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NUMERICAL ALGORITHM FOR FINDING A SOLUTION TO A NONLINEAR FILTRATION MATHEMATICAL MODEL WITH A RANDOM SHOWALTER–SIDOROV INITIAL CONDITION

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The article is devoted to the study of a nonlinear model of fluid filtration based on the stochastic Oskolkov equation. It is assumed that the experimental initial data is affected by "noise", which leads to the study of a stochastic model with the Nelson–Glicklich derivative. Sufficient conditions for the existence of solutions of the investigated model with the initial Showalter–Sidorov condition are constructed. An algorithm for the numerical solution method is constructed and a computational experiment is presented.

Keywords: Sobolev type equations; stochastic model of nonlinear filtration; Nelson–Glicklich derivative.

Dedicated to the 70th anniversary
of the Professor Alexander Leonidovich Shestakov

Introduction

In the study of natural phenomena, the development of the theory of random processes led to a transition from deterministic representations to probabilistic ones and, as a result, the emergence of a large number of works devoted to stochastic modelling in mathematical biology, chemistry, economics, etc. [1–3]. For a long time, the theory of stochastic models (in the finite-dimensional case) was developed in the framework of the now classical Ito–Stratonovich–Skorokhod direction [4]. The main problem solved here is cupping difficulties associated with differentiating a non-differentiable (in the "usual" sense) Wiener process. These difficulties are overcome by passing from the differential equation to the integral one and subsequent consideration of the integrals of Ito, Stratonovich, etc. The work [5] presents fundamental review of attempts to continue the Ito–Stratonovich–Skorokhod approach in the infinite-dimensional situation. The paper [6] contains applications of the results obtained in [5] to classical models of mathematical physics. Currently, researchers (see, for example, [4, 7]) actively develop a new approach to the study of stochastic equations, where "white noise" stands for the Nelson–Gliklikh derivative of the Wiener process. We also note that overcoming the differentiation of the Wiener process by integration is not the unique method of studying stochastic equations. At the school founded by I.V. Melnikova, a direction arose in which stochastic equations are considered in Schwartz spaces [8]. Here the white noise is the generalized derivative of the Wiener process. Let us also pay attention to the model of the Shestakov–Sviridyuk measuring device, in which the "white noise" is understood as the Nelson–Gliklikh derivative of the Wiener process [9, 10].

One of examples of problems with random influence is the problem of recovering the input signal, since the experimental data are distorted by random noise. Problems of finding observation values by experimental data arise when solving various problems [11]. This problem is a preliminary stage in solving problems of dynamic measurements. In the theory of dynamic measurements, an actual problem is the problem of restoring a measurement by an observation. The traditional approach to solution of this problem is an approach based on the theory of inverse problems. Another one, which is traditional approach as well, is the approach based on the theory of automatic control [12]. Also, researchers study the case when the observation is distorted by white noise, which is understood as the derivative of Brownian motion in Einstein's theory. The work [13] obtains sufficient conditions for the existence of a solution to the Cauchy–Dirichlet problem for a stochastic model of Sobolev type. Note that in the model of the measuring device Shestakov–Sviridyuk under the "white noise" is understood as the derivative of Nelson–Gliklich $\overset{o}{\eta}(t)$ [9]. The notion of the Nelson–Gliklikh derivative $\overset{o}{\eta}(t)$ was introduced in the monograph [4]. Moreover, this derivative coincides with the classical derivative if η is a function. Also, note that the Nelson–Gliklikh derivative is based on the concept of the derivative in the mean introduced by E. Nelson [10]. In order to study the Cauchy problem, we construct the spaces of K-"noises", i.e. the spaces of stochastic K-processes that are almost surely differentiable in the sense of Nelson–Gliklikh. This approach is based on the paper [7]. Note that this approach allows to transfer on the stochastic case the methods of functional analysis that are applied in the deterministic case [1–3]. The work [13] considers various mathematical models based on semilinear equations in evolutionary and dynamic form in the deterministic case. The work [7] is the first attempt to study the evolutionary model in the stochastic case. It should be noted that the main research methods of this model are transformed from the deterministic case [13].

This work is devoted to the numerical study of the Showalter–Sidorov problem for the stochastic equation of fluid filtration. Let $D \subset \mathbb{R}^n$ be a bounded domain with a boundary ∂D from the class C^∞. Often in experiments, "noises" can occur. Then, to study physical models, it is necessary to consider stochastic models. In the stochastic case, the mathematical model of nonlinear filtering has the form

$$\eta(s,t) = 0,\ (s,t) \in \partial D \times \mathbb{R}_+, \tag{1}$$

$$(\lambda - \Delta)\overset{o}{\eta} = \alpha\Delta\eta - |\eta|^{p-2}\eta,\ p \geq 2. \tag{2}$$

