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ANALYSIS OF THE PROBLEM FOR THE BIHARMONIC EQUATION

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For a biharmonic equation, we consider a mixed problem with the main boundary conditions. We continue the original problem along the boundary with the Dirichlet conditions into a rectangular domain. The continued problem is given as an operator equation. The method of iterative extensions is written out in the operator form when solving the continued problem. The operator continued problem is given on a finite-dimensional subspace. The method of iterative extensions is given for solving the operator continued problem on a finite-dimensional subspace. After discretization, the continued problem is written in the matrix form. The continued problem in the matrix form is solved by the method of iterative extensions in the matrix form. It is established that in the cases under consideration the method of iterative extensions has relative errors converging as a geometric progression in a norm stronger than the energy norm of the extended problem. In the applied iterative processes, the iterative parameters are selected on the basis of minimizing the residuals. We give conditions that guarantee the convergence of the iterative processes used. Also, we present an algorithm that implements the method of iterative extensions in the matrix form. The algorithm performs an independent selection of iterative parameters and provides a criterion for stopping if an estimate of the required accuracy is achieved. A computational example of using the method of iterative extensions on a computer is given.

Keywords: biharmonic equation; method of iterative extensions.

Introduction

Let us consider a boundary problem under the obligatory presence of homogeneous main boundary conditions for a biharmonic equation in a bounded flat domain. The main problems in solving the problem under consideration are due to the complexity of the geometry of the domain, the order of the equation, and the Dirichlet boundary conditions [1–5]. We assume that the proposed methods must be computationally stable with respect to rounding errors, be asymptotically optimal in terms of computational complexity, be quite universal and have a simple implementation in computer calculations. To fulfill these conditions in solving the original problem, we propose the method of iterative extensions as a development of the fictitious component method [4–7]. Note that to solve problems in a rectangular domain, which we obtain when solving the original problems, we can, for example, use the well-known marching methods, which are optimal in terms of computational complexity [8–10].

1. Boundary Problem

Suppose that there exists a first bounded domain. Select a second bounded domain.

$$\omega \in \{1,\ \mathrm{II}\}\,,\ \Omega_\omega \subset \mathrm{R}^2.$$

The intersection of the first and second domains is empty, and the union of the closures of these domains is the closure of the rectangular domain.

$$\Omega_1 \cap \Omega_{\mathrm{II}} = \emptyset,\ \bar{\Omega}_1 \cup \bar{\Omega}_{\mathrm{II}} = \bar{\Pi}.$$

For all domains, the boundary is the closure of the union of four non-intersecting open parts.

$$\partial\Pi = \bar{s},\ s = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,\ \Gamma_i \cap \Gamma_j = \emptyset,\ i \neq j,\ i,j = 0,1,2,3,$$

$$\partial\Omega_\omega = \bar{s}_\omega,\ s_\omega = \Gamma_{\omega,0} \cup \Gamma_{\omega,1} \cup \Gamma_{\omega,2} \cup \Gamma_{\omega,3},\ \Gamma_{\omega,i} \cap \Gamma_{\omega,j} = \emptyset,\ i \neq j,\ i,j = 0,1,2,3.$$

We assume that a non-empty intersection of the boundary of the first domain and the boundary of the second domain is a closure of the intersection of the corresponding parts of the boundaries of these domains.

$$\partial\Omega_1 \cap \partial\Omega_{\text{II}} = \bar{S},\ S = \Gamma_{1,0} \cap \Gamma_{\text{II},3} \neq \emptyset.$$

All parts of the boundaries of the domains are obtained as the union of a finite number of open non-intersecting arcs of smooth curves. The boundaries of the domains do not have self-contacts and self-intersections. In the first domain, we consider a mixed problem for the Sophie Germain equation. In the second domain, we present a mixed problem for the screened homogeneous Sophie Germain equation. The original problem is set on the first domain. On the second domain, a fictitious problem with a zero solution is given. Let us present the problem to be solved and an additional fictitious problem.

$$\begin{array}{c} \Delta^2 \breve{u}_\omega + a_\omega \breve{u}_\omega = \breve{f}_\omega,\ a_1 = 0,\ a_{\text{II}} \geq 0,\ \breve{f}_{\text{II}} = 0, \\ \breve{u}_\omega \big|_{\Gamma_{\omega,0}} = \frac{\partial \breve{u}_\omega}{\partial n} \big|_{\Gamma_{\omega,0}} = 0,\ \breve{u}_\omega \big|_{\Gamma_{\omega,1}} = l_1 \breve{u}_\omega \big|_{\Gamma_{\omega,1}} = 0, \\ \frac{\partial \breve{u}_\omega}{\partial n} \big|_{\Gamma_{\omega,2}} == l_2 \breve{u}_\omega \big|_{\Gamma_{\omega,2}} 0,\ l_1 \breve{u}_\omega \big|_{\Gamma_{\omega,3}} = l_2 \breve{u}_\omega \big|_{\Gamma_{\omega,3}} = 0, \end{array} \tag{1}$$

where we use differential operators with derivatives with respect to normals and tangents on the corresponding parts of the boundaries.

$$l_1 \breve{u}_\omega = \Delta \breve{u}_\omega + (1 - \sigma_\omega) n_1 n_2 \breve{u}_{\omega xy} - n_2^2 \breve{u}_{\omega xx} - n_1^2 \breve{u}_{\omega yy},$$

$$l_2 \breve{u}_\omega = \frac{\partial \Delta \breve{u}_\omega}{\partial n} \Delta \breve{u}_\omega + (1 - \sigma_\omega) \frac{\partial}{\partial s} (n_1 n_2 (\breve{u}_{\omega yy} - \breve{u}_{\omega xx}) + (n_1^2 - n_2^2) \breve{u}_{\omega xy}),$$

$$n_1 = -\cos(n, x),\ n_2 = -\cos(n, y),\ \sigma_\omega \in (0; 1).$$

Let us rewrite the above problems in the variational form. This is a representation for linear functionals as inner products on function spaces.

$$\breve{u}_\omega \in \breve{H}_\omega :\ \ \Lambda_\omega(\breve{u}_\omega, \breve{v}_\omega) = F_\omega(\breve{v}_\omega),\ \forall \breve{v}_\omega \in \breve{H}_\omega. \tag{2}$$

The solutions to such problems belong to the Sobolev space.

$$\breve{H}_\omega = \breve{H}_\omega(\Omega_\omega) = \left\{ \breve{v}_\omega \in W_2^2(\Omega_\omega) :\ \breve{v}\big|_{\Gamma_{\omega,0} \cup \Gamma_{\omega,1}} = 0,\ \frac{\partial \breve{v}_\omega}{\partial n}\bigg|_{\Gamma_{\omega,0} \cup \Gamma_{\omega,2}} = 0 \right\}.$$

The right parts of these problems are linear functionals.

$$F_\omega(\breve{v}_\omega) = (\breve{f}_\omega, \breve{v}_\omega),\ (\breve{f}_\omega, \breve{v}_\omega) = \int_{\Omega_\omega} \breve{f}_\omega \breve{v}_\omega d\Omega_\omega.$$

The left parts of these problems contain bilinear forms.

$$\Lambda_\omega(\breve{u}_\omega, \breve{v}_\omega) = \int\limits_{\Omega_\omega} (\sigma_\omega \Delta \breve{u}_\omega \Delta \breve{v}_\omega + (1 - \sigma_\omega)(\breve{u}_{\omega xx} \breve{v}_{\omega yy} + 2 \breve{u}_{\omega xy} \breve{v}_{\omega xy} + \breve{u}_{\omega yy} \breve{v}_{\omega yy}) + a_\omega \breve{u}_\omega \breve{v}_\omega) d\Omega_\omega.$$

We assume that bilinear forms define a norm equivalent to the norm of Sobolev spaces.

$$\exists c_1, c_2 > 0: \; c_1 \left\| \breve{v}_\omega \right\|^2_{W_2^2(\Omega_\omega)} \leq \Lambda_\omega(\breve{v}_\omega, \breve{v}_\omega) \leq c_2 \left\| \breve{v}_\omega \right\|^2_{W_2^2(\Omega_\omega)}, \; \forall \breve{v}_\omega \in \breve{H}_\omega.$$

These assumptions ensure the existence and uniqueness of solutions to each of these two problems [1]. The solution to the fictitious problem will be zero.

