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INVESTIGATION OF VARIOUS TYPES OF CONTROL PROBLEMS FOR ONE NONLINEAR MODEL OF FILTRATION

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A wide class of mathematical physics problems can be considered within the framework of semi-linear Sobolev type equations, which describe various processes (for example deformation processes, processes occurring in semiconductors, processes of oscillatory motion propagation in various media, and so on). The article is devoted to the study of control problems (optimal, start and rigid) of one mathematical model of Sobolev type, which is based on the equation describing the process of changing the concentration potential of a viscoelastic fluid filtered in a porous medium (the process of nonlinear diffusion of matter). We find the sufficient conditions under which there exists a solution to the control problem of the model under study. An algorithm for the numerical solution method is constructed and a computational experiment is presented.

Keywords: Sobolev type equations; nonlinear diffusion model; start control problem; optimal control problem; the problem of rigid control; mathematical modeling; projection method; decomposition method.

Introduction

Now there is a wealth of research experience devoted to nonlinear models of mathematical physics. In the study of such models, one can especially distinguish a class of models based on Sobolev type equations. A feature of these equations is, first of all, the possibility of degeneration of the basic equation for certain values of the parameters of the problem. This feature entails certain difficulties in the study of problems associated with these equations. Before proceeding to the construction of methods for their study, it is necessary to study the structure of the phase manifold and find conditions for the existence of solutions to initial-boundary value problems. In the 1990s G.A. Sviridyuk developed a method for studying degenerate equations, which was named the method of the phase space. This tools have opened to solving problems that had not been resolved in connection with a possible degeneration of the basic equation and should undertake a systematic study of previously known models. At the moment, in the theory of Sobolev type equations, there are many models that describe various processes (for example, deformation processes [1, 2], processes occurring in semiconductors [4], processes of oscillatory motion propagation in various media [5] and so on.

As classical works in the theory of optimal control, we note works written by J.-L. Lions. For example, the work [6] systematically studies optimal control problems for partial differential equations. Optimal control problems for linear [7, 8] and nonlinear [3] Sobolev type equations were widely studied. Namely, the optimal control problems for linear equations of Sobolev type were first studied by G.A. Sviridyuk and A.A. Efremov [7]. The review [9] is devoted to description mathematical model of the optimal dynamic measurement. In this article consider various types of control problems (such as optimal control, start control, and rigid control) for on the nonlinear diffusion model.

Consider the domain $\Omega \subset \mathbb{R}^n$ is bounded and has a smooth boundary for ease of consideration $\partial\Omega$ class C^∞. In $\Omega \times \mathbb{R}_+$. Consider the initial-boundary value problem

$$(\lambda - \Delta)(x(s,0) - x_0(s)) = 0,\ s \in \Omega, \tag{1}$$

$$x(s,t) = 0,\ (s,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2}$$

for partial differential equation

$$(\lambda - \Delta)x_t = \operatorname{div}(|\nabla x|^{p-2}\nabla x) + u. \tag{3}$$

The problem (1)–(3) describes the process of changing the concentration potential of a viscoelastic fluid filtering in a porous medium (the process of nonlinear diffusion of a substance) [10] and forms a mathematical model of nonlinear diffusion. The parameter $\lambda \in \mathbb{R}$ characterizes the viscosity of the liquid, and it was experimentally confirmed that the negative value of the parameter λ does not contradict the physical meaning of the model. Right side $u = u(s,t)$ of the equation (3) describes a given external influence. For the first time the question of existence and uniqueness of the solution to the problem (1)–(3) was considered by G.A. Sviridyuk [5]. He obtained conditions for the existence and uniqueness of a solution in the case of a local classical and weakly generalized solution in the case of non-negative definiteness of the operator at the time derivative (degenerate case). In case $\lambda \in \mathbb{R}_+$ (non-degenerate case) this problem was investigated by Liu Changchung. He obtained conditions for the existence of a weak solution [11], and revealed the asymptotic properties of the solution [12].

Let $\mathfrak{X}, \mathfrak{Y}$ is functional space, then, due to the possible degeneracy (1)–(3) mathematical model can be reduced to an abstract problem

$$L(x(0) - x_0) = 0 \tag{4}$$

for the semilinear operator equation

$$L\,\dot{x} + M(x) = u \tag{5}$$

with s-monotone and p-coercive operator M (the Showalter–Sidorov problem for an equation of Sobolev type). Consider control problems

$$J(x,u) \to \inf \tag{6}$$

solutions (1)–(3) in the weak generalized case. Here $J(x,u)$ is target functional constructed in different ways depending on the considered control problem; vector $u \in \mathfrak{U}_{ad}$ in case $\mathfrak{U}_{ad}$ is a given closed and convex subset in $\mathfrak{U}$.

1. Mathematical Model

We reduce the problem (1), (2) for

$$(\lambda - \Delta)x_t - \operatorname{div}(|\nabla x|^{p-2}\nabla x) = y \tag{7}$$

to an abstract problem. Next, we consider the function spaces $\mathcal{H} = L_2(\Omega)$, $\mathfrak{H} = \overset{0}{W}{}_2^1(\Omega)$, $\mathfrak{B} = \overset{0}{W}{}_p^1(\Omega)$, $\mathfrak{H}^* = W_2^{-1}(\Omega)$, $\mathfrak{B}^* = W_q^{-1}(\Omega)$. By construction space $\mathcal{H}$ is Hilbert if we introduce in it the classical scalar product

$$\langle x, y\rangle = \int\limits_{\Omega} xyds\, \forall x, y \in \mathcal{H}.$$

Due to the choice of function spaces $\mathfrak{H}$, $\mathfrak{B}$ and constructed conjugate spaces $\mathfrak{H}^*$, $\mathfrak{B}^*$ exists a dense and continuous embedding

$$\mathfrak{B} \hookrightarrow \mathfrak{H} \hookrightarrow \mathcal{H} \hookrightarrow \mathfrak{H}^* \hookrightarrow \mathfrak{B}^*.$$

