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NUMERICAL STUDY OF THE NON-UNIQUENESS OF SOLUTIONS TO THE SHOWALTER–SIDOROV PROBLEM FOR A MATHEMATICAL MODEL OF I-BEAM DEFORMATION

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The article is devoted to the question of the uniqueness or multiplicity of solutions of the Showalter–Sidorov–Dirichlet problem for the Hoff equation on a segment. The Hoff equation simulates the dynamics of deformation of an I-beam under constant load. To investigate the non-uniqueness of solutions to the Showalter–Sidorov problem, the phase space method will be used, which was developed by G.A. Sviridyuk to study the solvability of Sobolev-type equations. It was also previously shown that the phase space of the model under study contains features of type 2-Whitney assembly. The article presents the conditions of uniqueness or multiplicity of solutions to the Showalter–Sidorov problem depending on the system parameters. An algorithm for the numerical solution of the problem based on the Galerkin method. The results of computational experiments are presented.

Keywords: Sobolev type equations; Showalter–Sidorov problem; Hoff equation; non-uniqueness of solutions; phase space method; Galerkin method.

Introduction

The approach proposed in the work of Hoff [1] for the study of deformation under compression of the rod (buckling) extends to the case of creep of initial irregularities. Creep should be understood as the deformation of a solid body that takes place over a long time, under the influence of a constant load. In the process of creep, deformations increase – both on the concave and convex sides of the curved rod, which leads to a rapid increase in deflections or to the phenomenon of buckling of the rod. In his work, N.J. Hoff applied his approach to an «idealized» I-beam section consisting of two identical shelves connected by a thin wall of a certain height, which led to the creation of a model of buckling of an I-beam.

Generalized Hoff equation

$$(-\lambda-\Delta)u_t+\alpha_0 u+\alpha_1 u^3+\alpha_2 u^5+\ldots+\alpha_{k-1}u^{2k-1}+\alpha_k u^{2k+1}=y,\ t\in R_+,\ x\in\Omega \quad (1)$$

models the dynamics of deformation of an I-beam. The function $u=u(x,t),\ x\in\Omega,\ t\in\mathbb{R}_+$ shows the deviation of the beam from the equilibrium position. The parameter $\lambda\in R_+$ characterizes the longitudinal load on the beam, and the parameters $\alpha_i\in R, i=\overline{0,k}$ characterize the properties of the beam material, $y=y(x,t)$ is external load (lateral load for $n=1$).

Various approaches were used in the study of the Hoff model [2–4]. For example, in the work of S.A. Zagrebina a multipoint initial-final problem was studied for a linear Hoff model [5]. The work of N.A. Manakova is devoted to the study of optimal control of the deformation process [6]. And in the article by S.A. Zagrebina and P.O. Pivovarova was investigated the stability of the linear Hoff equations on the graph [7].

The review article [8] contains the results of many years of research on the non-uniqueness of the Showalter–Sidorov problem:

$$L\left(u(x,0)-u_0(x)\right)=0 \tag{2}$$

for semilinear Sobolev type equations

$$L\dot{u}=Mu+N(u) \tag{3}$$

and models are given in which the existence of several solutions to the Showalter–Sidorov problem is possible. The following works were devoted to the issues of non-uniqueness of solutions of equations and systems of equations that reduce to semilinear equations of the form (3) and the connection of non-uniqueness of solutions with the existence of assemblies and Whitney folds in the phase space of equations (3). A.F. Gilmutdinova find the conditions for the existence of non-uniqueness of the solution of the problem were identified for Plotnikov's mathematical model [9]. The work of T.A. Bokareva shows the existence of a 2-Whitney assembly and a 1-Whitney assembly for a model of nerve impulse propagation in a membrane and for a model of autocatalytic reaction with diffusion, respectively [10]. In the article by N.A. Manakova and O.V. Gavrilova reveals the conditions for the existence of a non-uniqueness of the solution of the problem for the model of propagation of a nerve impulse in a membrane [11].