One of the well-known initial value problems for the model under consideration is the weakened (in the sense of S.G. Kerin) Showalter–Sidorov problem

$$\lim_{t\to 0+}(\lambda - \Delta)(\eta(t) - \eta_0) = 0,\ s \in D. \tag{3}$$

For the first time, equation (2) without stochastic component was described by A.P. Oskolkov [15]. In general, equation (2) without stochastic component illustrates the dependence of the pressure of a viscoelastic incompressible fluid (for example, oil), filtering in a porous formation, on an external load (for example, the pressure of water injected through the wells into the formation). Problem (1), (2) can be reduced to the stochastic equation

$$L \overset{o}{\eta} = M\eta + N(\eta) \tag{4}$$

endowed with weakened Showalter–Sidorov condition (3). Solution (4) $\eta = \eta(t)$ is a stochastic K-process. Stochastic K-processes $\eta = \eta(t)$ and $\zeta = \zeta(t)$ are considered to be equal, if almost surely each trajectory of one of the processes coincides with a trajectory of other process.

1. Stochastic Mathematical Model of Nonlinear Filtration

Consider a complete probability space $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ and the set of real numbers $\mathbb{R}$ endowed with a Borel σ-algebra. A measurable mapping $\xi : D \to \mathbb{R}$ is called a *random variable*. The set of random variables having zero expectations (i.e. $\mathbf{E}\xi = 0$) and finite variance forms Hilbert space $\mathbf{L_2}$ (i.e. $\mathbf{D}\xi < +\infty$) with the inner product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$, where $\mathbf{E}$, $\mathbf{D}$ are the expectation and variance of the random variable, respectively.

Consider a set $\mathcal{I} \subset \mathbb{R}$ and the following two mappings. The first one, $f : \mathcal{I} \to \mathbf{L_2}$ associates each $t \in \mathcal{I}$ with the random variable $\xi \in \mathbf{L_2}$. The second one, $g : \mathbf{L_2} \times D \to \mathbb{R}$, associates each pair (ξ, ω) with the point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : \mathbb{R} \times D \to \mathbb{R}$ of the form $\eta = \eta(t, \omega) = g(f(t), \omega)$, where f and g are defined above, is called a *stochastic process*. A random process η is called *continuous*, if almost surely all its trajectories are continuous. The set of continuous stochastic processes forms a Banach space, which is denoted by $\mathbf{C}(\mathcal{I}, \mathbf{L}_2)$.

Consider a real separable Hilbert space $(\mathbf{H}, < \cdot, \cdot >)$ identified with its conjugate space with the orthonormal basis $\{\varphi_k\}$. Each element $x \in \mathbf{H}$ can be represented as $x = \sum\limits_{k=1}^{\infty} < x, \varphi_k > \varphi_k$. Next, choose a monotonely decreasing numerical sequence $K = \{\mu_k\}$ such that $\sum\limits_{k=1}^{\infty} \mu_k^2 < +\infty$. Consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ such that $\sum\limits_{k=1}^{\infty} \mu_k^2 \mathbf{D}\xi_k < +\infty$. Denote by $\mathbf{H}_K\mathbf{L}_2$ the Hilbert space of *random K-variables* of the form $\xi = \sum\limits_{k=1}^{\infty} \mu_k \xi_k \varphi_k$. Moreover, there exists a random K-variable $\xi \in \mathbf{H}_K\mathbf{L}_2$, if, for example, $\mathbf{D}\xi_k <$ const $\forall k$. Note that the space $\mathbf{H}_K\mathbf{L}_2$ is a Hilbert space with the scalar product $(\xi^1, \xi^2) = \sum\limits_{k=1}^{\infty} \mu_k^2 \mathbf{E}\xi_k^1\xi_k^2$. Consider a sequence of random processes $\{\eta_k\} \subset \mathbf{C}(\mathcal{I}, \mathbf{L}_2)$ and define the $\mathbf{H}$-*valued continuous stochastic K-process*

$$\eta(t) = \sum_{k=1}^{\infty} \mu_k \eta_k(t) \varphi_k, \tag{5}$$

if series (5) converges uniformly in the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact set in $\mathcal{I}$ and consider random variable

$$\eta_0 = \sum_{k=1}^{\infty} \mu_k \eta_{0k} \varphi_k. \tag{6}$$

Consider the Nelson–Gliklikh derivatives of the random K-process

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \mu_k \overset{o}{\eta}_k(t) \varphi_k$$

inclusively in the right-hand side, and all series converge uniformly in the norm $\mathbf{H}_K\mathbf{L}_2$ on any compact from $\mathcal{I}$. Next, consider the space $\mathbf{C}^1(\mathcal{I};\mathbf{H}_K\mathbf{L}_2)$ of continuous stochastic K-processes and the space $\mathbf{C}^1(\mathcal{I};\mathbf{H}_K\mathbf{L}_2)$ of stochastic K-processes whose trajectories are almost surely continuously differentiable by Nelson–Gliklikh.