2. Continued Problem

We present the formulation of the problem to be solved together with the fictitious problem in the variational and operator form. We call this problem the continued problem.

$$\begin{gathered} \breve{u} \in \breve{V}: \; \Lambda_1(\breve{u}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}, \breve{v}) = F_1(I_1\breve{v}), \; \forall \breve{v} \in \breve{V}, \\ \breve{u} \in \breve{V}: \; \breve{B}\breve{u} = \breve{f}, \end{gathered} \tag{3}$$

if we specify the right side and the operator in the continued problem in this way.

$$(\breve{B}\breve{u}, \breve{v}) = \Lambda_1(\breve{u}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}, \breve{v}), \; \forall \breve{u}, \breve{v} \in \tilde{V}, \quad (\breve{f}, \breve{v}) = F_1(I_1\breve{v}), \; \forall \breve{v} \in \tilde{V}.$$

$$F(\breve{v}) = (\breve{f}, \breve{v}), \; (\breve{f}, \breve{v}) = \int\limits_\Pi \breve{f}\breve{v} d\Pi.$$

The solution to the continued problem belongs to the following extended space of solutions.

$$\breve{V} = \breve{V}(\Pi) = \left\{ \breve{v} \in W_2^2(\Pi): \; \breve{v}|_{\Gamma_0 \cup \Gamma_1} = 0, \; \left. \frac{\partial \breve{v}}{\partial n} \right|_{\Gamma_0 \cup \Gamma_2} = 0 \right\}.$$

In the extended space of solutions, there exists a subspace that is the space of solutions to the continued problem. This is the space of solutions to the original problem on the first domain that is continued by zero on the second domain.

$$\breve{V}_1 = \breve{V}_1(\Pi) = \left\{ \breve{v}_1 \in \breve{V}: \; \breve{v}_1|_{\Pi \setminus \Omega_1} = 0 \right\}.$$

In the continued problem, we use an operator that projects the extended space of solutions to the continued problem onto the space of solutions to the continued problem.

$$I_1 : \breve{V} \mapsto \breve{V}_1, \; \breve{V}_1 = \operatorname{im} I_1, \; I_1 = I_1^2.$$

The extended space of solutions consists of the following subspaces.

$$\breve{V}_3 = \breve{V}_3(\Pi) = \left\{ \breve{v}_3 \in \breve{V}: \; \breve{v}_3|_{\Pi \setminus \Omega_{\mathrm{II}}} = 0 \right\},$$

$$\breve{V}_2 = \breve{V}_2(\Pi) = \left\{ \breve{v}_2 \in \breve{V}: \; \Lambda(\breve{v}_2, \breve{v}_1) = 0, \; \forall \breve{v}_1 \in \breve{V}_1, \; \Lambda(\breve{v}_2, \breve{v}_3) = 0, \; \forall \breve{v}_3 \in \breve{V}_3 \right\},$$

$$\breve{V} = \breve{V}_1 \oplus \breve{V}_{\mathrm{II}}, \; \breve{V}_{\mathrm{II}} = \breve{V}_2 \oplus \breve{V}_3.$$

We use the bilinear form as a sum of bilinear forms.

$$\Lambda(\breve{u}, \breve{v}) = \Lambda_1(\breve{u}, \breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}, \breve{v}), \; \forall \breve{u}, \breve{v} \in \breve{V}.$$

We assume that the bilinear form defines an equivalent normalization of the Sobolev space on the extended space.

$$\exists c_1, c_2 > 0: \ c_1 \|\breve{v}\|^2_{W_2^2(\Pi)} \le \Lambda(\breve{v},\breve{v}) \le c_2 \|\breve{v}\|^2_{W_2^2(\Pi)}, \ \forall \breve{v} \in \breve{V}.$$

Also, we assume that for the Sobolev spaces used, the continuation of functions takes place under the same norm. Let us use this statement in the usual way.

$$\exists \breve{\beta}_1 \in (0,1], \ \breve{\beta}_2 \in [\breve{\beta}_1, 1]: \ \breve{\beta}_1 \Lambda(\breve{v}_2,\breve{v}_2) \le \Lambda_{\mathrm{II}}(\breve{v}_2,\breve{v}_2) \le \breve{\beta}_2 \Lambda(\breve{v}_2,\breve{v}_2), \ \forall \breve{v}_2 \in \breve{V}_2,$$

$$\exists \breve{\beta}_1 \in (0,1], \ \breve{\beta}_2 \in [\breve{\beta}_1, 1]: \ \breve{\beta}_1 (\breve{\Lambda}\breve{v}_2,\breve{v}_2) \le (\breve{\Lambda}_{\mathrm{II}}\breve{v}_2,\breve{v}_2) \le \breve{\beta}_2 (\breve{\Lambda}\breve{v}_2,\breve{v}_2), \ \forall \breve{v}_2 \in \breve{V}_2,$$

where the considered operators are defined as follows:

$$\breve{\Lambda} = \breve{\Lambda}_{\mathrm{I}} + \breve{\Lambda}_{\mathrm{II}}, \ (\breve{\Lambda}\breve{u},\breve{v}) = \Lambda(\breve{u},\breve{v}), \ (\breve{\Lambda}_{\mathrm{I}}\breve{u},\breve{v}) = \Lambda_1(\breve{u},\breve{v}), \ (\breve{\Lambda}_{\mathrm{II}}\breve{u},\breve{v}) = \Lambda_{\mathrm{II}}(\breve{u},\breve{v}), \ \forall \breve{u},\breve{v} \in \breve{V}.$$

Then the continued problem has a unique solution. For the continued problem, the solution to the original problem on the first domain is extended by zero to the second domain, i.e. to the rest of the rectangular domain. Note that the solution to the original problem and the solution to the original problem under extension by zero are denoted in the same way as the initial function and the continued function, respectively.