In this paper we will consider the Showalter–Sidorov problem

$$\lambda\left(u(x,0)-u_0(x)\right)+\left(u_{xx}(x,0)-u_0(x)\right)=0,\ x\in(0,l) \tag{4}$$

for the Hoff equation (1) in the one - dimensional case for $k=1$

$$\lambda u_t+u_{xxt}=\alpha u+\beta u^3,\ t\in(0,T) \tag{5}$$

with the Dirichlet condition

$$u(x,t)=0,\ x\in\partial\Omega,t\in(0,T) \tag{6}$$

In the work [12] by G.A. Sviridyuk and V.O. Kazak it was shown that the phase space of the equation (5) is a simple Banach C^∞–manifold in the case $\alpha\beta>0$. In the case $\alpha\beta<0$ the phase space of the equation (5) it will no longer be a simple manifold, it lies on the 2-Whitney assembly as shown in the article [13].

In this paper, the Galerkin method is used to implement the numerical solution of the problem. The Galerkin method is most suitable in the case of degenerate semilinear equations, since it allows taking into account the degeneracy of equations for certain parameters. Approximate solutions of a model which coefficients satisfy a system of algebra-differential equations with corresponding initial conditions are constructed using the Galerkin method. For the first time for semilinear Sobolev type equations, this method was considered by G.A. Sviridyuk and T.G. Sukacheva [15]. In the case of degenerate semilinear equations for finding approximate solutions, the Galerkin method was used in the works of N.A. Manakova, A.A. Zamyshlyaeva, K.V. Perevozchikova and many others [16–19].

In this study, we will consider the non-uniqueness of solutions of the Showalter–Sidorov problem for the Hoff equation (5) in the case of $\alpha\beta < 0$. In this case, we restrict ourselves to the condition $\dim\ker(\lambda + \Delta) = 1$, which is obviously executed at $n = 1$.

1. Features of the Phase Space

Reducing the problem (4), (5) to the problem (2), (3). For that take $\mathfrak{U} = L_4(\Omega)$, $\mathfrak{h} = \overset{\circ}{W}{}_2^1$. Operators L, M, N are defined by formulas

$$\begin{aligned} \langle Lu, v\rangle &= \int\limits_0^l (\lambda uv - u_x v_x)dx, \quad \forall u, v \in \mathfrak{h}, \\ \langle Mu, v\rangle &= \alpha \int\limits_0^l uv\, dx, \\ \langle N(u), v\rangle &= \beta \int\limits_0^l u^3 v\, dx, \quad \forall u, v \in \mathfrak{U}, \end{aligned} \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is an inner product in $L_2(\Omega)$. Let $\mathfrak{F}, \mathfrak{U}^*$ be the conjugate of $\mathfrak{h}, \mathfrak{U}$ spaces with respect to duality $\langle\cdot,\cdot\rangle$. By $n = 1$ according to Sobolev's embedding theorem, all embeddings

$$\mathfrak{h} \hookrightarrow \mathfrak{U} \hookrightarrow L_2(\Omega) \hookrightarrow \mathfrak{U}^* \hookrightarrow \mathfrak{F}$$

are dense and continuous. Note that the operator $L \in \mathcal{C}l(\mathfrak{U}, \mathfrak{F})$, dom $L = \overset{\circ}{W}{}_2^1$, operators $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$ and $M \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$.

Lemma 1. *For any $\lambda \in \mathbb{R}, \alpha \in \mathbb{R}\backslash\{0\}$ the operator $M(L, 0)$ is bounded.*

Let $\lambda \in \sigma(-\Delta)$ then $\ker L = \operatorname{span}\{\varphi_g : \lambda_g = \lambda\}$, im $L = (\ker L)^\perp$, where φ_g is an orthonormal (in sense$L_2(\Omega)$) family of eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator $-\Delta$ in the domain Ω which correspond to the eigenvalues $\lambda_g = \sigma(-\Delta)$ indexed in the in the nonincreasing order, taking into account the multiplicity (orthogonality in the sense of $L_2(\Omega)$). Since L is a Fredholm operator, we can take $\varphi \in \ker L\backslash\{0\}$, i.e.

$$\varphi = a\varphi_g, \lambda_g = \lambda, |a| > 0,$$

at the same time φ_g, λ_g by $n = 1$ will take the form

$$\varphi_g = \sqrt{\frac{2}{\pi}} \sin\ gx,\ \lambda_g = g^2,\ g = 1, 2, \ldots$$