We define the operators L and M:

$$\langle Lx, y\rangle = \int\limits_{\Omega} (\nabla x \cdot \nabla y + \lambda xy)ds, \quad x,\, y \in \mathfrak{H},$$

$$\langle M(x), y\rangle = \int\limits_{\Omega} |\nabla x|^{p-2}\nabla x \cdot \nabla yds, \quad x,\, y \in \mathfrak{B}.$$

Further, in order to prove the existence of a solution, we need to determine the properties of the operators L and M. Let $\{\varphi_k\}$ is the sequence of eigenfunctions of the operator $(-\Delta)$ in the Sturm–Liouville problem with the homogeneous Dirichlet condition corresponding to the sequence $\{\lambda_k\}$. The properties of the operator L are described in the following lemma:

Lemma 1. [2] *(i) For all $\lambda \geq -\lambda_1$ the operator $L \in \mathcal{L}(\mathfrak{H}; \mathfrak{H}^*)$ is self-adjoint. In case $\lambda > -\lambda_1$ positive defined, and in case $\lambda = -\lambda_1$ a degenerate operator, and also has the Fredholm property.*

(ii) The operator $M \in C^1(\mathfrak{B}; \mathfrak{B}^*)$ is s-monotonous and p-coercive.

Consider

$$\ker L = \begin{cases} \{0\},\ \lambda > -\lambda_1, \\ \operatorname{span}\{\varphi_1\},\ \lambda = -\lambda_1 \end{cases}$$

and

$$\operatorname{coim} L = \begin{cases} \mathfrak{H}^*,\ \lambda > -\lambda_1, \\ \{x \in \mathfrak{H}^* : \langle x, \varphi_1\rangle = 0\},\ \lambda = -\lambda_1 \end{cases}$$

for operator L. Then there is a projector Q defined as follows

$$Q = \begin{cases} \mathbb{I},\ \lambda > -\lambda_1, \\ \mathbb{I} - \langle \cdot, \varphi_1\rangle,\ \lambda = -\lambda_1. \end{cases}$$

Following the phase space method, it is necessary to construct a phase manifold. Then the phase manifold (7) in operator form

$$\mathfrak{M} = \begin{cases} \mathfrak{B},\ \lambda > -\lambda_1, \\ \{x \in \mathfrak{B} : \langle M(x), \varphi_1\rangle = \langle y, \varphi_1\rangle\},\ \lambda = -\lambda_1 \end{cases}$$

or

$$\int_\Omega \operatorname{div}(|x|^{p-2}x)\varphi_1\, ds = \int_\Omega y\varphi_1\, ds.$$

By virtue of the obtained properties of the operators L, M, namely, such as the nonnegative definiteness, the Fredholm property of the operator L and the s-monotone, the p-coercive of the nonlinear operator M implies the simplicity of the phase manifold. If the phase manifold of the equation under study with the boundary condition is a simple manifold, then this fact implies the uniqueness of the solution to the initial problem.

Let the right side of the equation in (7) have the property

$$\int_\Omega y\varphi_1\, ds \tag{8}$$

does not depend on t. Then the following theorem is true.

Theorem 1. *For all fixed values of the parameter $\lambda \geq -\lambda_1$ and if the condition* (8) *is satisfied, then the set $\mathfrak{M}$ is a simple Banach C^1-manifold modeled by the subspace* $\operatorname{coim} L \cap \mathfrak{B}$.

Proof of the Theorem 1 is a consequence of the abstract case [2] for a semilinear equation with s-monotone and p-coercive operators.

Theorem 2. *For all fixed values of the parameter $\lambda \geq -\lambda_1$ and for all $x_0 \in \mathfrak{B}$, $T \in \mathbb{R}_+$, $y \in L_q(0,T;\mathfrak{B}^*)$ exists weak generalized solution there exists a unique weak generalized solution $x \in L_\infty(0,T;\operatorname{coim} L) \cap L_p(0,T;\mathfrak{B})$ problem* (1), (2), (7).

Proof of the existence of a weak generalized solution is based on the solvability of the Showalter–Sidorov problem for a semilinear operator equation with an s-monotone and p-coercive operator. Following the phase space method, we introduce in the subspace $\operatorname{coim} L$ the equivalent norm $|x|^2 = \langle Lx, x\rangle$. We construct a priori estimates for the solution of the problem, for this we scalarly multiply equation (7) on x and integrate the resulting result over $(0,t)$, and

$$\int_0^t \int_\Omega ((\lambda - \Delta)(x_\tau)x + |\nabla x|^{p-2}(\nabla x)^2) ds d\tau = \int_0^t \int_\Omega yx ds d\tau$$

$$\int_0^t \int_\Omega (\lambda \nabla(x_\tau)\nabla x + x_\tau x + |\nabla x|^p) ds d\tau \leq \int_0^t \|x\|_{\mathfrak{B}} \|y\|_{\mathfrak{B}^*} d\tau$$

$$\int_0^t \int_\Omega (\frac{1}{2}\frac{d}{d\tau}(\lambda(\nabla x)^2 + x^2) + |\nabla x|^p) ds d\tau \leq \int_0^t \|x\|_{\mathfrak{B}} \|y\|_{\mathfrak{B}^*} d\tau$$

$$\begin{gathered} C_3|x|_{L_\infty(0,t;\operatorname{coim}\ L)} + C_4\|x\|^p_{L_p(0,t;\mathfrak{B})} \leq \\ C_1\|y\|_{L_q(0,T;\mathfrak{B}^*)} + C_2|x(0)|^2_{L_\infty(0,t;\operatorname{coim}\ L)}, \quad C_i > 0,\ i = 1,...,4. \end{gathered} \tag{9}$$

□

2. Investigation of the Problems of Start Control

Let's move on to the consideration of the start control problem and the final observation or the start control problem for (7). On the lateral surface of the cylinder Q_T and at the initial moment of time, instead of the condition (1), consider the modified Showalter–Sidorov condition

$$(\lambda - \Delta)(x(s,0) - u(s)) = 0,\ s \in \Omega, \tag{10}$$

and the Dirichlet condition (2).

