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STOCHASTIC BARENBLATT–ZHELTOV–KOCHINA MODEL WITH NEUMANN CONDITION AND MULTIPOINT INITIAL-FINAL VALUE CONDITION

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The article deals with the stochastic Barenblatt–Zheltov–Kochina model with the Neumann condition. We prove trajectory-wise unique solvability of the multipoint initial-final value problem for the considered model in the domain. The article, in addition to the introduction and references, contains three parts. The first and second parts present theoretical information about deterministic and stochastic equations of Sobolev type with the multipoint initial-final value condition. The third part examines the solvability of the Bareblatt–Zheltov–Kochina model with the Neumann condition and the initial-final value condition.

Keywords: Sobolev type equations; additive white noise; relatively bounded operator; stochastic Barenblatt–Zheltov–Kochina model; Neumann condition; multipoint initial-final value condition.

Introduction

Let $G \subset \mathbb{R}^d$ be a bounded domain with the boundary ∂G of the class C^∞. In the cylinder $G \times \mathbb{R}$, consider the Barenblatt–Zheltov–Kochina equation

$$(\lambda - \Delta)u_t = \alpha\Delta u + f \tag{1}$$

with the Neumann condition

$$\frac{\partial u}{\partial \mathbf{n}} u(x,t) = 0, \quad (x,t) \in \partial G \times \mathbb{R}, \tag{2}$$

which models the dynamics of fluid motion in a fractured-porous medium [1]. Here $\mathbf{n} = \mathbf{n}(x)$, $x \in \partial G$, is the unit normal outward to the domain G. The parameters α, λ are real and characterize the environment; the parameter $\alpha \in \mathbb{R}_+$, and the parameter λ can also take negative values in cases where there is no contradiction to the physical meaning of the problem [2]. The function $f = f(x,t)$ describes an external influence. Depending on nature of the external influence, model (1), (2) is either stochastic or deterministic. In the first case, an example of an external influence can be a change in the ambient temperature due to natural phenomena, while in the second case, as a non-random external influence, we can consider a physical process during various methods of oil production, for example, during artificial pressure build-up in deep or hard-to-reach wells in oil fields. Great interest of researchers to equation (1) is also caused by the fact that the equation also describes other physical processes: the process of moisture transfer in soil [3], the process of heat conduction with «two temperatures» [4], and, in addition, the dynamics of some non-Newtonian fluids [5].

This article is devoted to the study of stochastic model (1), (2), which is considered together with the multipoint initial-final value condition [6]

$$P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{0, m}. \tag{3}$$

Here $\tau_j \in \mathbb{R}$, $j = \overline{0, m}$ are such that $\tau_{j+1} > \tau_j$, $\tau_0 \geq 0$; P_j are relatively spectral projectors, which we construct below.

The article, in addition to the introduction and references, contains three parts. The first and second parts present theoretical information about deterministic and stochastic equations of Sobolev type with the multipoint initial-final value condition. The third part examines the solvability of the Bareblatt–Zheltov–Kochina model with the Neumann condition and the initial-final value condition.

1. Sobolev Type Deterministic Equation with Multipoint Initial-Final Value Condition

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, the operators $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ (i.e. linear and continuous) and $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$ (i.e. linear, closed, and densely defined). In addition, suppose that the operator M is (L, p)-bounded [7], $p \in \mathbb{N}_0$ (here and below $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$), then there exist degenerate analytic groups of resolving operators

$$U^t = \frac{1}{2\pi i} \int_\gamma R^L_\mu(M) e^{\mu t} d\mu \quad \text{and} \quad F^t = \frac{1}{2\pi i} \int_\gamma L^L_\mu(M) e^{\mu t} d\mu$$