3. Operator Form of Method of Iterative Extensions

Consider the modified method of fictitious components as an iterative process, where at each step an extended problem arises with a bilinear form of the continued problem without a projection operator. The solution to such a problem belongs to the extending of the solution space for the continued problem, i.e. to the solution space for the extended problem.

$$\begin{aligned}
&\breve{u}^k \in \breve{V}: \Lambda(\breve{u}^k - \breve{u}^{k-1},\breve{v}) = -\tau_{k-1}(\Lambda_1(\breve{u}^{k-1}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}^{k-1},\breve{v}) - \\
&\qquad\qquad - F_1(I_1\breve{v})), \ \forall \breve{v} \in \breve{V}, \ k \in \mathbb{N}, \\
&\breve{u}^k \in \breve{V}: \breve{\Lambda}(\breve{u}^k - \breve{u}^{k-1}) = -\tau_{k-1}(\breve{B}\breve{u}^{k-1} - \breve{f}), \ k \in \mathbb{N}, \\
&\forall \breve{u}^0 \in \breve{V}, \tau_0 = 1, \tau_{k-1} = 2/(\breve{\beta}_1 + \breve{\beta}_2), \ k \in \mathbb{N} \setminus \{1\}.
\end{aligned} \tag{4}$$

Let us formulate the method of iterative extensions as a development of the modified method of fictitious components. We describe an iterative method, when at each step we solve an extended problem with the following bilinear form, with an operator generated by such a bilinear form.

$$C(\breve{u},\breve{v}) = \Lambda_1(\breve{u},\breve{v}) + \breve{\gamma}\Lambda_{\mathrm{II}}(\breve{u},\breve{v}), \ \forall \breve{u},\breve{v} \in \breve{V}, \ \breve{\gamma} \in (0;+\infty),$$

$$(\breve{C}\breve{u},\breve{v}) = C(\breve{u},\breve{v}), \ \forall \breve{u},\breve{v} \in \breve{V}.$$

The solution to such a problem belongs to the space of solutions to the extended problem, to the extended space of solutions to the continued problem.

$$\begin{aligned}
&\breve{u}^k \in \breve{V}: C(\breve{u}^k - \breve{u}^{k-1},\breve{v}) = -\tau_{k-1}(\Lambda_1(\breve{u}^{k-1}, I_1\breve{v}) + \Lambda_{\mathrm{II}}(\breve{u}^{k-1},\breve{v}) - \\
&\qquad\qquad - F_1(I_1\breve{v})), \ \forall \breve{v} \in \breve{V}, k \in \mathbb{N}, \\
&\breve{u}^k \in \breve{V}: \breve{C}(\breve{u}^k - \breve{u}^{k-1}) = -\tau_{k-1}(\breve{B}\breve{u}^{k-1} - \breve{f}), \ k \in \mathbb{N}, \\
&\forall \breve{u}^0 \in \breve{V}_1, \breve{\gamma} > \breve{\alpha}, \tau_0 = 1, \tau_{k-1} = (\breve{r}^{k-1},\breve{\eta}^{k-1})/(\breve{\eta}^{k-1},\breve{\eta}^{k-1}), \ k \in \mathbb{N} \setminus \{1\}.
\end{aligned} \tag{5}$$

To determine the iterative parameters, we calculate residuals, corrections, and equivalent residuals.

$$\breve{r}^{k-1} = \breve{B}\breve{u}^{k-1} - \breve{f},\ \breve{w}^{k-1} = \breve{C}^{-1}\breve{r}^{k-1},\ \breve{\eta}^{k-1} = \breve{B}\breve{w}^{k-1},\ k \in \mathbb{N}.$$

We suppose that the assumptions for the continuation of functions, which we write in the following form, take place.

$$\exists\breve{\delta}_1 \in (0,+\infty), \breve{\delta}_2 \in [\breve{\delta}_1,+\infty) : \breve{\delta}_1^2(\breve{C}\breve{v}_2,\breve{C}\breve{v}_2) \le (\breve{\Lambda}_{\mathrm{II}}\breve{v}_2,\breve{\Lambda}_{\mathrm{II}}\breve{v}_2) \le \breve{\delta}_2^2(\breve{C}\breve{v}_2,\breve{C}\breve{v}_2),\ \forall\breve{v}_2 \in \breve{V}_2,$$

$$\exists\breve{\alpha} \in (0,+\infty) : (\breve{\Lambda}_{\mathrm{I}}\breve{v}_2,\breve{\Lambda}_{\mathrm{I}}\breve{v}_2) \le \breve{\alpha}^2(\breve{\Lambda}_{\mathrm{II}}\breve{v}_2,\breve{\Lambda}_{\mathrm{II}}\breve{v}_2),\ \forall\breve{v}_2 \in \breve{V}_2,$$

if the introduced operators are defined in the following way.

$$\breve{C} = \breve{\Lambda}_{\mathrm{I}} + \gamma\breve{\Lambda}_{\mathrm{II}},\ (\breve{C}\breve{u},\breve{v}) = C(\breve{u},\breve{v}),\ (\breve{\Lambda}_{\mathrm{I}}\breve{u},\breve{v}) = \Lambda_{\mathrm{I}}(\breve{u},\breve{v}),\ (\breve{\Lambda}_{\mathrm{II}}\breve{u},\breve{v}) = \Lambda_{\mathrm{II}}(\breve{u},\breve{v}),\ \forall\breve{u},\breve{v} \in \breve{V}.$$

4. Operator Form of Continued Problem on Finite-Dimensional Subspace

Consider discretization of the continued problem with a specific type of boundary conditions. Consider the introduced rectangular domain with parts of its boundary in the rectangular coordinates

$$\begin{array}{l}\Pi = (0,b_1)\times(0,b_2), \Gamma_0 = \emptyset,\\ \Gamma_1 = \{b_1\}\times(0,b_2)\cup(0,b_1)\times\{b_2\},\\ \Gamma_2 = \{0\}\times(0,b_2)\cup(0,b_1)\times\{0\},\\ \Gamma_3 = \emptyset,\ b_1,b_2 \in (0,+\infty).\end{array}$$

Let us define a grid in a rectangular domain.

$$(x_i, y_j) = ((i-0.5)h_1, (j-0.5)h_2),$$

where $h_1 = b_1/(m+0.5)$, $h_2 = b_2/(n+0.5)$, $i = \overline{1,m}, j = \overline{1,n},\ m,n \in \mathbb{N}$. Introduce grid functions on the set of nodes of the considered grid

$$v_{i,j} = v(x_i,y_j) \in \mathrm{R},\ i = \overline{1,m},\ j = \overline{1,n},\ m,n \in \mathbb{N}.$$

Complete the grid functions using the parabolic basis functions.

$$\begin{array}{c}\Phi^{i,j}(x,y) = \Psi^{1,i}(x)\Psi^{2,j}(y),\ i = \overline{1,m}, j = \overline{1,n},\ m,n \in \mathbb{N},\\ \Psi^{1,i}(x) = [1/i]\Psi(x/h_1 - i + 3) + \Psi(x/h_1 - i + 2) - [i/m]\Psi(x/h_1 - i),\\ \Psi^{2,j}(y) = [1/j]\Psi(y/h_2 - j + 3) + \Psi(y/h_2 - j + 2) - [j/n]\Psi(y/h_2 - j),\\ \Psi(z) = \begin{cases} 0.5z^2, z \in [0,1],\\ -z^2 + 3z - 1.5, z \in [1,2],\\ 0.5z^2 - 3z + 4.5, z \in [2,3],\\ 0, z \notin (0,3).\end{cases}\end{array}$$

The basis functions are assumed to be equal to zero outside the considered rectangular domain

$$\Phi^{i,j}(x,y) = 0,\ (x,y) \notin \Pi, i = \overline{1,m}, j = \overline{1,n},\ m,n \in \mathbb{N}.$$

In the solution space for the extended problem, linear combinations of basis functions form a finite-dimensional subspace.

$$\tilde{V} = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} v_{i,j} \Phi^{i,j}(x;y) \right\} \subset \breve{V}.$$