As the control space, we take the space $\mathfrak{U} = \mathfrak{B}$ and fix in it a non-empty, closed, and convex subset $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Let's build the target functional and consider

– the problem of start control and final observation of solutions (2), (7), (10) and

$$J(x(T), u) = \vartheta\|x(T) - x_f\|^p_{\mathfrak{B}} + (1-\vartheta)\|u\|^p_{\mathfrak{B}} \to \inf, \quad \vartheta \in (0,1), \tag{11}$$

$x_f = x_f(s)$ is a fix state of the system, which must be achieved with the minimum initial action u after time $t = T$;

– the problem of start control of solutions (2), (7), (10) and

$$J(x, u) = \vartheta\|x(s,t) - x_f(s,t)\|^p_{L_p(0,T;\mathfrak{B})} + (1-\vartheta)\|u\|^p_{\mathfrak{B}} \to \inf, \quad \vartheta \in (0,1), \tag{12}$$

$x_f = x_f(s,t)$ is fix state of the system to be achieved with minimal initial impact u.

Definition 1. *The pair* $(\hat{x}(T),\ \hat{u}) \in \mathfrak{B} \times \mathfrak{U}_{ad}$ *called a solution* (2), (7), (10), (11), *if*

$$J(\hat{x}(T), \hat{u}) = \inf_{(x(T),u)} J(x(T), u),$$

provided that $(x, u) \in [L_\infty(0, T; \operatorname{coim} L) \cap L_p(0, T; \mathfrak{B})] \times \mathfrak{U}_{ad}$ *and* x *is weak generalized solution problem* (2), (7), (10). *Moreover, the vector function* $\hat{u}$ *called start control* (2), (7), (10), (11).

The solution to the problem (2), (7), (10), (12) is defined similarly.

Definition 2. *The pair* $(\hat{x}(T), \hat{u})$ *is admissible for problem* (2), (7), (10), (11), *if* $(x(T), u) \in \mathfrak{M} \times \mathfrak{U}_{ad}$ *and* $J(x(T), u) < +\infty$.

The admissible pair for the problem (2), (7), (10), (12) is defined similarly.

Remark 1. If the set is nonempty $\mathfrak{U}_{ad} \neq \emptyset$, for all fix $u \in \mathfrak{U}_{ad}$ by virtue of the existence Theorem, there is a unique solution $x = x(y, u)$ for problem (2), (7), (10). Therefore, the set of admissible elements of a control problem is not empty.

Theorem 3. *For all fix* $\lambda \geq -\lambda_1$, $T \in \mathbb{R}_+$, $y \in L_q(0, T, \mathfrak{B}^*)$ *exist solution* $(\hat{x}(T), \hat{u})$ *problem* (2), (7), (10), (11).

Proof. It follows from the existence theorem that the operator

$$(L\frac{d}{dt} + M) : [L_\infty(0, T; \operatorname{coim} L) \cap L_p(0, T; \mathfrak{B})] \to L_q(0, T, \mathfrak{H}^*)$$

is a homeomorphism. Then the functional (11) can be regarded as a functional that does not depend on the pair $(\hat{x}(T, s), \hat{u}(s))$, but only from $\hat{u}(s)$, that is

$$J(x(T), u) = J(u).$$

Let $\{u_m\} \subset \mathfrak{U}_{ad}$ minimizing sequence, that is

$$\lim_{m\to\infty} J(u_m) = \inf_{u\in\mathfrak{U}_{ad}} J(u).$$

Due to the existence of a limit, the sequence is bounded, which, due to the structure of the functional, entails the boundedness

$$\|u_m\|_{\mathfrak{B}} \leq \text{const}, \ \forall m \in \mathbb{N}.$$

Thus, due to the reflexivity of the constructed function spaces and the weak sequential closedness of the unit ball, from a converging sequence, one can choose a converging subsequence From the estimates above (possibly, passing to a sequence), we select weakly converging sequences $u_m \rightharpoonup \hat{u}$ in $\mathfrak{B}$. According to the Mazur theorem, the point $\hat{u} \in \mathfrak{U}_{ad}$. Let $x_m = x(u_m)$ weak generalized solution to the problem

$$x_m(s,t) = 0, \ (s,t) \in \partial\Omega \times (0,T), \tag{13}$$

$$(\lambda - \Delta)(x_m(s,0) - u(s)) = 0, \ s \in \Omega, \tag{14}$$

for

$$(\lambda - \Delta)x_{mt} - \operatorname{div}(|\nabla x_m|^{p-2}\nabla x_m) = y \tag{15}$$

Due to the reflexivity of space $L_p(0,T,\mathfrak{B})$ and inequality (9) exist weak limits

$$x_m \rightharpoonup \hat{x} \ *\text{-weakly in } L_\infty(0,T;\operatorname{coim} L);$$

$$x_m \rightharpoonup \hat{x} \text{ weakly in } L_p(0,T;\mathfrak{B}).$$

Due to the p-coercivity of the operator M, we obtain

$$\int_0^T \langle M(x_m), x_m\rangle\, d\tau \leq \int_0^T \|M(x_m)\|_{\mathfrak{B}^*}\|x_m\|_{\mathfrak{B}}\, d\tau \leq C^M \int_0^T \|x_m\|_{\mathfrak{B}}^p \|x_m\|_{\mathfrak{B}}\, d\tau,$$

and therefore $M(x_m)$ limited in $L_q(0,T;\mathfrak{B}^*)$. Due to the reflexivity of space $L_q(0,T;\mathfrak{B}^*)$

$$M(x_m) \rightharpoonup \mu \text{ weakly in } L_q(0,T;\mathfrak{B}^*).$$

Due to the compact embedding $\mathfrak{B} \hookrightarrow \mathcal{H}$ subsequence $M(x_m) \to \mu$ in $L_2(0,T;\mathcal{H})$, then

$$\mu = M(x_m).$$

Hence, passing to the limit in the equation state and initial condition, we get

$$L\frac{d\hat{x}}{dt} + M(\hat{x}) = y, \qquad L(\hat{x}(0) - \hat{u}) = 0.$$

Then $\hat{x} = \hat{x}(\hat{u})$ and $\liminf J(u_m) \geq J(\hat{u})$. So $\hat{u}$ is a start control.

