabinsk Region, project number 20-41-740010.

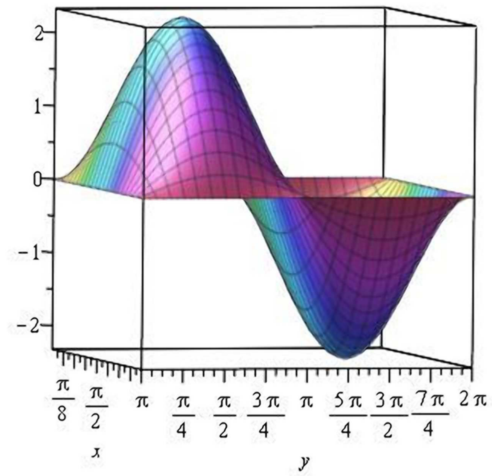

Fig. 2. The solution to problem in Example

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Received September 10, 2021

УДК 519.63 **DOI: 10.14529/jcem210404**

ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ КОШИ-ВЕНТЦЕЛЯ ДЛЯ МОДЕЛИ ДЗЕКЦЕРА В ОГРАНИЧЕННОЙ ОБЛАСТИ

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В терминах теории p-секториального оператора исследуется задача Коши для уравнения Дзекцера, описывающего эволюцию свободной поверхности фильтрующейся жидкости с чистыми граничными условиями Вентцеля. В частности, рассматривается относительный спектр в уравнении Дзекцера и строится разрешающая голоморфная полугруппа операторов в задаче Коши-Вентцеля. В статье эти проблемы решаются в предположении, что начальное пространство, в котором оператор Лапласа действует в ограниченной области, является пространством Лебега $L^2(\Omega)$. Целью данной работы является описание нового подхода к разрешимости этой задачи с граничными условиями Вентцеля, а именно, в соответствии с модифицированным методом Галеркина, описывается решение задачи Коши-Вентцеля.

Ключевые слова: уравнение Дзекцера; задача Коши–Вентцеля; метод Галеркина; численное моделирование.

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Поступила в редакцию 10 сентября 2021 г.