We present the continued problem on the introduced finite-dimensional subspace in the variational and operator form

$$\begin{aligned} &\tilde{u} \in \tilde{V}: \quad \Lambda_1(\tilde{u}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}) = F_1(I_1\tilde{v}), \ \forall \tilde{v} \in \tilde{V}, \\ &\tilde{u} \in \tilde{V}: \quad \tilde{B}\tilde{u} = \tilde{f}, \end{aligned} \tag{6}$$

if we define the right side and the operator in the continued problem in the following form.

$$(\tilde{B}\tilde{u}, \tilde{v}) = \Lambda_1(\tilde{u}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}), \ \forall \tilde{u}, \tilde{v} \in \tilde{V}, \ (\tilde{f}, \tilde{v}) = F_1(I_1\tilde{v}), \ \forall \tilde{v} \in \tilde{V}.$$

A finite-dimensional space contains a finite-dimensional subspace for solutions to the continued problem. This is a finite-dimensional subspace of solutions to the original problem on the first domain, which is continued by zero on the complement to the rectangular domain.

$$\tilde{V}_1 = \tilde{V}_1(\Pi) = \left\{ \tilde{v}_1 \in \tilde{V}: \ \tilde{v}_1|_{\Pi \setminus \Omega_1} = 0 \right\}.$$

We assume that the projection operator acts on the corresponding finite-dimensional subspaces in the same way as before. In conclusion, we assume that, on the space of solutions to the continued problem, the projection operator sets to zero the coefficients of the basis functions whose carriers do not completely belong to the first domain.

$$I_1 : \tilde{V} \mapsto \tilde{V}_1, \ \tilde{V}_1 = \operatorname{im} I_1, \ I_1 = I_1^2.$$

We also define finite-dimensional subspaces corresponding to the previously introduced subspaces.

$$\tilde{V}_3 = \tilde{V}_3(\Pi) = \left\{ \tilde{v}_3 \in \tilde{V} : \tilde{v}_3|_{\Pi \setminus \Omega_{\mathrm{II}}} = 0 \right\},$$

$$\tilde{V}_2 = \tilde{V}_2(\Pi) = \left\{ \tilde{v}_2 \in \tilde{V} : \mathrm{A}(\tilde{v}_2, \tilde{v}_1) = 0, \forall \tilde{v}_1 \in \tilde{V}_1, \mathrm{A}(\tilde{v}_2, \tilde{v}_3) = 0, \forall \tilde{v}_3 \in \tilde{V}_3 \right\}, \tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 \oplus \tilde{V}_3.$$

We suppose that for finite-dimensional subspaces, the assumptions for the continuation of functions in the same form are satisfied.

$$\exists \tilde{\beta}_1 \in (0,1], \ \tilde{\beta}_2 \in [\tilde{\beta}_1, 1]: \ \tilde{\beta}_1 \Lambda(\tilde{v}_2, \tilde{v}_2) \le \Lambda_{\mathrm{II}}(\tilde{v}_2, \tilde{v}_2) \le \tilde{\beta}_2 \Lambda(\tilde{v}_2, \tilde{v}_2) \ \forall \tilde{v}_2 \in \tilde{V}_2,$$

$$\exists \tilde{\beta}_1 \in (0,1], \ \tilde{\beta}_2 \in [\tilde{\beta}_1, 1]: \ \tilde{\beta}_1 (\tilde{\Lambda}\tilde{v}_2, \tilde{v}_2) \le (\tilde{\Lambda}_{\mathrm{II}}\tilde{v}_2, \tilde{v}_2) \le \tilde{\beta}_2 (\tilde{\Lambda}\tilde{v}_2, \tilde{v}_2) \ \forall \tilde{v}_2 \in \tilde{V}_2,$$

where the considered operators are defined in the following way.

$$\tilde{\Lambda} = \tilde{\Lambda}_{\mathrm{I}} + \tilde{\Lambda}_{\mathrm{II}}, (\tilde{\Lambda}\tilde{u}, \tilde{v}) = \Lambda(\tilde{u}, \tilde{v}), \ (\tilde{\Lambda}_{\mathrm{I}}\tilde{u}, \tilde{v}) = \Lambda_1(\tilde{u}, \tilde{v}), \ (\tilde{\Lambda}_{\mathrm{II}}\tilde{v}, \tilde{v}) = \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}) \ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

5. Operator Form of Method of Iterative Extensions on Finite-Dimensional Subspace

On a finite-dimensional subspace, we formulate the modified method of fictitious components both in the variational form and in the operator form.

$$\begin{gathered}\tilde{u}^k \in \tilde{V} : \Lambda(\tilde{u}^k - \tilde{u}^{k-1}, \tilde{v}) = -\tau_{k-1}(\Lambda_1(\tilde{u}^{k-1}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}^{k-1}, \tilde{v}) - \\ - F_1(I_1\tilde{v})),\ \forall \tilde{v} \in \tilde{V},\ k \in \mathbb{N}, \\ \tilde{u}^k \in \tilde{V} : \tilde{\Lambda}(\tilde{u}^k - \tilde{u}^{k-1}) = -\tau_{k-1}(\tilde{B}\tilde{u}^{k-1} - \tilde{f}),\ k \in \mathbb{N}, \\ \forall \tilde{u}^0 \in \tilde{V}_1, \tau_0 = 1, \tau_{k-1} = 2/(\tilde{\beta}_1 + \tilde{\beta}_2),\ k \in \mathbb{N} \setminus \{1\}.\end{gathered} \tag{7}$$

We consider the method of iterative extensions on a finite-dimensional subspace as a development of the method of fictitious components. We consider an iterative process at each step, for which we solve an extended problem with the introduced bilinear form, with the operator generated by this bilinear form on a finite-dimensional and approximating subspace.

$$C(\tilde{u}, \tilde{v}) = \Lambda_1(\tilde{u}, \tilde{v}) + \gamma\Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v} \in \breve{V}, \gamma \in (0; +\infty),$$
$$(\tilde{C}\tilde{u}, \tilde{v}) = C(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

The solution to such a problem belongs to the space of solutions to the extended problem, to the extended space of solutions to the continued problem, as well as in the finite-dimensional version.

$$\begin{gathered}\tilde{u}^k \in \tilde{V} : C(\tilde{u}^k - \tilde{u}^{k-1}, \tilde{v}) = -\tau_{k-1}(\Lambda_1(\tilde{u}^{k-1}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}^{k-1}, \tilde{v}) - \\ - F_1(I_1\tilde{v})), \forall \tilde{v} \in \tilde{V}, k \in \mathbb{N}, \\ \tilde{u}^k \in \tilde{V} : \tilde{C}(\tilde{u}^k - \tilde{u}^{k-1}) = -\tau_{k-1}(\tilde{B}\breve{u}^{k-1} - \tilde{f}), \quad k \in \mathbb{N}, \\ \tilde{u}^0 \in \tilde{V}_1, \gamma > \alpha, \tau_0 = 1, \tau_{k-1} = (\tilde{r}^{k-1}, \tilde{\eta}^{k-1})/(\tilde{\eta}^{k-1}, \tilde{\eta}^{k-1}),\ k \in \mathbb{N} \setminus \{1\}.\end{gathered} \tag{8}$$

When calculating iteration parameters, we need to calculate the residuals, corrections, and equivalent residuals, respectively.

$$\tilde{r}^{k-1} = \tilde{B}\tilde{u}^{k-1} - \tilde{f},\ \tilde{w}^{k-1} = \tilde{C}^{-1}\tilde{r}^{k-1},\ \tilde{\eta}^{k-1} = \tilde{B}\tilde{w}^{k-1},\ k \in \mathbb{N}.$$