Let's build a projector $\mathbb{I} - Q = \langle\cdot, \varphi\rangle$, where $\varphi \in \ker L$, $\|\varphi\|_{L_2(\Omega)} = 1$. Then the set $\mathfrak{B}$ can be represented in the following form:

$$\mathfrak{B} = \{u \in \mathfrak{U} : \langle Mu + N(u), \varphi\rangle = 0\}$$

Next we will assume that $\lambda = \lambda_1$, where λ_1 is the first eigenvalue of the homogeneous Dirichlet problem of the Laplace operator $-\Delta$ in the domain Ω accordingly. Note that the considered eigenfunctions φ_g correspond to single eigenvalues λ_g, satisfying the condition $\dim(1+\lambda\Delta)=1$.

Considering that $\alpha\beta<0$ and $\lambda=\lambda_1$ we represent vector u as $u=s_1\varphi_1+u_1^{\perp}$, where $u_1^{\perp}\in\mathfrak{U}_1^{\perp}=\{u_1\in\mathfrak{U}:\langle u_1,\varphi_1\rangle=0\}$ note that the set $\mathfrak{B}$ C^{∞} is diffeomorphic to the set

$$
\begin{aligned}
\mathfrak{B}=\{(s_1,u_1^{\perp})\in\mathbb{R}\times\mathfrak{U}: s_1^3\|\varphi_1\|_{\mathfrak{U}}^4+3s_1^2\int_0^l\varphi_1^3u_1^{\perp}\,dx+\\
+\,s_1\,(3\int_0^l\varphi_1^2(u_1^{\perp})^2\,dx+\alpha\beta^{-1}+\int_0^l\varphi_1(u_1^{\perp})^3\,dx=0\}
\end{aligned}
\tag{8}
$$

In the [13] set $\mathfrak{B}$ named 2-assemblies Whitney, in [12] it is shown that in the case of $\alpha\beta>0$ for any vector $u_1^{\perp}\in\mathfrak{U}_1^{\perp}$ there exists unique value $s_1\in\mathbb{R}$ such that $s_1\varphi_1+u_1^{\perp}\in\mathfrak{B}$.

The equation defining the set $\mathfrak{B}$ are cubic equation of general form

$$
as_1^3+bs_1^2+cs_1+d=0 \tag{9}
$$

According to Cardano's formulas, any cubic equation of general form with the help of replacement $s_1=y-\frac{b}{3a}$ can be reduced to canonical form $y^3+py+q=0$ with coefficients

$$
\begin{aligned}
&a=\|\varphi_1\|_{\mathfrak{U}}^4,\ b=3\int_0^l\varphi_1^3u_1^{\perp}\,dx,\\
&c=3\int_0^l\varphi_1^2(u_1^{\perp})^2\,dx+\alpha\beta^{-1}, d=\int_0^l\varphi_1(u_1^{\perp})^3dx,\\
&p=\frac{3ac-b^2}{9a^2},\\
&q=\frac{1}{2}\left(\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a}\right),\\
&Q_1(s_1,u)=p^3+q^2,\\
&R_1(s_1,u)=3s_1^2\|\varphi_1\|_{\mathfrak{U}}^4+6s_1\int_0^l\varphi_1^3u_1^{\perp}\,dx+3\int_0^l\varphi_1^2(u_1^{\perp})^2\,dx+\alpha\beta^{-1}.
\end{aligned}
\tag{10}
$$

For convenience of further consideration, introduce the following sets:

$$
\begin{aligned}
(\mathfrak{U}_1)_0^{\perp}&=\{u\in\mathfrak{U}_1^{\perp}:R_1(s_1,u)=0\},\\
(\mathfrak{U}_1)_+^{\perp}&=\{u\in\mathfrak{U}_1^{\perp}:Q_1(s_1,u)>0\},\\
(\mathfrak{U}_1)_-^{\perp}&=\{u\in\mathfrak{U}_1^{\perp}:Q_1(s_1,u)<0\}.
\end{aligned}
\tag{11}
$$

Lemma 2. *Let $\alpha\beta<0$ and $\lambda=\lambda_1$. Then*
(i) for any $u\in\mathfrak{U}_1^{\perp}\cap(\mathfrak{U}_1)_+^{\perp}$ there exists a unique solution of the equation (9);
(ii) for any $u\in\mathfrak{U}_1^{\perp}\cap(\mathfrak{U}_1)_+^{\perp}\cap(\mathfrak{U}_1)_0^{\perp}$ there exists two solutions of the equation (9);
(iii) for any $u\in\mathfrak{U}_1^{\perp}\cap(\mathfrak{U}_1)_-^{\perp}$ there exists three solutions of the equation (9).