defined on the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively, and $U^0 \equiv P$, $F^0 \equiv Q$ are projectors. Here γ is a contour bounding the domain D containing the L-spectrum $\sigma^L(M)$ of the operator M; $R^L_\mu(M) = (\mu L - M)^{-1} L$ is the *right*, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the *left L-resolvents* of the operator M. For a degenerate analytic group, we define a *kernel* $\ker U^\bullet = \ker P = \ker U^t$ ($\ker F^\bullet = \ker Q = \ker F^t$) for any $t \in \mathbb{R}$ and an *image* $\operatorname{im} U^\bullet = \operatorname{im} P = \operatorname{im} U^t$ ($\operatorname{im} F^\bullet = \operatorname{im} Q = \operatorname{im} F^t$) for any $t \in \mathbb{R}$. Denote $\mathfrak{U}^0 = \ker U^\bullet$, $\mathfrak{U}^1 = \operatorname{im} U^\bullet$, and $\mathfrak{F}^0 = \ker F^\bullet$, $\mathfrak{F}^1 = \operatorname{im} F^\bullet$, then $\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{U}$ and $\mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}$. Also, denote by L_k (M_k) the restriction of the operator L (M) to $\mathfrak{U}^k$ ($\operatorname{dom} M \cap \mathfrak{U}^k$), $k = 0, 1$.

Consider the following condition:

$$\left.\begin{array}{c} \sigma^L(M) = \bigcup\limits_{j=0}^{m} \sigma^L_j(M),\ m \in \mathbb{N},\ \text{where } \sigma^L_j(M) \neq \emptyset,\ \text{there exists} \\ \text{a cloused contour } \gamma_j \subset \mathbb{C} \text{ that bounds the domain } D_j \supset \sigma^L_j(M), \\ \text{and is such that } \overline{D_j} \cap \sigma^L_0(M) = \emptyset,\ \overline{D_k} \cap \overline{D_l} = \emptyset \text{ for all } j, k, l = \overline{1, m}, k \neq l. \end{array}\right\} \tag{4}$$

Then, the following theorem is true.

Theorem 1. [6] *Let the operator M be (L, p)-bounded, $p \in \mathbb{N}_0$, and condition* (4) *be satisfied. Then (i) there exist degenerate analytic groups*

$$U^t_j = \frac{1}{2\pi i} \int\limits_{\gamma_j} R^L_\mu(M) e^{\mu t} d\mu, \quad F^t_j = \frac{1}{2\pi i} \int\limits_{\gamma_j} L^L_\mu(M) e^{\mu t} d\mu, j = \overline{1, m};$$

(ii) $U^t U^s_j = U^s_j U^t = U^{s+t}_j$, $F^t F^s_j = F^s_j F^t = F^{s+t}_j$ for all s, $t \in \mathbb{R}$, $j = \overline{1, m}$;

(iii) $U_k^t U_l^s = U_l^s U_k^t = \mathbb{O}$, $F_k^t F_l^s = F_l^s F_k^t = \mathbb{O}$ *for all* $s, t \in \mathbb{R}$, $k, l = \overline{1,m}$, $k \neq l$;
(iv) $U_0^t = U^t - \sum\limits_{k=1}^{m} U_k^t$, $F_0^t = F^t - \sum\limits_{k=1}^{m} F_k^t$, *for* $t \in \mathbb{R}$.

Remark 1. By condition (4), construct the units $P_j \equiv U_j^0$, $Q_j \equiv F_j^0$, $j = \overline{0,m}$, of degenerate analytic groups $\{U_j^t : t \in \mathbb{R}\}$, $\{F_j^t : t \in \mathbb{R}\}$, $j = \overline{0,m}$. Obviously, $PP_j = P_jP = P_j$, $QQ_j = Q_jQ = Q_j$, $j = \overline{0,m}$, and $P_kP_l = P_lP_k = \mathbb{O}$, $Q_kQ_l = Q_lQ_k = \mathbb{O}$, $k, l = \overline{0,m}$, $k \neq l$. Therefore, $P_j \in \mathcal{L}(\mathfrak{U})$, $Q_j \in \mathcal{L}(\mathfrak{F})$ are projectors, $j = \overline{0,m}$, which are called *relatively spectral projectors.*