Now, we write the fulfilled assumptions about the continuation of functions in the appropriate form.

$$\exists \tilde{\delta}_1 \in (0, +\infty),\ \tilde{\delta}_2 \in [\tilde{\delta}_1, +\infty) :\ \tilde{\delta}_1^2(\tilde{C}\tilde{v}_2, \tilde{C}\tilde{v}_2) \le (\tilde{\Lambda}_{\mathrm{II}}\tilde{v}_2, \tilde{\Lambda}_{\mathrm{II}}\tilde{v}_2) \le \tilde{\delta}_2^2(\tilde{C}\tilde{v}_2, \tilde{C}\tilde{v}_2),\ \forall \tilde{v}_2 \in \tilde{V}_2,$$
$$\exists \tilde{\alpha} \in (0, +\infty) :\ (\tilde{\Lambda}_{\mathrm{I}}\tilde{v}_2, \tilde{\Lambda}_{\mathrm{I}}\tilde{v}_2) \le \tilde{\alpha}^2(\tilde{\Lambda}_{\mathrm{II}}\tilde{v}_2, \tilde{\Lambda}_{\mathrm{II}}\tilde{v}_2)\, \forall \tilde{v}_2 \in \tilde{V}_2,$$

where the considered operators are defined in the following way

$$\tilde{C} = \tilde{\Lambda}_{\mathrm{I}} + \gamma\tilde{\Lambda}_{\mathrm{II}},\ (\tilde{C}\tilde{u}, \tilde{v}) = C(\tilde{u}, \tilde{v}), (\tilde{\Lambda}_{\mathrm{I}}\tilde{u}, \tilde{v}) = \Lambda_1(\tilde{u}, \tilde{v}), (\tilde{\Lambda}_{\mathrm{II}}\tilde{u}, \tilde{v}) = \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

6. Matrix Form of Method of Iterative Extensions

Consider the matrix form of the continued problem. This system of equations is obtained after approximating the continued problem on a finite-dimensional subspace.

$$\bar{u} \in \mathrm{R}^N :\ B\bar{u} = \bar{f},\ \bar{f} \in \mathrm{R}^N. \tag{9}$$

We also assume that the operator of projection onto the solution space of the continued problem sets to zero the coefficients of the basis functions, whose carriers do not completely belong to the first domain. We obtain the matrix form of the continued problem by defining the continued right-hand side of the system and the continued matrix of the system.

$$\langle B\bar{u}, \bar{v}\rangle = \Lambda_1(\tilde{u}, I_1\tilde{v}) + \Lambda_{\mathrm{II}}(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v} \in \tilde{V}$$
$$\langle \bar{f}, \bar{v}\rangle = F_1(I_1\tilde{v}),\ \forall \tilde{v} \in \tilde{V},$$
$$\langle \bar{f}, \bar{v}\rangle = (\bar{f}, \bar{v})h_1h_2 = \bar{f}\bar{v}h_1h_2, \bar{v} = (v_1, v_2, ..., v_N) \in \mathrm{R}^N,\ N = mn.$$

For example, let us present the numbering of the grid nodes, the basis functions at these nodes, the coefficients at these basis functions.

$$v_{n(i-1)+j} = v_{i,j}, \Phi_{n(i-1)+j} = \Phi^{i,j}(x_i, y_j), \quad i = \overline{1, m}, j = \overline{1, n},$$
$$\tilde{v} = \sum_{i=1}^{m}\sum_{j=1}^{n} v_{i,j}\Phi^{i,j}(x, y) = \sum_{l=1}^{N} v_l\Phi_l.$$

First, we enumerate the basis functions, whose carriers belong completely to the first domain. Then, we enumerate the basis functions, whose carriers cross the boundary of the first and the second domains, together. We finish the numbering by the basis functions, whose carriers belong completely to the second domain. With this numbering, the resulting vectors have the following structure.

$$\bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')', \bar{u} = (\bar{u}_1', \bar{0}', \bar{0}')', \bar{f} = (\bar{f}_1', \bar{0}', \bar{0}')'.$$

We calculate the components of the vector from the right side, the elements of the matrix for the previously specified system.

$$b_{ij} = h_1^{-1}h_2^{-1}(\Lambda_1(\Phi_i, I_1\Phi_j) + \Lambda_{\mathrm{II}}(\Phi_i, \Phi_j)), f_i = h_1^{-1}h_2^{-1}F_1(I_1\Phi_i), \quad i, j = \overline{1, N}.$$

Now, consider the modified method of fictitious components in the matrix form. The well-known method of fictitious components in the matrix form is obtained by approximating the modified method of fictitious components in the variational form on a finite-dimensional subspace with the previously indicated projection operator.

$$\begin{aligned} &\bar{u}^k \in \mathrm{R}^N : \Lambda(\bar{u}^k - \bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1} - \bar{f}), \quad k \in \mathbb{N}, \\ &\forall \bar{u}^0 \in \bar{V}_1, \tau_0 = 1, \tau_{k-1} = 2/(\tilde{\beta}_1 + \tilde{\beta}_2), \quad k \in \mathbb{N} \setminus \{1\}. \end{aligned} \tag{10}$$

At each step of this iterative process, an extended problem is obtained in the matrix form and with an extended matrix.

$$\langle \Lambda\bar{u}, \bar{v}\rangle = \Lambda(\tilde{u}, \tilde{v}),\ \forall \tilde{u}, \tilde{v} \in \tilde{V}.$$

Find the elements of this matrix.

$$l_{ij} = h_1^{-1}h_2^{-1}\Lambda(\Phi_i, \Phi_j),\ i, j = \overline{1, N}.$$

The resulting matrices have the well-known structure, namely,

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}, \ B = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{12} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}.$$

We introduce a subspace of vectors.

$$\bar{V}_1 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \bar{v}_2 = \bar{0}, \bar{v}_3 = \bar{0} \right\}.$$

In addition, we define subspaces of vectors, as we previously introduced the corresponding finite-dimensional subspaces.

$$\bar{V}_3 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \bar{v}_1 = \bar{0}, \bar{v}_2 = \bar{0} \right\},$$

$$\bar{V}_2 = \left\{ \bar{v} = (\bar{v}_1', \bar{v}_2', \bar{v}_3')' \in \mathrm{R}^N : \mathrm{A}_{11}\bar{v}_1 + \mathrm{A}_{12}\bar{v}_2 = \bar{0}, \mathrm{A}_{23}\bar{v}_2 + \mathrm{A}_{33}\bar{v}_3 = \bar{0} \right\}, \mathrm{R}^N = \bar{V}_1 \oplus \bar{V}_2 \oplus \bar{V}_3.$$

In the method of fictitious components, the continued problem is usually solved in the matrix form.

$$B\bar{u} = \bar{f}, \quad \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{0} \\ \bar{0} \end{bmatrix}.$$

The solution to the continued problem in the matrix form contains the solution to the original problem in the matrix form and the zero solution to the fictitious problem in the matrix form.

$$\Lambda_{11}\bar{u}_1 = \bar{f}_1, \ \begin{bmatrix} \Lambda_{02} & \Lambda_{23} \\ \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, \ \begin{bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.$$

We introduce norms without a matrix, with an extended matrix, and with a squared extended matrix.

$$\|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle}, \ \|\bar{v}\|_\Lambda = \sqrt{\langle \Lambda\bar{v}, \bar{v} \rangle}, \ \|\bar{v}\|_{\Lambda^2} = \sqrt{\langle \Lambda^2\bar{v}, \bar{v} \rangle}, \quad \forall \bar{v} \in \mathrm{R}^N.$$