Theorem 1. [14] *Let $n=1,\alpha\beta<0$ and $\lambda=\lambda_1$. Then*
(i) for any $u_0\in(\mathfrak{U}_1)^{\perp}\cap(\mathfrak{U}_1)_-^{\perp}$ there exists three solutions of the problem (4)–(6);
(ii) for any $u_0\in(\mathfrak{U}_1)^{\perp}\cap(\mathfrak{U}_1)_+^{\perp}\cap(\mathfrak{U}_1)_0^{\perp}$ there exists two solutions of the problem (4)–(6);
(iii) for any $u_0\in(\mathfrak{U}_1)^{\perp}\cap(\mathfrak{U}_1)_+^{\perp}$ there exists a unique solution of the problem (4)–(6).

2. Algorithm for a Numerical Method for Finding a Solution to the Showalter–Sidorov Problem

Let us describe an algorithm for the numerical solution of the problem (4)–(6) on the interval $(0, \pi)$. The algorithm allows you to find approximate solutions to the problem using the Galerkin method. Denote by λ_g the set of eigenvalues, numbered by non-increment, taking into account multiplicity and φ_g is orthonormal (in the sense of $L_2(\Omega)$) family eigenfunctions of the homogeneous Dirichlet problem in the domain Ω of the Laplace operator $-\Delta$.

Following the Galerkin method, we search for approximate solution for an approximate solution of the problem under consideration as sums

$$u(x,t) = \sum_{j=1}^{m} u_j(t)\varphi_j(x). \tag{12}$$

Substitute Galerkin sums in (4). Then multiply the resulting equation scalar in $L_2(\Omega)$ on eigenfunctions $\varphi_j(x), j = \overline{1,m}$, and get a system of equations with respect to the unknowns $u_j(t)$. At the same time, depending on the parameter λ, the equations in this system can be differential or algebraic. Consider these cases in more detailes:

(i) If $\lambda < \lambda_1$, then in this case all the equations of the system will be ordinary differential equations of the first order. To solve this system relatively $u_j(t), j = \overline{1,m}$, from the conditions (6), multiplying them scalar in $L_2(\Omega)$ on eigenfunctions $\varphi_j(x), j = \overline{1,m}$, we find m initial conditions. Next, the resulting system of nonlinear first-order differential equations with initial conditions is solved numerically, and unknown functional coefficients are found $v_j(t), j = \overline{1,m}$ in an approximate solution $u(x,t)$.

(ii) If $\lambda = \lambda_1$, then the first equation is algebraic, and the rest ones are differential. Consider separately a system composed of first-order differential equations and an algebraic equation. To solve a system of first-order ordinary differential equations with respect to $u_j(t), j = \overline{2,m}$ from the conditions (6), multiplying them scalar in $L_2(\Omega)$ on eigenfunctions $\varphi_j(x), j = \overline{2,m}$, we find $(m-1)$ initial conditions. Let us proceed to the numerical solution of a system of algebra-differential equations with initial conditions $(m-1)$.

Check the uniqueness or multiplicity of the solution of the Showalter–Sidorov problem for given initial functions u_0:

(i) In the case when $Q < 0$, the required problem has three solutions $u^1(x,t), u^2(x,t), u^3(x,t)$, consequently, the system of algebra-differential equations have three solutions and three sets $u_j(t)$ for each of the solutions, respectively. In this case, all subsequent steps must be done three times for each of the sets $u_j(t)$.

(ii) In the case when $R = 0$, the required problem has two solutions $u^1(x,t), u^2(x,t)$, consequently, the system of algebra-differential equations have two solutions and two sets $u_j(t)$ for each of the solutions, respectively. In this case, all subsequent steps must be done twice for each of the sets $u_j(t)$.