Let us introduce the subspaces $\mathfrak{U}^{1j} = \operatorname{im} P_j$, $\mathfrak{F}^{1j} = \operatorname{im} Q_j$, $j = \overline{0,m}$. By construction,

$$\mathfrak{U}^1 = \bigoplus_{j=0}^{m} \mathfrak{U}^{1j} \text{ and } \mathfrak{F}^1 = \bigoplus_{j=0}^{m} \mathfrak{F}^{1j}.$$

Denote by L_{1j} the restriction of the operator L to $\mathfrak{U}^{1j}$, $j = \overline{0,m}$, and denote by M_{1j} the restriction of the operator M to $\operatorname{dom} M \cap \mathfrak{U}^{1j}$, $j = \overline{0,m}$. Since, as it is easy to show, $P_j\varphi \in \operatorname{dom} M$, if $\varphi \in \operatorname{dom} M$, then the domain $\operatorname{dom} M_{1j} = \operatorname{dom} M \cap \mathfrak{U}^{1j}$ is dense in $\mathfrak{U}^{1j}$, $j = \overline{0,m}$.

Theorem 2. *(Generalised spectral theorem)* [6]. *Suppose that the operators* $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ *and* $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, *while the operator* M *is* (L,p)*-bounded,* $p \in \mathbb{N}_0$, *and condition* (4) *is satisfied. Then*
(i) the operators $L_0 \in \mathcal{L}(\mathfrak{U}^0;\mathfrak{F}^0)$, $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{F}^0)$, $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$;
(ii) the operators $L_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{F}^1)$, $L_{1j} \in \mathcal{L}(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$;
(iii) the operators $M_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{F}^1)$, $M_{1j} \in \mathcal{L}(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$, $j = \overline{0,m}$;
(iv) there exist the operators $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$, $L_{1j}^{-1} \in \mathcal{L}(\mathfrak{F}^{1j};\mathfrak{U}^{1j})$, $j = \overline{0,m}$.

Let $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$, $S_j = L_{1j}^{-1}M_{1j} \in \mathcal{L}(\mathfrak{U}^{1j})$, $j = \overline{0,m}$.

So, let condition (4) be satisfied. Fix $\tau_j \in \mathbb{R}$, $(\tau_j < \tau_{j+1})$, the vectors $u_j \in \mathfrak{U}$, $j = \overline{0,m}$, the vector function $f \in C^\infty(\mathbb{R};\mathfrak{F})$ and consider the linear inhomogeneous equation of Sobolev type

$$L\dot{u} = Mu + f. \tag{5}$$

The vector-function $u \in C^\infty(\mathbb{R};\mathfrak{U})$ satisfying equation (5) is called a *solution to equation* (5). Solution $u = u(t)$ to equation (5), $t \in \mathbb{R}$, satisfying the conditions

$$P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{0,m}, \tag{6}$$

is said to be a *solution to the multipoint initial-final value problem for equation* (5).

Based on Theorem 2, we reduce equation (5) to the system

$$\begin{cases} H\dot{u}^0 = u^0 + M_0^{-1}(\mathbb{I} - Q)f, \\ \dot{u}^{1j} = S_j u^{1j} + L_{1j}^{-1} Q_j f, \quad j = \overline{0,m}, \end{cases} \tag{7}$$

where $u^0 = (\mathbb{I} - P)u$, $u^{1j} = P_j u$, $j = \overline{0,m}$, and each equation is defined on «its own» subspace. From the first equation of (7), by differentiation of the equation and multiplication from the left by H, due to the nilpotency of the operator H we obtain

$$u^0(t) = -\sum_{k=0}^{p} H^K M_0^{-1}(\mathbb{I} - Q) f^{(k)}(t). \tag{8}$$

For the remaining equations of (7), conditions (6) become the Cauchy conditions

$$u^{1j}(\tau_j) = P_j u_j, \quad j = \overline{0, m}. \tag{9}$$

Solving these problems step by step, we obtain

$$u^{1j}(t) = U_j^{t-\tau_j} u_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j f(s) ds, \ j = \overline{0, m}. \tag{10}$$