Lemma 1. *In method of fictitious components (10), the following estimate takes place:*

$$\left\| \bar{u}^1 - \bar{u} \right\|_{\Lambda^2} \le 2 \left\| \bar{u}^0 - \bar{u} \right\|_{\Lambda^2}.$$

Lemma 2. *In method of fictitious components (10), the following inequality with a positive value is satisfied:*

$$\left\| \bar{u}^1 - \bar{u} \right\|_{\Lambda} \le d \left\| \bar{u}^1 - \bar{u} \right\|_{\Lambda^2},$$

where

$$d \approx (\lambda_{1,1}^2 + a_{\mathrm{II}})^{1/2} \lambda_{1,1}^{-2}, \ h_1, h_2 \to 0, \ \lambda_{1,1} = \pi^2(b_1^{-2} + b_2^{-2})/4.$$

Theorem 1. *In method of fictitious components (10), the absolute error is estimated as follows:*

$$\left\| \bar{u}^k - \bar{u} \right\|_{\Lambda} \le \varepsilon \left\| \bar{u}^0 - \bar{u} \right\|_{\Lambda^2} = \varepsilon \left\| \bar{f}^0 - \bar{f} \right\|,$$
$$\varepsilon = cq^{k-1}, \ c = 2d \in (0, +\infty), \ k \in \mathbb{N}, \ \bar{f}^0 = \mathrm{A}\bar{u}^0, \ 0 \le q = (\beta_2 - \beta_1)/(\beta_1 + \beta_2) < 1,$$
$$d \approx (\lambda_{1,1}^2 + a_{\mathrm{II}})^{1/2} \lambda_{1,1}^{-2}, \ h_1, h_2 \to 0, \ \lambda_{1,1} = \pi^2(b_1^{-2} + b_2^{-2})/4.$$

The absolute error of the method of fictitious components has a rate of convergence in the energy norm no worse than the rate of convergence of a geometric progression.

In the matrix form, the method of iterative extensions is a development of the method of fictitious components in the matrix form. To solve problem (9), we use a new method. Let us define the matrices used in what follows.

$$\langle \Lambda_{\mathrm{I}}\bar{u},\ \bar{v}\rangle = \Lambda_1(\tilde{u},\tilde{v}),\ \langle \Lambda_{\mathrm{II}}\bar{u},\bar{v}\rangle = \Lambda_{\mathrm{II}}(\tilde{u},\tilde{v}),\ \forall \tilde{u},\tilde{v} \in \tilde{V}.$$

These matrices have the following structure:

$$\Lambda_{\mathrm{I}} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Lambda_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix}.$$

We apply the new definition of the extended matrix, using an additional positive parameter.

$$C = \Lambda_{\mathrm{I}} + \gamma\Lambda_{\mathrm{II}},\ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{21} & \Lambda_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{02} & \Lambda_{23} \\ 0 & \Lambda_{32} & \Lambda_{33} \end{bmatrix},\ \gamma \in (0,+\infty).$$

Let us write the assumptions for finite-dimensional subspaces on the continuation of functions in the matrix form.

$$\exists \delta_1 \in (0,+\infty), \delta_2 \in [\delta_1,+\infty):\ \delta_1^2 \langle C\bar{v}_2, C\bar{v}_2\rangle \le \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle \le \delta_2^2 \langle C\bar{v}_2, C\bar{v}_2\rangle, \quad \forall \bar{v}_2 \in \bar{V}_2,$$

$$\exists \alpha \in (0,+\infty): \langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{I}}\bar{v}_2\rangle \le \alpha^2 \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle, \quad \forall \bar{v}_2 \in \bar{V}_2.$$

Next, we use the last inequality.

$$\begin{aligned} &\langle C\bar{v}_2, C\bar{v}_2\rangle = \langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{I}}\bar{v}_2\rangle + 2\gamma \langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle + \gamma^2 \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle \le \\ &\le 2\langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{I}}\bar{v}_2\rangle + 2\gamma^2 \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle \le 2(\alpha^2+\gamma^2) \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle. \end{aligned}$$

We obtain an estimate for the constant of the first inequality.

$$0,5(\alpha^2+\gamma^2)^{-1} \le \delta_1^2.$$

We assume the existence of an asymptotic equality.

$$\langle C\bar{v}_2, C\bar{v}_2\rangle \approx \langle \Lambda_{\mathrm{I}}\bar{v}_2, \Lambda_{\mathrm{I}}\bar{v}_2\rangle + \gamma^2 \langle \Lambda_{\mathrm{II}}\bar{v}_2, \Lambda_{\mathrm{II}}\bar{v}_2\rangle,\ h_1, h_2 \to 0.$$

We obtain asymptotic estimates of the constants of the first and second inequalities.

$$\delta_1^2 \approx (\alpha^2+\gamma^2)^{-1}, \delta_2^2 \approx \gamma^{-2},\ h_1, h_2 \to 0.$$

Now, to solve problem (9), we consider the method of iterative extensions as a generalization of the method of fictitious components by introducing an additional parameter for the extended matrix. The method of fictitious components is obtained with a single value of this parameter from the method of iterative extensions, if the choice of iterative parameters is not taken into account.

$$\begin{aligned} &\bar{u}^k \in \mathrm{R}^N : C(\bar{u}^k - \bar{u}^{k-1}) = -\tau_{k-1}(B\bar{u}^{k-1} - \bar{f}), \quad k \in \mathbb{N}, \\ &\forall \bar{u}^0 \in \bar{V}_1, \gamma > \alpha, \tau_0 = 1, \tau_{k-1} = \left\langle \bar{r}^{k-1}, \bar{\eta}^{k-1}\right\rangle / \left\langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1}\right\rangle, \quad k \in \mathbb{N} \setminus \{1\}. \end{aligned} \tag{11}$$

When calculating iterative parameters, we need to calculate residuals, corrections, equivalent residuals.

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f},\ \bar{w}^{k-1} = C^{-1}\bar{r}^{k-1},\ \eta^{k-1} = B\bar{w}^{k-1},\ k \in \mathbb{N}.$$

Lemma 3. *In method of iterative extensions (11), the following estimate takes place.*

$$\left\|\bar{u}^1 - \bar{u}\right\|_{C^2} \leq 2\left\|\bar{u}^0 - \bar{u}\right\|_{C^2}.$$

Theorem 2. *For method of iterative extensions (5), the following estimate takes place.*

$$\left\|\breve{u}^k - \breve{u}\right\|_{\breve{C}^2} \leq \varepsilon\left\|\breve{u}^0 - \breve{u}\right\|_{\breve{C}^2}, \varepsilon = 2(\breve{\delta}_2/\breve{\delta}_1)(\breve{\alpha}/\breve{\gamma})^{k-1}, \quad k \in \mathbb{N},$$

where the relative errors are estimated by an infinitely decreasing geometric progression in the norm generated by the square of the operator of the extended problem

$$\|\breve{v}\|_{\breve{C}^2} = \sqrt{(\breve{C}\breve{v}, \breve{C}\breve{v})},\ \forall\breve{v} \in \breve{V}.$$

Proof. In the iterative process, errors and residuals satisfy equalities.

$$\breve{\psi}^k = \breve{\psi}^{k-1} - \tau_k\breve{C}^{-1}\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^{k-1}, \breve{r}^k = \breve{r}^{k-1} - \tau_k\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, k \in \mathbb{N} \setminus \{1\}.$$