□

The existence of a solution to the start control problem is proved in a similar way:

Theorem 4. *For all fix parameter values $\lambda \geq -\lambda_1$, $T \in \mathbb{R}_+$, $y \in L_q(0,T,\mathfrak{B}^*)$ exist solution $(\hat{x},\hat{u})$ problem* (2), (7), (10), (12).

3. Optimal Control Problem

Consider the space $\mathfrak{U} = L_q(0,T;\mathfrak{B}^*)$ and fix in it a non-empty, closed, and convex subset $\mathfrak{U}_{ad} \subset \mathfrak{U}$. Consider optimal control problems

$$J(x,u) = \vartheta \|x - x_f\|^p_{L_p(0,T;\mathfrak{B})} + (1-\vartheta)\|u - u_f\|^p_{\mathfrak{U}} \to \inf, \quad \vartheta \in (0,1), \tag{16}$$

for initial-boundary value problem (1), (2) for partial differential equation (3). Here $x_f = x_f(s,t)$ is fix system state, $u_f = u_f(s,t)$ is a fix state of external influence on the system. The problem is to find such a distribution of the state so that the external influence is the closest to the given one with the best achievement of the required state.

Remark 2. If the set is nonempty $\mathfrak{U}_{ad} \neq \emptyset$, for all $u \in \mathfrak{U}_{ad}$ by virtue of the existence Theorem, there is a unique solution $x = x(y,u)$ problem (1)–(3), (16). From which it follows that the set of admissible elements of the control problem is not empty.

Problem (1)–(3), (16) belongs to the class of problems with s-monotone and p-coercive operators. The proof of the Theorem is based on reducing the problem to an abstract setting, on the monotonicity method, Mazur's theorem, and weak limit transitions. The existence of a solution to problem (1)–(3), (16) is based on the same methods as in the proof of the previous Theorem.

Theorem 5. *For all fix parameter values $\lambda \geq -\lambda_1$, $T \in \mathbb{R}_+$ there is a solution $(\hat{x},\hat{u})$ problem* (1)*–*(3)*,* (16)*.*

4. Algorithm of a Numerical Method to Find Solution for Control Problems

Based on the existence and uniqueness Theorem for the solution of the Showalter–Sidorov problem, the convergence of a sequence of approximate solutions to the exact solution, existence Theorems for the solution of the control problems under study, we construct a numerical method algorithm based on decomposition and penalty methods for the nonlinear diffusion model. In the final qualifying work, three main types of control problem are investigated:

– the problem of optimal control,
– the problem of rigid control,
– the problem of start control.

A distinctive feature of the studied problems is the type of control action. In the first and second cases, control is characterized by an external action $u = u(s,t)$ and enters the right side of the system as an unknown external action. In the third case, the control $u = u(s)$ corresponds to the form of the state of the system at the initial moment of time and enters the initial condition of the problem as an unknown state. Therefore, it is necessary to modify the existing algorithms.

We denote by σ the spectrum of the operator $(-\Delta)$ with the homogeneous Dirichlet condition, by $\{\lambda_k\}$ the set of eigenvalues, and $\{\varphi_k\}$ is a family of corresponding eigenfunctions orthonormal with respect to the scalar product $< \cdot,\cdot >$ from $L_2(\Omega)$. We find a numerical solution to the problem in the form of Galerkin approximations

$$\tilde{x}(s,t)=\sum_{k=1}^{K}x_k(t)\varphi_k(s),$$

where $K\in\mathbb{N}$, need to $K>1$, since the first equation of the system can turn out to be algebraic. Then

– in the case of study the problem of optimal or control, the right side of equation is an unknown function, to form a system of differential or algebraic differential equations, we represent the right side of equation (control) (3) in the form

$$\tilde{u}(s,t)=\sum_{k=1}^{K}u_k(t)\varphi_k(s),$$

– in the case of study the problem of start control, the right side of the equation is a known function; form a system of differential or algebraic-differential equations, we represent the right side of equation (7) in the form

$$\tilde{y}(s,t)=\sum_{k=1}^{K}y_k(t)\varphi_k(s).$$

The initial state of the system from condition (10) function $u(s)$ can also be represented in the form

$$\tilde{u}=\sum_{k=1}^{K}u_k\varphi_k(s).$$

We apply the decomposition method and linearize the original equation (3) or (7) and

– in the case of study the problem of optimal or boundary control, we introduce into the equation (3) unknown function $f=f(s,t)$ such that

$$(\lambda-\Delta)x_t-\operatorname{div}(|\nabla f|^{p-2}\nabla f)=u,\quad x(s,t)=f(s,t),\tag{17}$$

– in the case of study the start control problem, we introduce into equation (7), the unknown function $f=f(s,t)$ such that

$$(\lambda-\Delta)x_t-\operatorname{div}(|\nabla f|^{p-2}\nabla f)=y,\quad x(s,t)=f(s,t).\tag{18}$$

We represent the unknown function $f(s,t)$ in form

$$\tilde{f}(s,t)=\sum_{k=1}^{K}f_k(t)\varphi_k(s).$$

By virtue of the second equation of the system in (17) or (18) target functional

– in an optimal control problem with a compromise functional is equivalent to

$$J(x,u)=\theta\cdot\beta\int_0^T dt\int_\Omega|\nabla x(s,t)-\nabla x_d(s,t)|^p\,ds+$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega|\nabla f(s,t)-\nabla x_d(s,t)|^p\,ds+$$
$$+(1-\theta)\int_0^T dt\int_\Omega|\nabla u(s,t)-\nabla u_d(s,t)|^p\,ds,$$