Consider dual pairs of reflexive Banach spaces $(\mathfrak{H},\mathfrak{H}^*)$ and $(\mathcal{B},\mathcal{B}^*)$, where $\mathfrak{N} = \overset{\circ}{W}{}_2^1(D)$, $\mathfrak{B} = L_p(D)$, $\mathbf{H} = L_2(D)$ defined in the domain D such that the embeddings

$$\mathfrak{H} \hookrightarrow \mathcal{B} \hookrightarrow \mathbf{H} \hookrightarrow \mathcal{B}^* \hookrightarrow \mathfrak{H}^* \tag{7}$$

are dense and continuous. The operators L, M and N are defined as follows:

$$(L\eta, z) = \lambda(\eta, z) + (\nabla\eta, \nabla z) \ \ \forall\ \eta, z \in \mathfrak{H}_K\mathbf{L}_2,$$

$$(M\eta, z) = -\alpha(\nabla\eta, \nabla z)\ \forall\ \eta, z \in \mathfrak{H}_K\mathbf{L}_2,$$

$$(N(\eta), z) = -\left(|\eta|^{p-2}\eta, z\right)\ \ \forall\ \eta, z \in \mathfrak{B}_K\mathbf{L}_2,$$

where $(\cdot,\cdot)$ is the dot product in $\mathbf{H}_K\mathbf{L}_2$. Similarly, we construct the spaces $\mathfrak{H}_K\mathbf{L}_2$ and $\mathfrak{B}_K\mathbf{L}_2$. As a system of functions $\{\varphi_k\}$ consider sequence of eigenfunctions of the homogeneous Dirichlet problem for the Laplace operator $(-\Delta)$ in the domain D, and denote by $\{\lambda_k\}$ the corresponding sequence of eigenvalues numbered in non-decreasing order taking into account the multiplicity.

Lemma 1. [14] (i) *For all $\lambda \geq -\lambda_1$, the operator $L \in \mathcal{L}(\mathfrak{H}_K\mathbf{L}_2;\mathfrak{H}^*_K\mathbf{L}_2)$ is self-adjoint, Fredholm and non-negative definite.*

(ii) *For all $\alpha \in \mathbb{R}_+$, the operator $M \in \mathcal{L}(\mathfrak{H}_K\mathbf{L}_2;\mathfrak{H}^*_K\mathbf{L}_2)$ is symmetric and the operator $(-M)$ is 2-coercive.*

(iii) *The operator $N \in C^1(\mathfrak{B}_K\mathbf{L}_2;\mathfrak{B}^*_K\mathbf{L}_2)$ is dissipative and the operator $(-N)$ is p-coercive.*

Taking into account that the operator L is self-adjoint and Fredholm, we identify $\mathfrak{H} \supset \ker L \equiv \operatorname{coker} L \subset \mathfrak{H}^*$. We use the subspace $\ker L$ in order to construct the subspace $[\ker L]_K\mathbf{L}_2 \subset \mathbf{H}_K\mathbf{L}_2$ and, similarly, the subspace $[\mathrm{coker\,L}]_\mathrm{K}\mathbf{L}_2 \subset \mathbf{H}^*_\mathrm{K}\mathbf{L}_2$. Taking into account that embeddings (7) are continuous and dense, we construct the spaces $\mathfrak{H}^*_K\mathbf{L}_2 = [\operatorname{coker} L]_K\mathbf{L}_2 \oplus [\operatorname{im} L]_K\mathbf{L}_2$ and $\mathcal{B}^*_K\mathbf{L}_2 = [\operatorname{coker} L]_K\mathbf{L}_2 \oplus [\operatorname{im} L \cap \mathcal{B}^*]_K\mathbf{L}_2$. We use the subspace $\operatorname{coim} L \subset \mathfrak{H}$ in order to construct the subspace $[\operatorname{coim} L]_K\mathbf{L}_2$ such that the space $\mathfrak{H}_K\mathbf{L}_2 = [\ker L]_K\mathbf{L}_2 \oplus [\operatorname{coim} L]_K\mathbf{L}_2$. Consider $[\ker L]_K\mathbf{L}_2 \equiv \mathcal{B}^0_K\mathbf{L}_2$ such that the space $[\operatorname{coim} L]_K\mathbf{L}_2$ in order to construct the set $\mathcal{B}^1_K\mathbf{L}_2$, then $\mathcal{B}_K\mathbf{L}_2 = \mathcal{B}^0_K\mathbf{L}_2 \oplus \mathcal{B}^1_K\mathbf{L}_2$. Notice, that $L \in \mathcal{L}(\mathfrak{H};\mathfrak{H}^*)$ is a linear, continuous, self-adjoint, non-negatively defined and Fredholm operator, then $L \in \mathcal{L}(\mathfrak{H}_K\mathbf{L}_2;\mathfrak{H}^*_K\mathbf{L}_2)$ is a linear, continuous, self-adjoint, non-negatively defined and Fredholm operator, and

$$\mathfrak{H}_K\mathbf{L}_2 \supset [\ker L]_K\mathbf{L}_2 \equiv [\operatorname{coker} L]_K\mathbf{L}_2 \subset \mathfrak{H}^*_K\mathbf{L}_2$$

if

$$\mathfrak{H} \supset \ker L \equiv \operatorname{coker} L \subset \mathfrak{H}^*.$$