Let us minimize residuals.

$$0 \leq \left(\breve{r}^k, \breve{r}^k\right) = \tau_k^2\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right) - 2\tau_k\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right) + \left(\breve{r}^{k-1}, \breve{r}^{k-1}\right).$$

We determine the iterative parameters under the condition of minimizing the residuals.

$$\tau_{k-1} = \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)} = \frac{\left(\breve{r}^{k-1}, \breve{\eta}^{k-1}\right)}{\left(\breve{\eta}^{k-1}, \breve{\eta}^{k-1}\right)}.$$

We get the equality

$$\tau_{k-1} = \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)} = \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}\right)}.$$

Let us introduce the notation.

$$\breve{\Lambda}_{\mathrm{I}}\breve{w}^{k-1} = \breve{a},\ \breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1} = \breve{b}.$$

Note that the iteration parameters are positive.

$$\tau_k = \frac{\left(\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)}{\left(\breve{b}, \breve{b}\right)} = \breve{\gamma} - \frac{\left(\breve{a}, \breve{b}\right)}{\left(\breve{b}, \breve{b}\right)} \geq \breve{\gamma} - \frac{(\breve{a}, \breve{a})^{1/2}\left(\breve{b}, \breve{b}\right)^{1/2}}{\left(\breve{b}, \breve{b}\right)} \geq \breve{\gamma} - \frac{(\breve{a}, \breve{a})^{1/2}}{\left(\breve{b}, \breve{b}\right)^{1/2}} \geq \breve{\gamma} - \breve{\alpha} > 0.$$

We write out the inner products for the residuals with the obtained iterative parameters.

$$\left(\breve{r}^k, \breve{r}^k\right) = \left(\breve{r}^{k-1}, \breve{r}^{k-1}\right) - \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)^2}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)}.$$

We determine the ratio of inner products of residuals during iterations.

$$q_k^2 = \frac{\left(\breve{r}^k, \breve{r}^k\right)}{\left(\breve{r}^{k-1}, \breve{r}^{k-1}\right)} = 1 - \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{r}^{k-1}\right)^2}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{C}^{-1}\breve{r}^{k-1}\right)\left(\breve{r}^{k-1}, \breve{r}^{k-1}\right)} =$$

$$= \frac{\left(\breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}\right)\left(\breve{C}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right) - \left(\breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)^2}{\left(\breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{w}^{k-1}\right)\left(\breve{C}\breve{w}^{k-1}, \breve{C}\breve{w}^{k-1}\right)} =$$

$$= \frac{\left(\breve{b}, \breve{b}\right)\left(\breve{a} + \breve{\gamma}\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right) - \left(\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)^2}{\left(\breve{b}, \breve{b}\right)\left(\breve{a} + \breve{\gamma}\breve{b}, \breve{a} + \breve{\gamma}\breve{b}\right)}.$$

We introduce the notation

$$(\breve{a}, \breve{a}) = a, \ \left(\breve{b}, \breve{b}\right) = b, \ \left(\breve{a}, \breve{b}\right) = z,$$

then

$$q_k^2 = \frac{ab - z^2}{b(a + \breve{\gamma}^2 b + 2\breve{\gamma}z)} \leq \max_{|z|\leq\sqrt{ab}} q_k^2(z) = q_k^2\left(\frac{-a}{\breve{\gamma}}\right) = \frac{a}{\breve{\gamma}^2 b} \leq \frac{\breve{\alpha}^2}{\breve{\gamma}^2} = q^2.$$

Taken into account

$$q_k^2 \geq 0, \left(q_k^2(z)\right)'_z = \frac{-2\breve{\gamma}(z + a/\breve{\gamma})(z + \breve{\gamma}b)}{b(a + \breve{\gamma}^2 b + 2\breve{\gamma}z)^2}, \ -\breve{\gamma}b < \frac{a + \breve{\gamma}^2 b}{2\breve{\gamma}} < -\sqrt{ab} < -\frac{a}{\breve{\gamma}} < \sqrt{ab},$$

we get the inequalities.

$$\left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k\right) \leq q^2\left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^{k-1}, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^{k-1}\right), \ k \in \mathbb{N} \setminus \{1\},$$

$$\left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k\right) \leq q^{2(k-1)}\left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^1, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^1\right), \ k \in \mathbb{N} \setminus \{1\}.$$

Since

$$\left\langle\breve{C}\breve{\psi}^k, \breve{C}\breve{\psi}^k\right\rangle \leq \breve{\delta}_1^{-2}\left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^k\right), \left(\breve{\Lambda}_{\mathrm{II}}\breve{\psi}^1, \breve{\Lambda}_{\mathrm{II}}\breve{\psi}^1\right) \leq \breve{\delta}_2^2\left(\breve{C}\breve{\psi}^1, \breve{C}\breve{\psi}^1\right) \leq 4\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^0, \breve{C}\breve{\psi}^0\right),$$

$$\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^1, \breve{C}\breve{\psi}^1\right) \leq 4\breve{\delta}_2^2\left(\breve{C}\breve{\psi}^0, \ \breve{C}\breve{\psi}^0\right).$$

We have an inequality that gives an estimate of convergence in the iterative process of the method of iterative extensions. Additionally, we take into account the passage to the limit in the following inequality.

$$\left(\breve{C}\breve{\psi}^1, \breve{C}\breve{\psi}^1\right) \approx \left\langle C\bar{\psi}^1, C\bar{\psi}^1\right\rangle \leq 4\left\langle C\bar{\psi}^0, C\bar{\psi}^0\right\rangle \approx 4\left(\breve{C}\breve{\psi}^0, \breve{C}\breve{\psi}^0\right), \ h_1, h_2 \to 0.$$

Theorem 3. *For method of iterative extensions considered in the case of a finite-dimensional subspace (8), the following estimate of convergence holds:*

$$\left\|\tilde{u}^k - \tilde{u}\right\|_{\tilde{C}^2} \leq \varepsilon \left\|\tilde{u}^0 - \tilde{u}\right\|_{\tilde{C}^2}, \varepsilon = 2(\tilde{\delta}_2/\tilde{\delta}_1)(\tilde{\alpha}/\tilde{\gamma})^{k-1}, \quad k \in \mathbb{N},$$

where the relative errors are estimated by an infinitely decreasing geometric progression in the norm generated by the square of the operator of the extended problem on a finite-dimensional and approximating subspace

$$\|\tilde{v}\|_{\tilde{C}^2} = \sqrt{(\tilde{C}\tilde{v}, \tilde{C}\tilde{v})},\ \forall \tilde{v} \in \tilde{V}.$$

We assume that the properties are satisfied during the approximation.

$$(\tilde{\Lambda}_{\mathrm{I}}\tilde{v}, \tilde{\Lambda}_{\mathrm{I}}\tilde{v}) \approx (\breve{\Lambda}_{\mathrm{II}}\breve{v}, \breve{\Lambda}_{\mathrm{II}}\breve{v}), (\tilde{\Lambda}_{\mathrm{II}}\tilde{v}, \tilde{\Lambda}_{\mathrm{II}}\tilde{v}) \approx (\breve{\Lambda}_{\mathrm{II}}\breve{v}, \breve{\Lambda}_{\mathrm{II}}\breve{v}), \quad h_1, h_2 \to 0.$$

In this version, the previous theorem follows from the last theorem after passing to the limit.