– in the rigid control problem is equivalent to

$$J(x,u) = \theta\cdot\beta\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^p\,ds +$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega |\nabla f(s,t) - \nabla x_d(s,t)|^p\,ds,$$

– in the problem of start control is equivalent to

$$J(x,u) = \theta\cdot\beta\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^p\,ds +$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega |\nabla f(s,t) - \nabla x_d(s,t)|^p\,ds + (1-\theta)\int_\Omega |\nabla u(s)|^p\,ds,$$

– in the problem of start control and final observation is equivalent to

$$J(x,u) = \theta\cdot\beta\int_\Omega |\nabla x(s,T) - \nabla x_d(s)|^p\,ds + \theta\cdot\beta\int_\Omega |\nabla f(s,T) - \nabla x_d(s)|^p\,ds +$$
$$+(1-\theta)\int_\Omega |\nabla u(s)|^p\,ds.$$

To find an approximate solution, we will use the penalty method. Let's move on from considering the original functional of the problem to the functional of the penalty

– in the optimal control problem with a compromise target functional

$$J_\varepsilon(x,f,u) = \theta\cdot\beta\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^p\,ds +$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega |\nabla f(s,t) - \nabla f(s,t)|^p\,ds +$$
$$+(1-\theta)\int_0^T dt\int_\Omega |\nabla u(s,t) - \nabla u_d(s,t)|^p\,ds +$$
$$+r_\varepsilon\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^2\,ds;$$

– in the problem of rigid control

$$J_\varepsilon(x,f,u) = \theta\cdot\beta\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^p\,ds +$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega |\nabla f(s,t) - \nabla x_d(s,t)|^p\,ds + r_\varepsilon\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla f(s,t)|^2\,ds;$$

– in the start control problem

$$J_\varepsilon(x,f,u) = \theta\cdot\beta\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla x_d(s,t)|^p\,ds +$$
$$+\theta\cdot\beta\int_0^T dt\int_\Omega |\nabla f(s,t) - \nabla x_d(s,t)|^p\,ds + (1-\theta)\int_\Omega |\nabla u(s)|^p\,ds +$$
$$+r_\varepsilon\int_0^T dt\int_\Omega |\nabla x(s,t) - \nabla f(s,t)|^2\,ds;$$

– in the problem of start control and final observation

$$J_\varepsilon(x,f,u)=\theta\cdot\beta\int\limits_\Omega|\nabla x(s,T)-\nabla x_d(s)|^p\,ds+\theta\cdot\beta\int\limits_\Omega|\nabla f(s,T)-\nabla x_d(s)|^p\,ds+$$
$$+(1-\theta)\int\limits_\Omega|\nabla u(s)|^p\,ds+r_\varepsilon\int\limits_0^T dt\int\limits_\Omega|\nabla x_s(s,t)-\nabla f(t,s)|^2\,ds,$$

where parameter $r_\varepsilon\to+\infty$ при $\varepsilon\to 0+$. The last term directs the approximate functions $\tilde{x}$ and $\tilde{f}$ to each other. Thus, finding a solution to the starting control problem is reduced to finding a triplet of minimizing functions $(\tilde{x}(T),\tilde{f}(T),u)$.

5. Program Description

On the basis of the algorithm described in the previous paragraph, a complex of programs is built, consisting of 5 modules. Each module finds an approximate solution to the corresponding control problem. The program is written in the Maple language and implemented in the Maple 17 computer mathematics system.

The logical structure of each program module
The program includes the following steps:
– data input;
– decomposition of the original equation;
– formation of an approximate solution;
– building a quality functional;
– minimization of the quality functional;
– finding an approximate solution;
– solution output and plotting.

Let's describe each of the steps:
Step 1. Entering the parameters of the equation, the number of terms in the approximate solution, eigenfunctions and eigenvalues of the homogeneous Dirichlet problem for the operator $(-\Delta)$, the length of the segment.
Step 2. Formation of an approximate solution and control in the form of Galerkin approximate is carried out using the loop ≪**for()**≫ from 1 to K.
Step 3. Substitute the expansion of the solution into the equation using the procedure **subs**.
Step 4. Multiplying the resulting solution by the eigenfunctions $\varphi_k(s)$ we generate a system of equations. We make a check for possible degeneracy of the first equation of the system based on the procedure **"if ()"**.
Step 5. Find the solution of the system of differential algebraic equations with initial conditions using the procedure ≪**dsolve()**≫ with respect to the unknowns $x_k(t)$.
Step 6. Generate the quality functional using the procedure ≪**subs()**≫. Unknown controls are approximated by polynomials.
Step 7. Using the ≪**Optimization**≫ package and the ≪**NPLSolve()**≫ procedure, we find the minimum of the functional, then we compose the solution to the problem.
Step 8. The resulting solution is displayed and a graph is built.

Let's consider the results of the program using the following examples.

6. Computational Experiments

It is required to find a solution to the control problems considered in the previous sections for a mathematical model of nonlinear diffusion, which in case $n = 1$ and domain $\Omega = (0, \pi)$ take the form

$$x(0,t) = x(\pi,t) = 0,\ t \in (0,T), \tag{19}$$

for

$$\lambda x_t - x_{sst} - (|x_s|^{p-2}x_s)_s = u \tag{20}$$

or

$$\lambda x_t - x_{sst} - (|x_s|^{p-2}x_s)_s = u. \tag{21}$$

The initial condition for the optimal control problem be considered in the form

$$\lambda(x(s,0) - x_0(s)) = x_{ss}(s,0) - x_{0ss}(s),\ s \in (0,\pi),$$

where the initial distribution of the process $x_0(s)$ is known, or in the form

$$\lambda(x(s,0) - u(s)) = x_{ss}(s,0) - u_{ss}(s),\ s \in (0,\pi),$$

for problems of start control, where the initial distribution of the process $u(s)$ is to be found.