(iii) In the case when $Q > 0$, the required problem has one solution $u^1(x,t)$, therefore, the system of algebra-differential equations have one solution and one set $u_j(t)$.

The algorithm for finding an approximate solution to the Showalter–Sidorov problem is reduced to 5 stages:

Stage 1. Input of the parameters of the Hoff equation, initial and boundary conditions, as well as the number of Galerkin approximations. Based on these data, an approximate solution is formed in the form of a Galerkin sum.

Stage 2. Checking the parameter λ.

Stage 3. Checking the uniqueness or multiplicity of solutions of systems of algebra-differential equations.

Stage 4. Calculation of an approximate solution with an initial condition using the Galerkin method (depending on the cases (i), (ii), (iii)).

Stage 5. Plotting an approximate solution.

3. Description of the Operation of Computer Program

The described algorithm was implemented in the Maple 2017 computer mathematics system for Windows 7, 8.1, 10 as a set of programs. The program is intended for numerical investigation of the non-uniqueness of solutions of the Showalter–Sidorov problem for the Hoff equation (at $k = 1$) on a segment. The program implements the Galerkin method.

The following data is submitted to the program input. The parameters of the domain Ω (segment boundaries l_1,l_2), eigenvalue λ_g, coefficients of the equation α, β. At the output, the program gives approximate solutions $u(x, t)$ and build their graphs. The block diagram of the program is shown in Figure 1. The following steps are performed while the program is running.

Step 1. Enter parameters of the domain Ω, equation parameters, eigenfunctions and eigenvalues of the homogeneous Dirichlet problem for the operator $-\Delta$, as well as the number of Galerkin approximations.

Step 2. An approximate solution is formed: multiplication of eigenvalue λ_g on basic functions φ_g. Then the approximate solution is substituted into the original differential equation and integrated in the loop in the domain under consideration Ω using the procedure **int**. A system of algebra-differential equations is being compiled.

Step 3. Next, a check is made for the degeneracy of the equations, that is, whether λ is the eigenvalue of the operator $-\Delta$. If the verification condition is met, we solve the resulting algebraic equation with respect to the unknowns $u_j, j = \overline{1, m}$ using built-in procedures **subs, solve**. Otherwise, we find a solution to the system of differential equations using the built-in procedure **dsolve**. Also at this step, you should check for the presence of several solutions.

If $Q < 0$, then the algebraic equation has three different solutions. For $Q > 0$, then the equation has one solution, and in the case of $R = 0$ this equation has two solutions.

Step 4. In the case when the equation has one solution, we get one set u_j. In the case when the equation has two solutions there are two sets u_j for each solution and, accordingly, for three solutions there are three sets u_j. To realize the possibility of finding different solutions in other modules using the built-in procedure **save**: the initial conditions are saved to a file **usl.mw**, every solution found $u_0(s)$ saved to files **resh1.mw** or **resh2.mw** or **resh3.mw** depending on the number of solutions received.

Step 5. Using the built-in procedure **read** the initial conditions and one of the solutions are read $u_0(s)$. In the loop **for** j **to** 1 **do** m **end do** the right part of the solution

$u_0(s)$ multiplied by eigenfunctions φ_j and is integrated in the domain under consideration Ω using the procedure **int**. The resulting system is solved using the built-in procedure **dsolve**.

Step 6. A table of values of Galerkin approximations u_j is compiled and a graph of the solution $u(x,t)$ is plotted by the built-in procedure **plot3d**.

4. Numerical Experiment

Let us consider examples of numerical investigation of the non-uniqueness of solutions to the Showalter–Sidorov problem for the I-beam deformation model based on the implementation of the algorithm and program described above.

Example 1. It is required to find a numerical solution to the Showalter–Sidorov problem

$$(u(x,0)-u_0(x))+(u_{xx}(x,0)-u_0(x))=0,\ x\in\Omega \tag{13}$$

for the equation

$$u_t+u_{xxt}=-u+u^3,\ t\in(0,T) \tag{14}$$

with Dirichlet boundary condition

$$u(x,t)=0,\ x\in\partial\Omega,\ t\in(0,T) \tag{15}$$

if $\Omega=(0,\pi),\ T=1,\ u_0(x)=\sqrt{\frac{2}{\pi}}\sin(2x)$.