There exists a projector Q of the space $\mathfrak{H}^*_K\mathbf{L}_2$ on $[\operatorname{coim} L]_K\mathbf{L}_2$ along $[\operatorname{coker} L]_K\mathbf{L}_2$ (likewise a projector P of the space $\mathfrak{H}_K\mathbf{L}_2$ on $[\ker L]_K\mathbf{L}_2$ along $[\operatorname{coim} L]_K\mathbf{L}_2$).

Suppose that $\mathcal{I} \equiv (0, +\infty)$. We use the space $\mathbf{H}$ in order to construct the spaces of K-" noises", spaces $\mathbf{C}^k(\mathcal{I}; \mathbf{H}_K\mathbf{L}_2)$ and $\mathbf{C}^k(\mathcal{I}; \mathfrak{H}_K\mathbf{L}_2)$, $k \in \mathbf{N}$. Consider stochastic Sobolev type equation (4).

Let $\lambda \geq -\lambda_1$

$$\ker L = \begin{cases} \{0\}, \text{ if } \lambda > -\lambda_1; \\ \operatorname{span}\{\varphi_1\}, \text{ if } \lambda = -\lambda_1. \end{cases}$$

Then

$$[\operatorname{im} L]_K\mathbf{L}_2 = \begin{cases} \mathfrak{H}_K^*\mathbf{L}_2, \text{ если } \lambda > -\lambda_1; \\ \{\eta \in \mathfrak{H}_K^*\mathbf{L}_2 : \langle \eta, \varphi_1 \rangle = 0\}, \text{ if } \lambda = -\lambda_1, \end{cases}$$

$$[\operatorname{coim} L]_K\mathbf{L}_2 = \begin{cases} \mathfrak{H}_K\mathbf{L}_2, \text{ if } \lambda > -\lambda_1; \\ \{\eta \in \mathfrak{H}_K\mathbf{L}_2 : \langle \eta, \varphi_1 \rangle = 0\}, \text{ if } \lambda = -\lambda_1. \end{cases}$$

Hence the projectors

$$P = Q = \begin{cases} \mathbb{I}, \text{ if } \lambda > -\lambda_1; \\ \mathbb{I} - \langle \cdot, \varphi_1 \rangle, \text{ if } \lambda = -\lambda_1. \end{cases}$$

Definition 1. *A stochastic K-process $\eta \in C^1(\mathcal{I}; \mathcal{B}_K\mathbf{L}_2)$ is called a solution to equation (4), if almost surely all trajectories of η satisfy equation (4) for all $t \in \mathcal{I}$. A solution $\eta = \eta(t)$ to equation (4) that satisfies the initial value condition*

$$\lim_{t \to 0+} L(\eta(t) - \eta_0) = 0 \tag{8}$$

is called a solution to Showalter–Sidorov problem (4), (8) if the solution satisfies condition (8) for some random K-variable $\eta_0 \in \mathcal{H}_K\mathbf{L}_2$.

Fix $\omega \in \Omega$, since the solution of the problem is considered trajectory. The [13] has obtained sufficient conditions for the existence of a solution to the Cauchy–Dirichlet problem for a Sobolev type stochastic model. The difficult problem is that the solution lies in a subset, while in the Showalter–Sidorov problem the solution lies in the space.

Theorem 1. *Let $\lambda \geq -\lambda_1$, $\alpha \in \mathbb{R}_+$, $n = 2$, $\forall p$ or $n \geq 3$, $2 \leq p \leq 2 + \dfrac{4}{n-2}$, then for any $\eta_0 \in \mathfrak{H}_K\mathbf{L}_2)$, there exists a unique solution $\eta \in \mathbf{C}^k(\mathcal{I}; \mathfrak{H}_K\mathbf{L}_2)$ to problem* (1) – (3).

Proof. So $\omega \in D$ is fixed that the proof of the Theorem is equivalent to the deterministic case [14].

2. Algorithm for a Numerical Method for Finding a Solution to the Showalter–Sidorov Problem

Based on the theoretical results and the modified Galerkin method, an algorithm was developed for the numerical method of solving the Showalter–Sidorov problem for the Oskolkov model, which allows one to find approximate solutions on a segment for given initial values and values of the coefficients α, λ, and also obtain a graph of the approximate solutions.