Therefore, we arrive at

Theorem 3. [6] *Let the operator M be (L, p)-bounded, $p \in \mathbb{N}_0$, and condition* (4) *be satisfied. Then for any $f \in C^{\infty}(\mathbb{R}; \mathfrak{F})$, $u_j \in \mathfrak{U}$, $j = \overline{0, m}$, there exists a unique solution to problem* (5), (6) *of the form*

$$u(t) = -\sum_{k=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q) f^{(k)}(t) + \sum_{j=0}^{m} \left(U_j^{t-\tau_j} u_j + \int_{\tau_j}^{t} U_j^{t-s} L_{1j}^{-1} Q_j f(s) ds \right). \tag{11}$$

2. Sobolev-Type Stochastic Equation with Multipoint Initial-Final Value Condition

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space with the probability measure $\mathbf{P}$ associated with the σ-algebra $\mathcal{A}$ of subsets of the set Ω. Consider a real separable Hilbert space $\mathfrak{U} \equiv (\mathfrak{U}, \langle \cdot, \cdot \rangle)$ with an orthonormal basis $\{\varphi_k\}$ endowed with a Borel σ-algebra.

A measurable mapping $\xi : \Omega \to \mathfrak{U}$ is said to be a *($\mathfrak{U}$-valued) random variable*. Denote by $\mathcal{V} \equiv \mathcal{V}(\Omega; \mathfrak{U})$ the space of ($\mathfrak{U}$-valued) random variables. In the space $\mathcal{V}$, we consider the subspace [8]

$$\mathbf{L_2} \equiv \mathbf{L_2}(\Omega; \mathfrak{U}) = \left\{ \xi \in \mathcal{V} : \int_{\Omega} ||\xi(\omega)||^2 \, d\mathbf{P}(\omega) < +\infty \right\},$$

where $||\xi||^2 = \langle \xi, \xi \rangle$. The space $\mathbf{L_2}$, in particular, contains all *normally distributed* (i.e. Gaussian) random variables from $\mathcal{V}$.

Take the set $\mathfrak{I} \subset \mathbb{R}$ that is some interval. Consider the mapping $f : \mathfrak{I} \to \mathcal{V}$, which associates each $t \in \mathfrak{I}$ with a random variable $\xi \in \mathcal{V}$. Also, consider the mapping $g : \mathcal{V} \times \Omega \to \mathfrak{U}$, which associates each pair (ξ, ω) with the point $\xi(\omega) \in \mathfrak{U}$. The mapping $\eta : \mathfrak{I} \times \Omega \to \mathfrak{U}$ of the form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is said to be a *($\mathfrak{U}$-valued) random process*. Therefore, for each fixed $t \in \mathfrak{I}$, the random process $\eta = \eta(t, \cdot)$ is a random variable, i.e. $\eta(t, \cdot) \in \mathfrak{U}$, which we call a *section* of the random process at the point $t \in \mathfrak{I}$. For each fixed $\omega \in \Omega$, the function $\eta = \eta(\cdot, \omega)$ is called a *(sample) trajectory* of the random process corresponding to the elementary outcome $\omega \in \Omega$. Trajectories are also called *implementations* or *sample functions* of a random process. Usually, when it does not lead to ambiguity, the dependence of $\eta(t, \omega)$ on ω is not indicated and the random process is simply denoted by $\eta(t)$. A random process η is called *continuous* if its trajectories are a.s. (almost sure) continuous, i.e. for a.a. (almost all) $\omega \in \Omega$ the trajectory $\eta(t, \omega)$ is continuous on $\mathfrak{I}$.