Approximate solutions of the problem (13)–(15) on the interval $(0,\pi)$ can be represented in the form of $u(x,t)=u_1(t)\varphi_1(x)+u_2(t)\varphi_2(x)$, where $\varphi_g(x)=\sqrt{\frac{2}{\pi}}\sin\ gx$, $g=1,2$. Since under the conditions of this experiment, λ coincides with the first eigenvalue $\lambda_1=1$ homogeneous Dirichlet problem for $(-\Delta)$, we obtain a system of algebra-differential equations:

$$\begin{cases} u_1(t)(-3u_1^2(t)-6u_2^2(t)+2\pi)=0, \\ -6u_1^2(t)u_2(t)-3u_2^3(t)+6\pi u_2(t)dt+2\pi u_2(t)=0. \end{cases} \tag{16}$$

Using formulas (7), we find $Q=-0.0000311696<0$, from which it follows that this system of equations has three solutions. Having solved the algebraic equation of the system at the initial moment of time, we obtain three initial conditions u_{01},u_{02},u_{03}. Solving the resulting system by the Runge–Kutta method, we obtain three numerical solutions. Numerical solution corresponding to the initial condition u_{01} presented in Table 1 and Fig. 2, the numerical solution corresponding to the initial condition u_{02} is shown in Fig. 3 and in Table 2, the numerical solution corresponding to the initial condition u_{03} is shown in Fig. 4 and in Table 3.

Example 2. It is required to find a numerical solution to the Showalter–Sidorov problem