Theorem 4. *There exists an estimate for the method of iterative extensions in matrix form (11)*

$$\left\|\bar{u}^k - \bar{u}\right\|_{C^2} \leq \varepsilon \left\|\bar{u}^0 - \bar{u}\right\|_{C^2},\ \varepsilon = 2(\delta_2/\delta_1)(\alpha/\gamma)^{k-1}, \quad k \in \mathbb{N},$$

where the relative errors are estimated by an infinitely decreasing geometric progression in the norm generated by the square of the operator of the extended problem in the matrix form

$$\|\bar{v}\|_{C^2} = \sqrt{(C\bar{v}, Cv)},\ \forall \bar{v} \in \mathrm{R}^N.$$

Remark 1. If we use the passage to the limit, then Theorem 2 follows from Theorems 3 and 4. We can say that Theorem 3 and Theorem 4 coincide practically up to their notation. The proof of Theorem 4 is practically similar to the proof of Theorem 2 and does not use the passage to the limit in the inequality, which is obtained at the first iteration.

7. Algorithmic Implementation of Method of Iterative Extensions in Matrix Form

We choose a zero initial approximation and apply the method of minimum residuals to select the iterative parameters.

I. Calculate square of the norm of the initial absolute error.

$$\mathrm{E}_0 = \left\langle \bar{f}, \bar{f} \right\rangle.$$

II. Find the first approximation

$$\bar{u}^1 = C^{-1}\bar{f}.$$

III. Calculate the residual

$$\bar{r}^{k-1} = B\bar{u}^{k-1} - \bar{f} = \Lambda_{\mathrm{II}}\bar{u}^{k-1}, \quad k \in \mathbb{N} \setminus \{1\}.$$

IV. Calculate square of the norm of the absolute error

$$\mathrm{E}_{k-1} = \left\langle \bar{r}^{k-1}, \bar{r}^{k-1} \right\rangle, \quad k \in \mathbb{N} \setminus \{1\}.$$

V. Find the correction

$$\bar{w}^{k-1} = C^{-1}\bar{r}^{k-1}, \quad k \in \mathbb{N} \setminus \{1\}.$$

VI. Calculate the equivalent residual

$$\bar{\eta}^{k-1} = B\bar{w}^{k-1} = \Lambda_{\mathrm{II}}\bar{w}^{k-1}, \quad k \in \mathbb{N} \setminus \{1\}.$$

VII. Calculate the iterative parameter

$$\tau_{k-1} = \langle \bar{r}^{k-1}, \bar{\eta}^{k-1} \rangle / \langle \bar{\eta}^{k-1}, \bar{\eta}^{k-1} \rangle, \quad k \in \mathbb{N} \setminus \{1\}.$$

VIII. Calculate the next approximation

$$\bar{u}^{k} = \bar{u}^{k-1} - \tau_{k-1}\bar{w}^{k-1}, \quad k \in \mathbb{N} \setminus \{1\}.$$

IX. Check the condition for stopping the iterations

$$\mathrm{E}_{k-1} \leq \mathrm{E}_0\mathrm{E}, \; k \in \mathbb{N} \setminus \{1\}, \quad \mathrm{E} \in (0,1).$$

If the condition for stopping iterations is not satisfied, all calculations are repeated from Step III.

8. Example of Using Method of Iterative Extensions in Matrix Form

Consider the problem for the following domains:

$$\Pi = (0,8) \times (0,b), \; \Omega_1 = (0,8) \times (0,4), \; \Omega_{\mathrm{II}} = (0,8) \times (4,b).$$

We assume that the domains have the boundaries

$$\Gamma_0 = \emptyset, \; \Gamma_1 = (0,8) \times \{b\}, \; \Gamma_2 = (0,8) \times \{0\} \cup \{0,8\} \times (0,b), \; \Gamma_3 = \emptyset,$$
$$\Gamma_{1,0} = (0,8) \times \{4\}, \; \Gamma_{1,1} = \emptyset, \Gamma_{1,2} = (0,8) \times \{0\} \cup \{0,8\} \times (0,4), \; \Gamma_{1,3} = \emptyset,$$
$$\Gamma_{\mathrm{II},0} = \emptyset, \; \Gamma_{\mathrm{II},1} = (0,8) \times \{b\}, \; \Gamma_{\mathrm{II},2} = \{0,8\} \times (4,b), \; \Gamma_{\mathrm{II},3} = (0,8) \times \{4\}.$$

We take the right side with the coefficient of the equation.

$$\breve{f}_1(x,y) = 6, \; (x,y) \in (0,8) \times (0,4), \; a_{\mathrm{II}}(x,y) = 1, \; (x,y) \in (0,8) \times (4,b).$$

We present a solution to the problem.

$$\breve{u}_1(x,y) = (y+4)^2(y-4)^2/4, \; (x,y) \in (0,8) \times (0,4).$$

When discretizing, we choose

$$h_1 = h_2 = 8/(n+2), \; b = 4(2n+1)/(n+2), \; n = 36, 42, \ldots, 102.$$

When calculating, under the zero initial approximation by the method of iterative extensions, under a single relative error, the iterative process stops at the sixth iteration, if the estimate for the relative error is considered to be one ten thousandth.

$$n = 102, \, \mathrm{E} = 0,0001, u_{i,j}^1 \geq u_{i,j}^2 \approx u_{i,j}^3 \approx u_{i,j}^4 \approx u_{i,j}^5 \approx u_{i,j}^6 \approx u_{i,j} \geq u_{i,j}^0 = 0, \; k = 6.$$

On the smallest of the grids used, at the last iteration, there exist estimates that demonstrate the accuracy of the approximate solution for solving the original problem.

$$n = 102, \, \mathrm{E} = 0,0001, \max_{(x_i,y_j)\in\Omega_1} \frac{|u_{i,j}^k - u_{i,j}|}{|u_{i,j}|} \leq 0,004, \frac{\max\limits_{(x_i,y_j)\in\Omega_1} |u_{i,j}^k - u_{i,j}|}{\max\limits_{(x_i,y_j)\in\Omega_1} |u_{i,j}|} \leq 0,00004, \; k = 6.$$

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АНАЛИЗ ЗАДАЧИ ДЛЯ БИГАРМОНИЧЕСКОГО УРАВНЕНИЯ

А. Л. Ушаков

Для бигармонического уравнения рассматривается смешанная задача с главными краевыми условиями. Делается продолжение исходной задачи по границе с условиями Дирихле в прямоугольную область. Продолженная задача приводится как операторное уравнение. Метод итерационных расширений выписывается в операторной форме при решении продолженной задачи. Операторная продолженная задача приводится на конечномерном подпространстве. Метод итерационных расширений приводится для решения операторной продолженной задачи на конечномерном подпространстве. Продолженная задача после дискретизации записывается в матричной форме. Продолженная задача в матричной форме решается методом итерационных расширений в матричной форме. Устанавливается, что в рассматриваемых случаях метод итерационных расширений имеет относительные ошибки, сходящиеся как геометрическая прогрессия в более сильной норме, чем энергетическая норма у расширенной задачи. Итерационные параметры в применяемых итерационных процессах выбираются на основе минимизации невязок. Приводятся условия гарантирующие сходимости используемых итерационных процессов. Приводится алгоритм, реализующий в матричной форме метод итерационных расширений. В алгоритме выполняется самостоятельный выбор итерационных параметров и приводится критерий для остановки, если достигнута оценка необходимой точности. Приводится вычислительный пример использования метода итерационных расширений на ЭВМ.

Ключевые слова: бигармоническое уравнение; метод итерационных расширений.

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