To linearize the equation, following the proposed decomposition method. We introduce into (21) the required function $f = f(s,t)$:

$$\lambda x_t - x_{sst} - (|f_s|^{p-2}f_s)_s = y,\ x(s,t) = f(s,t). \tag{22}$$

An approximate solution (19), (22) is searched in the form

$$\tilde{x}(t,s) = \sum_{k=1}^{K} x_k(t)\varphi_k(s), \quad \tilde{f}(t,s) = \sum_{k=1}^{K} f_k(t)\varphi_k(s),\ K > 1, \tag{23}$$

where $\{\varphi_k(s)\}$ are solutions to the eigenvalue problem.

$$\lambda X(s) = -X_{ss}(s),\ s \in (0,\pi),$$

$$X(0) = X(\pi) = 0.$$

The considered spectral problem is solvable and has a countable set of eigenvalues $\{\lambda_k\}$, the functions $\{\varphi_k\}$ form orthonormal with weight $\sqrt{\frac{2}{\pi}}$ system of functions

$$\frac{2}{\pi}\int_0^l \varphi_k\varphi_l ds =< \varphi_k, \varphi_l >= \begin{cases} 1,\ l = k, \\ 0,\ l \neq k. \end{cases}$$

Here $\varphi_k = \varphi_k(s) = \sqrt{\frac{2}{\pi}}\sin(ks)$ and $\lambda_k = k^2$.

Multiplying scalarly (22) taking into account the condition (19) by the eigenfunctions φ_k, $k = 1, 2, ..., K$, we obtain a system of ordinary differential or algebraic-differential equations

$$\frac{-\sqrt{2}\pi^{3/2}t - 54f_3^2(t)f_1(t) - 36f_3(t)f_2^2(t) - 9f_3(t)f_1^2(t) - 24f_2^2(t)f_1(t) - 3f_1^3(t)}{2\pi} =$$
$$= \frac{(\lambda-\lambda_1)\frac{v_1(t)}{dt}2\pi}{2\pi} + < y(s,t), \varphi_1(s) >,$$
$$\frac{108f_3^2(t)f_2(t) + 36f_3(t)f_2(t)f_1(t) + 24f_2^3(t) + 12f_2(t)f_1^2(t) + (\lambda-\lambda_2)\frac{v_2(t)2\pi}{dt}\pi}{\pi} =$$
$$=< y(s,t), \varphi_2(s) >,$$
$$\frac{243f_3^3(t) + 216f_3(t)f_2^2(t) + 54f_3(t)f_1^2(t) + 36f_2^2(t)f_1(t) + 3f_1^3(t) + (\lambda-\lambda_2)\frac{v_3(t)}{dt}2\pi}{2\pi} =$$
$$=< y(s,t), \varphi_3(s) >,$$
$$\ldots$$

Using the decomposition method, we linearize the system of equations. From this system, we obtain the unknown Galerkin coefficients $x_k(t)$, which can be obtained as functions $f_k(t)$. These functions were found from the known right side of the equation $y(s,t)$ and the initial condition x_{0k} or u_k. Then, following the algorithm, the target functional is introduced and the necessary control coefficients are found. Let us consider several examples of various control problems.

Example 1. Consider the problem of start control (2), (7), (10),

$$J(x,u) = \theta\int_0^T dt\int_0^\pi |x_s(s,t) - x_{ds}(s,t)|^4\,ds + (1-\theta)\int_0^\pi |u_s(s)|^4\,ds \to \min, \qquad (24)$$

at $T = 1$, $K = 3$, $\theta = \frac{1}{3}$, $\beta = \frac{99}{100}$, $r_\varepsilon = 100$, right side of the system of equations $y(s,t) = t\sin(s)$, the required state of the system is given in the form

$$x_d(s,t) = \sqrt{\frac{2}{\pi}}(\sin(s) + t\sin(2s) + t^2\sin(3s)).$$

Set the target function as follows

$$J(x,u) = \beta\theta\int_0^T dt\int_0^\pi |x_s(s,t) - x_{ds}(s,t)|^4\,ds + (1-\beta)\theta\int_0^T dt\int_0^\pi |f_s(s,t) - x_{ds}(s,t)|^4\,ds +$$
$$+(1-\theta)\int_0^\pi |u_s(s)|^4\,ds + r_\varepsilon\int_0^T dt\int_0^\pi |x_s(s,t) - f_s(s,t)|^2\,ds.$$

As a result of «Software module of numerical solution of the start control problem» an approximate solution of the considered control problem is found, at that functional $J = 0.48755110$.