Consider the Oskolkov equation (2) with the boundary condition (1) and the Showalter–Sidorov condition (3). Here is an algorithm for finding an approximate solution to the problem (1) – (3):

Stage 0. Choose a monotonely decreasing numerical sequence $K = \{\mu_k\}$ such that $\sum\limits_{k=1}^{\infty} \mu_k^2 < +\infty$.

Stage 1. Finding the eigenvalues and eigenfunctions of the eigenvalues of the homogeneous Dirichlet problem for the Laplace operator $(-\Delta)$.

Stage 2. Representation of the required functions in the form of a Galerkin sum

$$\eta_N(s,t) = \sum_{k=1}^{N} \mu_k \eta_k(t) \varphi_k(s)$$

and substitution in (2).

Stage 3. Scalarly multiplying the equation obtained at the previous step by the eigenfunctions $\varphi_i(s)$, $k = 1, ..., m,$, we form a system of algebraic differential equations

$$\begin{gathered} \mathbf{E}\left(\int\limits_D \left[\lambda \eta_{kt} w + \nabla \eta_{kt} \cdot \nabla \varphi_i(s) + \alpha \nabla \eta_k \cdot \nabla \varphi_i(s) + |\eta_k|^{p-2} \eta_k \varphi_i(s)\right] \varphi_i ds\right) = 0, \\ w \in W_2^1(D), \quad i = 1, ..., k. \end{gathered} \tag{9}$$

Stage 4. Showalter–Sidorov initial conditions are set in the form

$$\mathbf{E}\left(\int\limits_D \left[\lambda\left(\eta_k(s,0) - \eta_0(s)\right)\varphi_i(s) + \nabla\left(\eta_k(s,0) - \eta_0(s)\right) \cdot \nabla \varphi_i(s)\right] ds\right) = 0, \tag{10}$$

where the function is $\eta_0 = \sum\limits_{k=1}^{\infty} \mu_k \eta_{0k} \varphi_k(s)$ whose coefficients are independent Gaussian random variables such that their variances are bounded ($\mathbf{D}\eta_{0k} \leq C$, $k \in \mathbb{N}$).

Stage 5. We find the solution of the system of algebraic-differential equations (9) with initial conditions (10) by the Runge–Kutta method of 4-5 orders.

Let us present the algorithm of a program that implements the Galerkin projection method. The program was written in the Maple 2017 application package, operated on an Intel (UX64) platform personal computer, and runs under Microsoft Windows. Based on the results of generating η_{0k} random variables, we obtain η_0. Fig. 1 shows the block diagram of the program. Let's describe its work step by step.

Step 1. We introduce the coefficients of the Oskolkov equation α, λ, the parameters of the domain D, the number of Galerkin approximations N, the parameter T of the time interval $[0, T]$, the parameters of the random influences – mathematical expectation and standard deviation, random variables are generated, which are part of the expansion for the initial function, using the function **stats[random, normald[μ,σ]](1)**.

Step 2. A separate procedure is used to find the eigenvalues and eigenfunctions of the homogeneous Dirichlet problem for the operator $(-\Delta)$.

Step 3. Using the **for i to 1 do N end do** loop, we form an approximate solution in the form of a Galerkin sum and substitute it into the (2) equation.

Step 4. A separate procedure, in the **for i to 1 do N end do** loop, forms a system of differential equations and a system of algebraic equations.

Step 5. The first equation of the system, which is algebraic, is solved at the initial time and η_{01} is found.

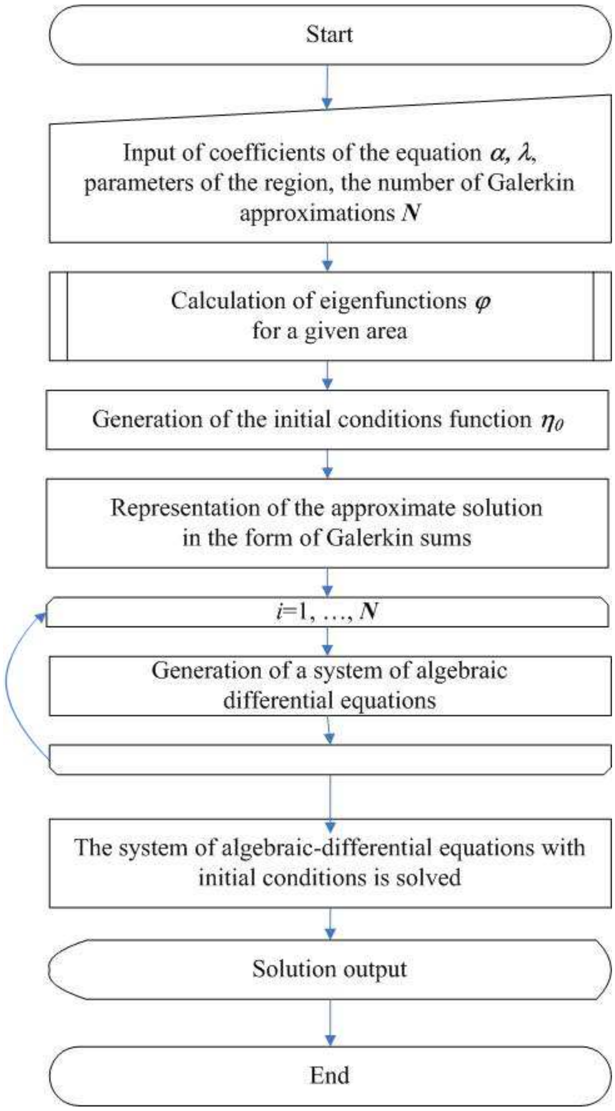


Fig. 1. Block diagram of the program for studying the Oskolkov model using the Galerkin projection method