Denote the space of random processes by $\mathcal{P} \equiv \mathcal{P}(\mathfrak{I}\times\Omega;\mathfrak{U})$. In $\mathcal{P}$, we consider a subspace $\mathbf{CL_2}$ of continuous random processes whose random variables belong to $\mathbf{L_2}$, i.e. $\eta \in \mathbf{CL_2}$ if $\eta(t) \in \mathbf{L_2}$ for all $t \in \mathfrak{I}$. Note that the space $\mathbf{CL_2}$ contains, in particular, random processes such that all of their trajectories are a.s. continuous, and all (independent) random variables are Gaussian.

Consider a monotonic sequence $K = \{\lambda_k\} \subset \mathbb{R}_+$ such that [9]

$$\sum_{k=1}^{\infty} \lambda_k^2 < +\infty, \tag{12}$$

and also a sequence $\{\xi_k\} = \{\xi_k(\omega)\} \subset \mathbf{L}_2$ of random variables such that $\|\xi_k\|_{\mathbf{L}_2} \leq C$, for some constant $C \in \mathbb{R}_+$ and for all $k \in \mathbb{N}$.

Then we can construct a $\mathfrak{U}$*-valued random* K*-variable* $\xi(\omega) = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k(\omega)\varphi_k$.

Let us introduce the sequence $\{\beta_k(t)\}$, $t \in \overline{\mathbb{R}}_+$ (here and below $\overline{\mathbb{R}}_+ \equiv \{0\} \cup \mathbb{R}_+$) of independent one-dimensional (standard) Wiener processes $\beta_k(t) \equiv \beta_k(t,\omega)$, $\beta_k : \overline{\mathbb{R}}_+ \times \Omega \to \mathbb{R}$, which are also called *Brownian motions*. In the Einstein-Smoluchowski model, the Wiener process describing Brownian motion

$$\beta(t,\omega) = \sum_{k=0}^{\infty} \xi_k(\omega) \sin \frac{\pi(2k+1)}{2} t, \ t \in \overline{\mathbb{R}}_+,$$

is a continuous stochastic process. Here the coefficients $\{\xi_k = \xi_k(\omega)\} \subset \mathbf{L}_2$ are pairwise uncorrelated Gaussian random variables such that $\mathbf{D}\xi_k^2 = \left[\frac{\pi}{2}(2k+1)\right]^{-2}$, $k \in \mathbb{N}_0$.

Definition 1. A random process

$$W(t) \equiv W(t,\omega) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) \varphi_k, \ t \in \overline{\mathbb{R}}_+, \tag{13}$$

is called *a ($\mathfrak{U}$-valued) Wiener* K*-process.*

In Definition 1, the dependence of the Wiener K-process $W = W(t)$ both on the sequence K and on the set of the motion sequence $\{\beta_k(t)\}$ is obvious. Next, we present a number of properties of the Wiener K-process that hold for any sequences K (with the properties described above) and $\{\beta_k(t)\}$:

(W1) $W(0) = 0$ a.a. on Ω, and trajectories are a.s. continuous on $\overline{\mathbb{R}}_+$.

(W2) Trajectories of Wiener K-process are a.s. non-differentiable at any point $t \in \overline{\mathbb{R}}_+$ and have unlimited variation on any interval $\mathfrak{I} \subset \overline{\mathbb{R}}_+$.

(W3) Wiener K-process is Gaussian.