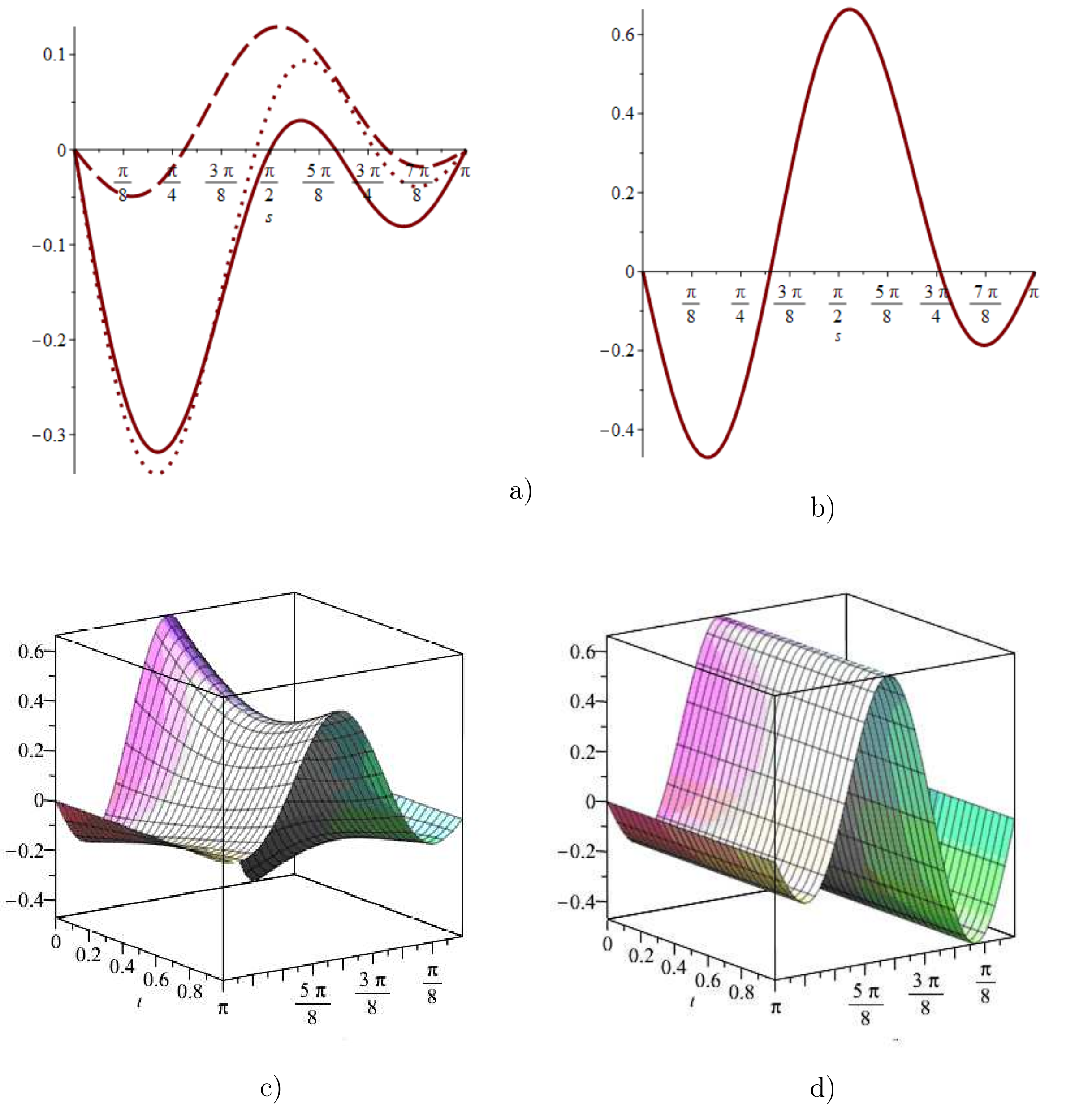

Fig. 1. a) Approximate solutions $\tilde{x}$, $\tilde{f}$, required state x_d at time $t = T$; b) the control function $\tilde{u}$ at time $t = T$; c) the approximate solution $\tilde{x}$ at time $t \in (0, 1)$; d) the control function $\tilde{u}$ at time $t \in (0, 1)$

Thus, approximate solution $(\tilde{x}, \tilde{f})$ has the form

$$\tilde{x}(s,t) = -0.6095389046\sin(s)(-0.4143453768t^2 + (3.3373466258t^{1}0 + 4.2523276614t^9 + \\ +1.5057931918t^8 + 1.5466228124t^7 + 1.3941901369t^6 + 0.2257921405t^5 + 0.2616133t^4 + \\ +0.1807035538t^3 - 0.0204324140t^2 + 0.0543035764t - 0.43736242)\cos^2(s) + \\ +(2.1733710027t^{1}0 + 2.6973985097t^9 + 0.9059160557t^8 + 1.0349831691t^7 + 0.8977797t^6 + \\ +0.1280656218t^5 + 0.1806120223t^4 + 0.1093883847t^3 - 0.01157808604t^2 + 0.0341217t - \\ -0.2255654656)\cos(s) + 0.1416949879t^{10} + 0.309165886113t^9 + 0.1564465748t^8 + \\ +0.1962756811t^7 + 0.2046397397t^6 + 0.0392128429t^5 + 0.0465780010t^4 + 0.0354851t^3 + \\ +0.0119926549t - 0.2095088798),$$

$\tilde{f}(s,t) = 1.595769121((1.7952775t^3 + 0.7083824t^2 - 0.0555775t + 0.1670601)\cos^2(s)+$
$+(0.887926088t^3 + 0.2493363702t^2 - 0.0128052001490951417t + 0.0861596612)\cos(s)+$
$+0.0206231663t^3 + 0.1103540259t^2 - 0.0121640234t + 0.0800264972)\sin(s).$

The obtained approximate solutions $\tilde{x}$, $\tilde{f}$ together with the required state x_d at time $t = T$ are shown in Fig. 1a. Note that the solid line shows the required state x_d, the dotted line $\tilde{x}$, the line of points $\tilde{f}$. The control function $\tilde{u}$ shown in Fig. 1b. Note that the solid line shows the required state x_d, the dotted line $\tilde{x}$, the line of points $\tilde{f}$. The approximate solution $\tilde{x}$, the control function $\tilde{u}$ are shown in Fig. 1c and Fig. 1d when $t \in (0,1)$ changes, respectively.