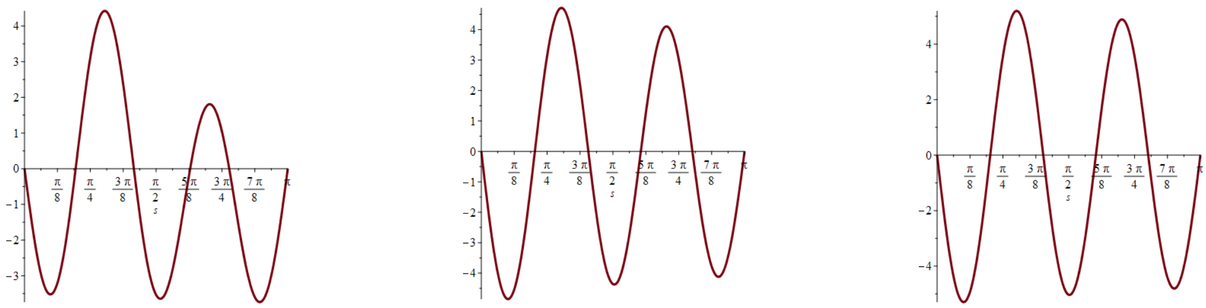
Step 6. A system consisting of algebraic and differential equations is solved, taking into account the expansion of the initial function and the η_{01} obtained at step 5. The solution is found using the **dsolve** built-in procedure.

Step 7. Functions for solving the problem are formed at times from 0 to T with a step frequency of $0,01T$.

Step 8. Using the **plot, plot3d** built-in procedures, 2D and 3D plots of the functions obtained in step 7 are displayed.

The study of a stochastic model requires a lot of computational experiments. Each of the experiments uses the above algorithm, where for the software implementation of this stochastic model, a generator of a random normally distributed variable with specified parameters of mathematical expectation and variance is used. Let us present the results of one of the computational experiments carried out. The following coefficients of the Oskolkov equation $\alpha = 1$ are given, $\lambda = -1$, the number of Galerkin approximations $N = 5$, the parameter $T = 1$ of the time interval [0, T], the parameters of the random action are the mathematical expectation equal to 0 and the standard deviation equal to 2. Consider the domain D as an interval from 0 to π. The generation of random variables included in the decomposition for the initial function gave the following results:

$$\eta_0 = -0,212286293540\sqrt{2}\sin(s) + 0,735283722265\sqrt{2}\sin(2s) - 0,113735508924\sqrt{2}\sin(3s) - -0,379582956388\sqrt{2}\sin(4s) - 2,40515897478\sqrt{2}\sin(5s).$$



$t = 0$ $\qquad$ $t = 0,5$ $\qquad$ $t = 1$

Fig. 2. Results of a computational experiment using the Galerkin projection method

As a result of the fifth step of the algorithm, we obtain a system of algebraic-differential equations of the form (9), which contains four differential equations and one algebraic equation - the first

$$3(\eta_1(t))^3 - 3(\eta_1(t))^2\eta_3(t) + 6\eta_1(t)(\eta_2(t))^2 - 6\eta_1(t)\eta_2(t)\eta_4(t) + 6\eta_1(t)(\eta_3(t))^2 - -6\eta_1(t)\eta_3(t)\eta_5(t) + 6\eta_1(t)(\eta_4(t))^2 + 6\eta_1(t)(\eta_5(t))^2 + 3(\eta_2(t))^2\eta_3(t) - 3(\eta_2(t))^2\eta_5(t) + +6\eta_2(t)\eta_3(t)\eta_4(t) + 6\eta_2(t)\eta_4(t)\eta_5(t) + 3(\eta_3(t))^2\eta_5(t) - 2\eta_1(t)\pi = 0,$$

$$6(\eta_1(t))^2\eta_2(t) - 3(\eta_1(t))^2\eta_4(t) + 6\eta_1(t)\eta_2(t)\eta_3(t) - 6\eta_1(t)\eta_2(t)\eta_5(t) + +6\eta_1(t)\eta_3(t)\eta_4(t) + 6\eta_1(t)\eta_4(t)\eta_5(t) + 3(\eta_2(t))^3 + 6\eta_2(t)(\eta_3(t))^2 + 6\eta_2(t)(\eta_4(t))^2 + +6\eta_2(t)(\eta_5(t))^2 + 3(\eta_3(t))^2\eta_4(t) + 6\eta_3(t)\eta_4(t)\eta_5(t) + 6\pi(\eta_{2t}^{'}(t))^2 - 8\eta_2(t)\pi = 0,$$