However, these properties clearly imply

Theorem 4. [8] *For any sequence* K *satisfying (12) and a sequence of Brownian motions* $\{\beta_k(t)\}$*, the Wiener* K*-process* $W \in \mathbf{CL_2}$*.*

Suppose that $\mathfrak{U}$ and $\mathfrak{F}$ are real separable Hilbert spaces, K satisfies (12), the operators $L,\ M,\ N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, and the operator M is (L,p)-bounded, $p \in \mathbb{N}_0$. Consider the linear stochastic equation

$$L d\eta = M\eta dt + N\delta W, \tag{14}$$

where, on the right side, δW denotes the generalized differential of a ($\mathfrak{U}$-valued) Wiener K-process. Let condition (4) be satisfied, then we construct the integrals

$$P_j = \frac{1}{2\pi i}\int_{\gamma_j} R^L_\mu(M)d\mu, \quad Q_j = \frac{1}{2\pi i}\int_{\gamma_j} L^L_\mu(M)d\mu, \ j = \overline{1,m}. \tag{15}$$

By virtue of Theorem 1 and Remark 1, in formula (15), the integrals P_j and Q_j, $j = \overline{1,m}$, are projectors in the spaces $\mathfrak{U}$ and $\mathfrak{F}$, respectively. As in Remark 1, we construct the projectors

$$P_0 = P - \sum_{j=1}^{m} P_j \ \text{ and } Q_0 = Q - \sum_{j=1}^{m} Q_j.$$

Next, on the half-interval $\overline{\mathbb{R}}_+$, we choose points $\tau_j \in \mathbb{R}$, $j = \overline{0,m}$, such that $\tau_j < \tau_{j+1}$, $0 \le \tau_0$, and ($\mathfrak{U}$-valued) pairwise independent random variables $\xi_j \in \mathbf{L_2}$, $j = \overline{0,m}$. Similarly to Section 1, we formulate the multipoint initial-final value problem as follows: find a random process $\eta \in \mathbf{CL_2}$ that satisfies equation (14) and the conditions

$$P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{0,m}. \tag{16}$$

If the condition

$$QN = N \tag{17}$$

holds, then, by virtue of Theorem 3, it is easy to construct a unique "formal" solution $\eta = \eta(t)$ to problem (14), (16)

$$\begin{aligned}\eta(t) = \sum_{j=0}^{m}\Big(U_j^{t-\tau_j}P_j\xi_j + L_{1j}^{-1}Q_j(W(t) - W(\tau_j))\Big) + \\ + \sum_{j=0}^{m}\int_{\tau_j}^{t} U_j^{t-s}S_jL_{1j}^{-1}Q_jW(s)ds,\end{aligned} \tag{18}$$

where, as well as above, U_j^t, L_{1j}, S_j, $j = \overline{0,m}$, are the same as in Section 1, see Theorem 2. The "formality" of the obtained solution is that the integrands are, generally speaking, non-integrable vector functions, therefore integration is "formal".

Theorem 5. *Let the operator M be (L,p)-bounded, $p \in \mathbb{N}_0$, and let conditions (4), (17) be satisfied. Let the random variables $\xi_j \in \mathbf{L_2}$, $j = \overline{0,m}$, be pairwise independent. Then the random process η defined by formula (18) belongs to $\mathbf{CL_2}(\overline{\mathbb{R}}_+)$.*

Remark 2. The requirement for pairwise independence of the random variables ξ_j, $j = \overline{0,m}$ is redundant. It is sufficient to have pairwise independence of their projectors $P_j\xi_j$, $j = \overline{0,m}$.

Definition 2. Let the operator M be (L,p)-bounded, $p \in \mathbb{N}_0$, conditions (4) and (17) be satisfied. Let the random variables $\xi_j \in \mathbf{L_2}$ (or their projectors $P_j\xi_j \in \mathbf{L_2}$), $j = \overline{0,m}$. Then for any ($\mathfrak{U}$-valued) Wiener K-process $W \in \mathbf{CL_2}$ the random process η defined by formula (18) is called a *solution to problem (14), (16).*

Remark 3. In modern mathematical literature, such a solution is often called "mild" (mild solution). It is clear that if we restrict ourselves to the "classical" interpretation of the derivative, then due to Property (W2) we cannot count on a smoother solution.