$$-(u(x,0)-u_0(x))+(u_{xx}(x,0)-u_0(x))=0,\ x\in\Omega \tag{17}$$

for the equation

$$-u_t+u_{xxt}=u-u^3,\ t\in(0,T) \tag{18}$$

with Dirichlet boundary condition

$$u(x,t)=0,\ x\in\partial\Omega,t\in(0,T) \tag{19}$$

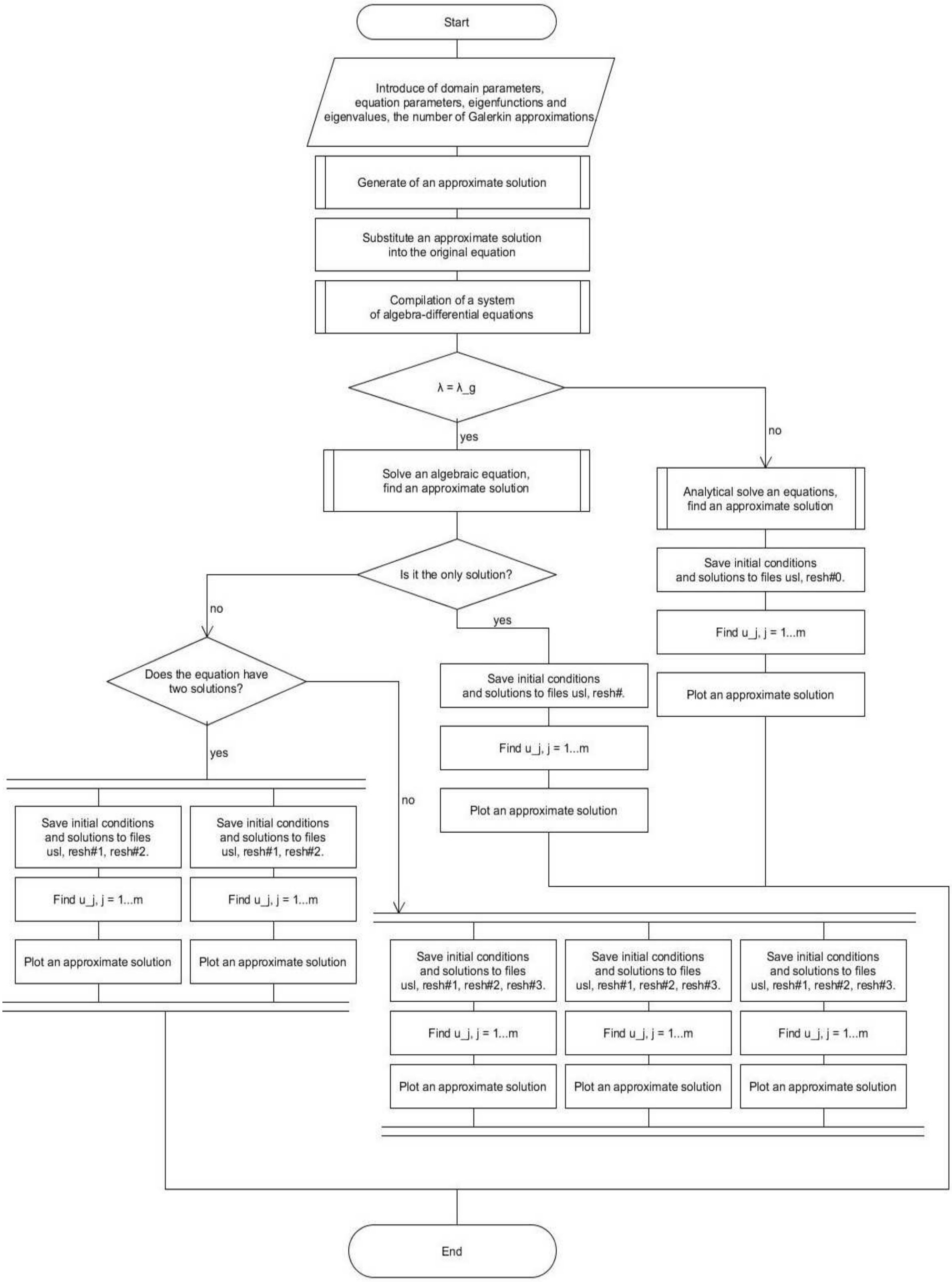

Fig. 1. Diagram of the algorithm

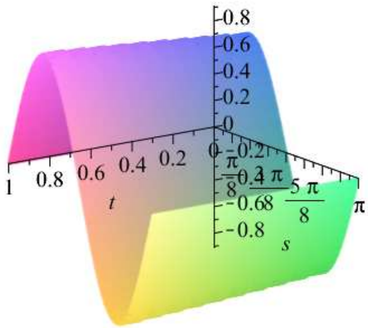

Fig. 2. Numerical solution of the problem (13) – (15) with the initial condition u_{01}

Table 1

Numerical solution of the problem (13)–(15) with the initial condition u_{01}

t	$u_1(t)$	$u_2(t)$
0	0	1
0.1	0	1.0172881386
0.2	0	1.0343005989
0.3	0	1.0510140109
0.4	0	1.0674061411
0.5	0	1.0834564817
0.6	0	1.0991463464
0.7	0	1.1144588705
0.8	0	1.1293790109
0.9	0	1.1438935842
1.0	0	1.1579913920

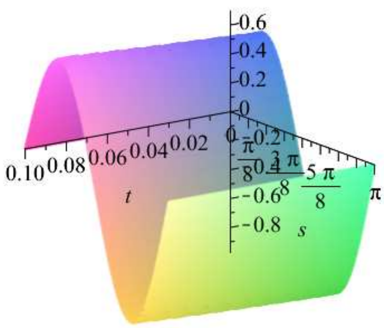

Fig. 3. Numerical solution of the problem (13)–(15) with the initial condition u_{02}

Table 2

Numerical solution of the problem (13)–(15) with the initial condition u_{02}

t	$u_1(t)$	$u_2(t)$
0	−0,3072378596	1
0.01	−0.2976471122	1.0014492745
0.02	−0.2876099716	1.0029146245
0.03	−0.2770761929	1.0043962938
0.04	−0.2659840256	1.0058945301
0.05	−0.2542576421	1.0074095774
0.06	−0.2418024366	1.0089416947
0.07	−0.2284957567	1.0104911417
0.08	−0.2141764598	1.0120581865
0.09	−0.1986212588	1.0136431008
0.1	−0.1815091065	1.0152461601

if $\Omega = (0, \pi), T = 1, u_0(x) = \sqrt{\frac{2}{\pi}} \sin(2x)$.