$$-(\eta_1(t))^3 + 6(\eta_1(t))^2\eta_3(t) - 3(\eta_1(t))^2\eta_5(t) + 3\eta_1(t)(\eta_2(t))^2 + 6\eta_1(t)\eta_2(t)\eta_4(t) + +6\eta_1(t)\eta_3(t)\eta_5(t) + 6(\eta_2(t))^2\eta_3(t) + 6\eta_2(t)\eta_3(t)\eta_4(t) + 6\eta_2(t)\eta_4(t)\eta_5(t) + 3(\eta_3(t))^3 + +6\eta_3(t)(\eta_4(t))^2 + 6\eta_3(t)(\eta_5(t))^2 + 3(\eta_4(t))^2\eta_5(t) + 16\pi(\eta_{2t}^{'}(t))^3 - 18\eta_3(t)\pi = 0,$$

$$-3(\eta_1(t))^2\eta_2(t) + 6(\eta_1(t))^2\eta_4(t) + 6\eta_1(t)\eta_2(t)\eta_3(t) + 6\eta_1(t)\eta_2(t)\eta_5(t) + 6(\eta_2(t))^2\eta_4(t) + +3\eta_2(t)(\eta_3(t))^2 + 6\eta_2(t)\eta_3(t)\eta_5(t) + 6(\eta_3(t))^2\eta_4(t) + 6\eta_3(t)\eta_4(t)\eta_5(t) + 3(\eta_4(t))^3 + +6\eta_4(t)(\eta_5(t))^2 + 30\pi(\eta_{2t}^{'}(t))^4 - 32\eta_4(t)\pi = 0,$$

$$-3(\eta_1(t))^2\eta_3(t) + 6(\eta_1(t))^2\eta_5(t) - 3\eta_1(t)(\eta_2(t))^2 + 6\eta_1(t)\ eta_2(t)\eta_4(t) + 3\eta_1(t)(\eta_3(t))^2 + +6(\eta_2(t))^2\eta_5(t) + 6\eta_2(t)\eta_3(t)\eta_4(t) + 6(\eta_3(t))^2\eta_5(t) + 3\eta_3(t)(\eta_4(t))^2 + 6(\eta_4(t))^2\eta_5(t) + +3(\eta_5(t))^3 + 48\pi(\eta_{2t}^{'}(t))^5 - 50\eta_5(t)\pi = 0$$

with Showalter–Sidorov conditions

$$\eta_2(0) = 1,30325646504, \eta_3(0) = -0,201590940777,$$

$$\eta_4(0) = -0,672793272788, \eta_5(0) = -4,26303328689.$$

Then an approximate solution of the initial-boundary value problem is found, the result which is graphically presented in Fig. 2 in the form of three-dimensional graphs at different times t through equal time gaps.