3. Stochastic Barenblatt–Zheltov–Kochina Model with Neumann Condition and Multipoint Initial-Final Value Condition

Let $G \subset \mathbb{R}^d$ be a bounded domain with the boundary ∂G of the class C^∞. Let us define the spaces [10]

$$\mathfrak{U} = \{u \in W_2^{l+2}(G) : \frac{\partial u(x)}{\partial \mathbf{n}} = 0, \, x \in \partial G\}, \quad \mathfrak{F} = W_2^l(G), \quad l \in \mathbb{N}.$$

Define a monotonic sequence $K = \{\lambda_k\} \subset \mathbb{R}_+$ satisfying condition (12) such that $\lambda_k^2 = \nu_k^{-2d}$. By formula (13), we define the Wiener K-process, where $\{\varphi_k\}$ is the orthonormal basis of $\mathfrak{U}$. Note that the Laplace operator $-\Delta : \mathfrak{U} \to \mathfrak{F}$ is a toplinear isomorphism. We construct the space

$$\mathcal{V} = \{u \in W_2^{l+2d} : \, (-1)^l \Delta^l u(x) = 0, \, \frac{\partial u(x)}{\partial \mathbf{n}} = 0, \, x \in \partial G, \; l = \overline{0, d-1}.\}$$

Finally, the formulas $L = \lambda - \Delta$ and $M = \alpha\Delta$ define the continuous linear operators L, $M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, which are Fredholm, $\alpha \in \mathbb{R} \setminus \{0\}$. A detailed discussion of this range of questions can be found in Triebel's fundamental reference book.

We look for a random process $\eta = \eta(x,t)$ satisfying the stochastic equation

$$Ldu = udt + N\delta W, \; L = (\lambda - \Delta), \; M = \alpha\Delta \tag{19}$$

and the Neumann condition

$$\frac{\partial u(x,t)}{\partial \mathbf{n}} = 0, \; (x,t) \in \partial G \times \mathbb{R}_+ \tag{20}$$

in the cylinder $G \times \mathbb{R}_+$.

Let $\{\mu_k\}$ be the eigenvalues of the spectral problem $-\Delta\varphi_k = \mu_k\varphi_k$ in the domain G with condition (20). Then we arrive at

Lemma 1. *For any $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$ the operator M is $(L,0)$-bounded.*

Proof. The statement is trivial if $-\lambda \notin \{\mu_k\}$. The kernel $\ker L = \operatorname{span}\{\varphi_k : \mu_k = -\lambda\}$ if $-\lambda \in \{\mu_k\}$. Take the vector $\psi = \sum\limits_{-\lambda = \nu_k} a_k\varphi_k \in \ker L$. Then $M\psi = -\alpha\lambda\psi \notin \operatorname{im} L$, i.e. the operator L has no M-adjoined vectors. Reference to Theorem 1.1.2 [7] completes the proof.

Further note that

$$R_\mu^L(M) = \sum_{-\lambda \neq \mu_k} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu + \alpha\mu_k(\lambda + \mu_k)^{-1}}, \quad L_\mu^L(M) = \sum_{-\lambda \neq \mu_k} \frac{[\cdot, \varphi_k]\varphi_k}{\mu + \alpha\mu_k(\lambda + \mu_k)^{-1}},$$

where $[\cdot,\cdot]$ is the inner product in $\mathfrak{F}$. By the formulas

$$P = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)d\mu \text{ and } Q = \frac{1}{2\pi i}\int_\gamma L_\mu^L(M)d\mu,$$

construct the projectors

$$P = \sum_{-\lambda \neq \mu_k} \langle \cdot, \varphi_k \rangle \varphi_k, \; Q = \sum_{-\lambda \neq \mu_k} [\cdot, \varphi_k]\varphi_k.$$