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References

1. Sviridyuk G.A., Shemetova V.V. Hoff Equations on Graphs. *Differential Equations*, 2006, vol. 42, no. 1, pp. 139–145. DOI: 10.1134/S0012266106010125
2. Manakova N.A., Sviridyuk G.A. Non-Classical Equations of Mathematical Physics. Phase Spaces of Semilinear Sobolev Equations. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2016, vol. 8, no. 3, pp. 31–51. DOI: 10.14529/mmph160304 (in Russian)
3. Manakova N.A. Mathematical Models and Optimal Control of The Filtration and Deformation Processes. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2015, vol. 8, no. 3, pp. 5–24. (in Russian) DOI: 10.14529/mmp150301
4. Manakova N.A., Vasiuchkova K.V. Research of one Mathematical Model of the Distribution of Potentials in a Crystalline Semiconductor. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2019, vol. 12, no. 2, pp. 150–157. DOI: 10.14529/mmp190213
5. Sviridyuk G.A. A Problem for the Generalized Boussinesq Filtration Equation. *Soviet Mathematics (Izvestiya VUZ. Matematika)*, 1989, vol. 33, no. 2, pp. 62–73.
6. Lions J.-L. *Quelques maerthodes de resolution des problemes aux limites non lineaires.* Paris, Dunod, 1968. (in French)
7. Sviridyuk G.A., Efremov A.A. Optimal Control Problem for One Class of Linear Sobolev Type Equations. *Russian Mathematics*, 1996, vol. 40, no. 12, pp. 60–71.
8. Sagadeeva M.A., Zagrebina S.A., Manakova N.A. Optimal Control of Solutions of a Multipoint Initial-Final Problem for Non-Autonomous Evolutionary Sobolev Type Equation. *Evolution Equations and Control Theory*, 2019, vol. 8, no. 3, pp. 473–488. DOI: 10.3934/eect.2019023
9. Shestakov A.L., Sviridyuk G.A., Hudyakov Yu.V. Dinamic Measurement in Spaces of ≪Noise≫. *Bulletin of the South Ural State University. Series: Computer Technologies, Automatic Control, Radioelectronics*, 2013, vol. 13, no. 2, pp. 4–11.
10. Davis P.L. A Quasilinear Parabolic and a Related Third Order Problem. *Journal of Mathematical Analysis and Applications*, 1972, vol. 40, no. 2, pp. 327–335.

11. Changchung Liu. Weak Solutions for a Viscous p-Laplacian Equation. *Electronic Journal of Differential Equations*, 2004, vol. 15, pp. 1–11.

12. Changchung Liu. Some Properties of Solutions of the Pseudo-Parabolic Equation. *Lobachevskii Journal of Mathematics*, 2003, vol. 63, pp. 3–10.

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ИССЛЕДОВАНИЕ РАЗЛИЧНЫХ ТИПОВ ЗАДАЧ УПРАВЛЕНИЯ ДЛЯ ОДНОЙ МОДЕЛИ НЕЛИНЕЙНОЙ ФИЛЬТРАЦИИ

К. В. Перевозчикова, Н. А. Манакова, А. С. Купцова

Обширный класс задач математической физики можно рассматривать в рамках полулинейных уравнений соболевского типа, которые описывают различные физические процессы (например, процессы деформации, процессы, протекающие в полупроводниках, процессы распространения колебательного движения в различных средах и т.д.). Статья посвящена исследованию задач управления (оптимального, стартового и жесткого) одной математической модели соболевского типа, в основе которой лежит уравнение, описывающее процесс изменения потенциала концентрации вязкоупругой жидкости, фильтрующейся в пористой среде (процесс нелинейной диффузии вещества). Найдены достаточные условия, при которых существует решения задач управления исследуемой модели. Построен алгоритм численного метода решения и проведен вычислительный эксперимент.

Ключевые слова: уравнения соболевского типа; модель нелинейной диффузии; задача стартового управления; задача оптимального управления; задача жесткого управления; математическое моделирование; проекционный метод; метод декомпозиции.

References

1. Свиридюк, Г.А. Уравнения Хоффа на графах / Г.А. Свиридюк, В.В. Шеметова // Дифференциальные уравнения. – 2006. – Т. 42, №. 1. – С. 139–145.
2. Манакова, Н.А. Неклассические уравнения математической физики. Фазовые пространства полулинейных уравнений соболевского типа / Н.А. Манакова, Г.А. Свиридюк // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2016. – Т. 8, № 3. – С. 31–51.
3. Манакова, Н.А. Математические модели и оптимальное управление процессами фильтрации и деформации / Н.А. Манакова // Вестник ЮУрГУ. Серия: Математического моделирования и программирования. – 2015. – Т. 8, № 3. – С. 5–24.
4. Манакова, Н.А. Исследование одной математической модели распределения потенциалов в кристаллическом полупроводнике / Н.А. Манакова, К.В. Васючкова // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2019. – Т. 12, №. 2. – С. 150–157.
5. Свиридюк, Г.А. Одна задача для обобщенного фильтрационного уравнения Буссинеска / Г. А. Свиридюк // Известия вузов. Серия: Математика. – 1990. – № 2. – С. 55–61.
6. Lions J.-L. Quelques maerthodes de resolution des problemes aux limites non lineaires / J.-L. Lions. – Paris: Dunod, 1968.
7. Sviridyuk G.A. Optimal Control Problem for One Class of Linear Sobolev Type Equations / G.A. Sviridyuk, A.A. Efremov // Russian Mathematics. – 1996. – V. 40, № 12. – P. 60–71.
8. Sagadeeva M.A. Optimal Control of Solutions of a Multipoint Initial-Final Problem for Non-Autonomous Evolutionary Sobolev Type Equation / M.A. Sagadeeva, S.A. Zagrebina, N.A. Manakova // Evolution Equations and Control Theory. – 2019. – V. 8, № 3. – P. 473–488.
9. Шестаков А.Л. Динамические измерения в пространствах «шумов» / А.Л. Шестаков, Г.А. Свиридюк, Ю.В. Худяков // Вестник ЮУрГУ. Серия: Компьютерные технологии, управление, радиоэлектроника. – 2013. – Т. 13, № 2. – С. 4–11.
10. Davis P.L. A Quasilinear Parabolic and a Related Third Order Problem / P.L. Davis // Journal of Mathematical Analysis and Applications. – 1972. – V. 40, № 2. – P. 327–335.
11. Changchung Liu. Weak Solutions for a Viscous p-Laplacian Equation Electronic / Liu Changchung // Journal of Differential Equations. – 2004. – V. 15. – P. 1–11.
12. Changchung Liu. Some Properties of Solutions of the Pseudo-Parabolic Equation / Liu Changchung // Lobachevskii Journal of Mathematics. – 2003. – V. 63. – P. 3–10.

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