Approximate solutions of the problem (17)–(19) on the interval $(0, \pi)$ can be represented in the form of $u(x,t) = u_1(t)\varphi_1(x) + u_2(t)\varphi_2(x)$, где $\varphi_g(x) = \sqrt{\frac{2}{\pi}} \sin\ gx$, $g = 1, 2$. Since under the conditions of this experiment $\lambda = -1$ does not coincide with the first eigenvalue $\lambda_1 = 1$ of the homogeneous Dirichlet problem for $(-\Delta)$, we obtain a system of ordinary differential equations:

$$\begin{cases} 3u_1^3(t) + 6u_1(t)u_2^2(t) - 4u_1(t)dt - 2\pi u_1(t) = 0, \\ 6u_1^2(t)u_2(t) + 3u_2^3(t) - 10u_2(t)dt - 2\pi u_2(t) = 0. \end{cases} \tag{20}$$

This system has one numerical solution, represented in Table 4 and Fig. 5.

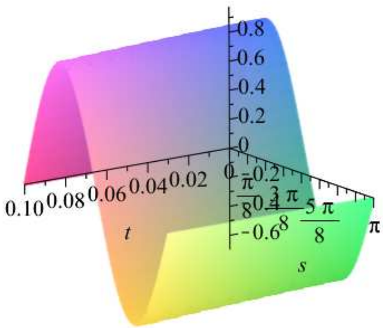

Fig. 4. Numerical solution of the problem (13)–(15) with the initial condition u_{03}

Table 3

Numerical solution of the problem (13)–(15) with the initial condition u_{03}

t	$u_1(t)$	$u_2(t)$
0	0.3072378596	1
0.01	0.2976471122	1.0014492745
0.02	0.2876099716	1.0029146245
0.03	0.2770761929	1.0043962938
0.04	0.2659840256	1.0058945301
0.05	0.2542576421	1.0074095774
0.06	0.2418024366	1.0089416948
0.07	0.2284957567	1.0104911418
0.08	0.2141764598	1.0120581865
0.09	0.1986212588	1.0136431008
0.1	0.1815091065	1.0152461601

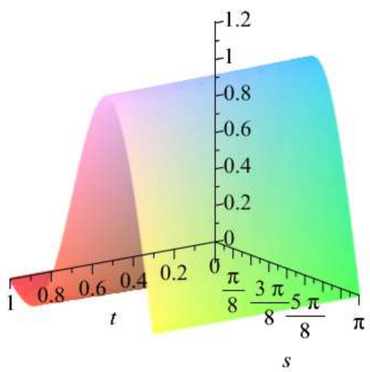

Fig. 5. Numerical solution of the problem (17)–(19) with the initial condition u_{01}

Table 4

Numerical solution of the problem (17)–(19) with the initial condition u_{01}

t	$u_1(t)$	$u_2(t)$
0	1	−0,7397280254
0.1	1.0001103494	−0.7365518125
0.2	1.0004323044	−0.7334156809
0.3	1.0009529363	−0.7303130835
0.4	1.0016602646	−0.7272379018
0.5	1.0025432573	−0.7241844462
0.6	1.0035918297	−0.7211474554
0.7	1.0047968348	−0.7181221005
0.8	1.0061499869	−0.7151039536
0.9	1.0076436087	−0.7120888734
1.0	1.0092705871	−0.7090729879

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ВОПРОСА НЕЕДИНСТВЕННОСТИ РЕШЕНИЯ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ДЕФОРМАЦИИ ДВУТАВРОВОЙ БАЛКИ

О. В. Гаврилова, Н. Г. Николаева

Статья посвящена вопросу о единственности или множественности решений задачи Шоуолтера – Сидорова – Дирихле для уравнения Хоффа на отрезке. Уравнение Хоффа моделирует динамику деформации двутавровой балки, находящейся под постоянной нагрузкой. Для исследования вопроса неединственности решений задачи Шоуолтера – Сидорова будет использован метод фазового пространства, который был разработан Г.А. Свиридюком для исследования разрешимости уравнений соболевского типа. Также ранее было показано, что фазовое пространство исследуемой модели содержит особенности типа 2–сборки Уитни. В статье представлены условия единственности или множественности решений задачи Шоуолтера – Сидорова в зависимости от параметров системы. Построен алгоритм численного решения задачи на основе метода Галеркина и представлены вычислительные эксперименты.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; уравнение Хоффа; неединственность решений; метод фазового пространства; метод Галеркина.

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