For the sake of simplicity, we set the operator $N = P$. Then, firstly, the operator $N \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ (moreover, N is compact!) due to the dense and continuous (moreover, compact!) embedding $\mathfrak{U} \hookrightarrow \mathfrak{F}$ (Sobolev – Kondrashev's theorem). Secondly, condition (17) holds automatically. Since, in this situation, $\sigma_k = -\alpha\mu_k(\lambda + \mu_k)^{-1}$ for $-\lambda \neq \mu_k$ represent points of the L-spectrum $\sigma^L(M)$ of the operator M, which in turn satisfies condition (4), we can construct the projectors

$$P_j = \sum_{\sigma_k \in \sigma_j^L(M)} \langle \cdot, \varphi_k \rangle \varphi_k, \quad Q_j = \sum_{\sigma_k \in \sigma_j^L(M)} [\cdot, \varphi_k] \varphi_k, \ j = \overline{0, m}.$$

Therefore, the reduction of Barenblatt – Zheltov – Kochina equation (19) with condition (20) to equation (14) with additive white noise is completed.

Let us proceed to the construction of a "mild" solution (18) (see Remark 3). First of all, note that, in Remark 2, the condition $P_j\xi_j = \xi_j$, $j = \overline{1, m}$, on the initial random variables ξ_j from (16) is equivalent to the condition

$$\langle \xi_j, \varphi_k \rangle = 0, \ -\lambda = \mu_k. \tag{21}$$

Next, in this situation, the first term in (17) has the form

$$U_j^t \xi_j = \sum_{\sigma_k \in \sigma_j^L(M)} \langle \xi_j, \varphi_k \rangle e^{\sigma_k t} \varphi_k, \ j = \overline{0, m}. \tag{22}$$

The second term in (17) can also be easily calculated as follows:

$$L_{1j}^{-1} N W(t) = \sum_{\sigma_k \in \sigma_j^L(M)} \frac{\beta_k(t)}{(\lambda + \mu_k)\nu_k^{2d}} \varphi_k. \tag{23}$$

Finally, the last term in (17) can be found as

$$\sum_{\sigma_k \in \sigma_j^L(M)} \int_{\tau_j}^t U_j^{t-s} S_j L_1^{-1} N W(s) ds = \sum_{\sigma_k \in \sigma_j^L(M)} \sum_{-\lambda \neq \mu_k} \int_0^t \frac{\beta_k(s) e^{\sigma_k (t-s)} ds}{(\lambda + \mu_k)\nu_k^{2m-1}} \varphi_k. \tag{24}$$

Therefore, the following theorem is proved.

Theorem 6. *For any $-\lambda \in \{\mu_k\}$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\xi_j \in \mathbf{L_2}$ such that (21) holds, there exists a unique solution $u \in \mathbf{CL_2}$ to problem (16) for the stochastic Barenblatt – Zheltov – Kochina equation with additive white noise and condition (20), and the solution has the form (17), where the terms are represented by formulas (22) – (24).*

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СТОХАСТИЧЕСКАЯ МОДЕЛЬ БАРЕНБЛАТТА–ЖЕЛТОВА–КОЧИНОЙ С УСЛОВИЕМ НЕЙМАНА И МНОГОТОЧЕЧНЫМ НАЧАЛЬНО-КОНЕЧНЫМ УСЛОВИЕМ

Л. А. Ковалева, А. С. Конкина, С. А. Загребина

В статье рассматривается стохастическая модель Баренблатта – Желтова – Кочиной с условием Неймана. Доказывается потраекторная однозначная разрешимость многоточечной начально-конечной задачи для рассматриваемой модели в области. Статья, кроме введения и списка литературы, содержит три части. В первой и второй частях приводятся теоретические сведения о детерминированных и стохастических уравнениях соболевского типа и многоточечным начально-конечным условием. В третьей части исследуется разрешимость модели Бареблатта – Желтова – Кочиной с условием Неймана и начально-конечным условием.

Ключевые слова: уравнения соболевского типа; аддитивный белый шум, относительно ограниченный оператор; стохастическая модель Баренблатта – Желтова – Кочиной; условие Неймана; многоточечная начально-конечное условие.

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