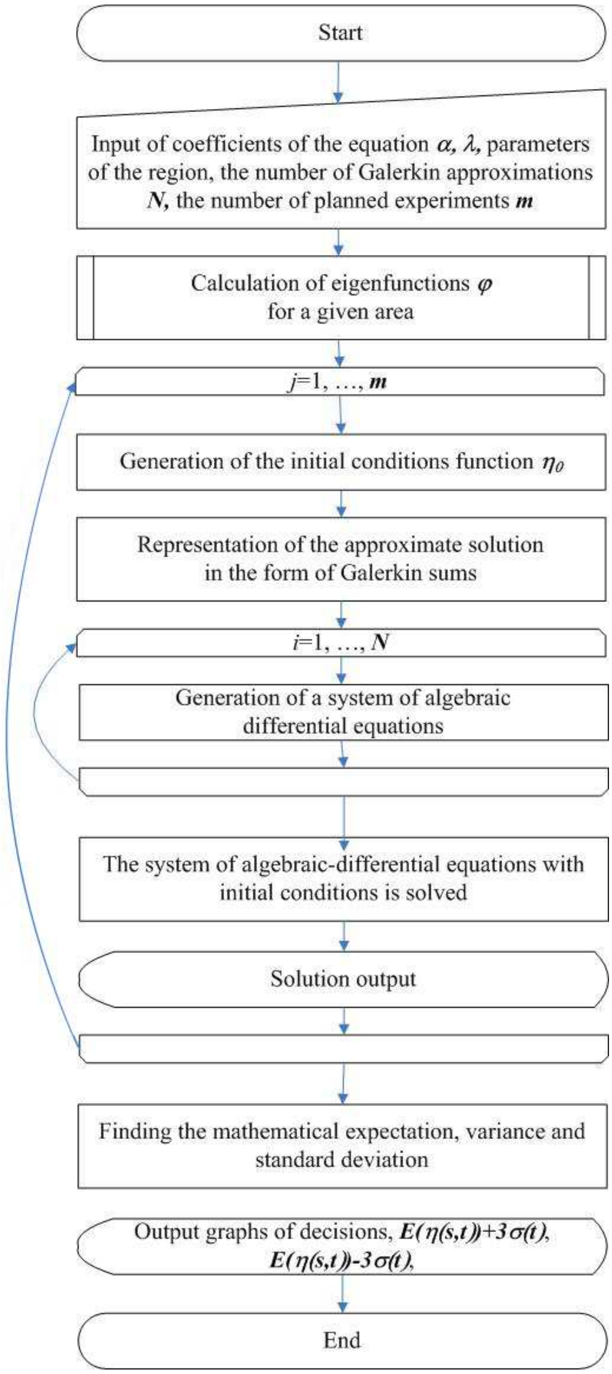
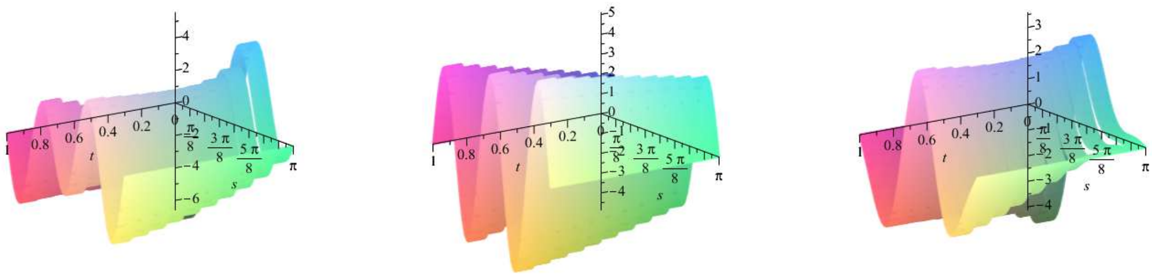


Fig. 3. Block diagram of the information processing algorithm for the numerical study of the Oskolkov model

3. Information Processing and Algorithm of Its Software Implementation. Results of Computational Experiments

The study of a stochastic model involves m computational experiments, each of which uses a generator of a random normally distributed variable with given parameters of mathematical expectation and variance for modeling. After the generation of random variables, the first three stages of the numerical solution of the Showalter–Sidorov–Dirichlet problem for the stochastic Oskolkov equation (section 2) are implemented. For subsequent processing of the results, a loop is run on i, which allows processing the results of m experiments in one program. Each cycle will allow you to get several implementations of the solution. The algorithm for the numerical study of the Oskolkov stochastic model is shown in Fig. 3.



Experiment 1 | Experiment 5 | Experiment 10

Fig. 4. Graphs of the function $\tilde{\eta}(s,t)$, $i = 1, 5, 10$

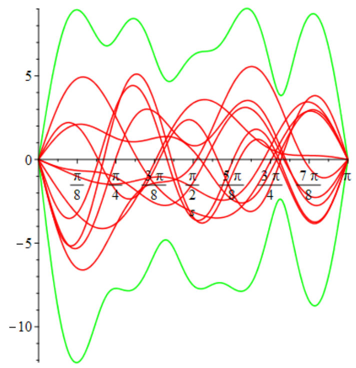


Fig. 5. Expectation $\mathbf{E}(\tilde{\eta}(s,0))$ and its estimate

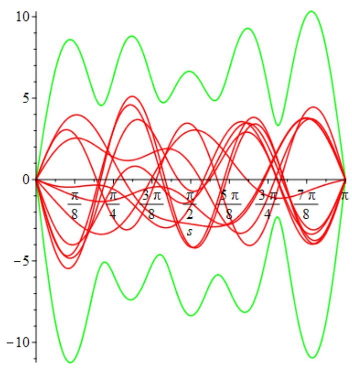


Fig. 6. Expectation $\mathbf{E}(\tilde{\eta}(s,0.5))$ and its estimate

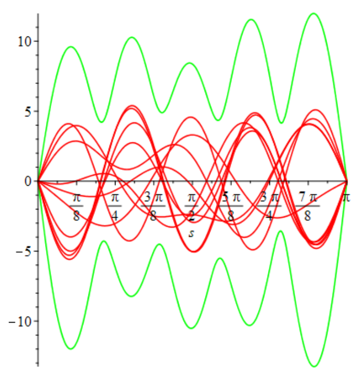


Fig. 7. Expectation $\mathbf{E}(\tilde{\eta}(s,1))$ and its estimate

According to the experimental results, the mathematical expectation will be the function $\mathbf{E}(\tilde{\eta}(s,t))$, which for any value of t is equal to the mathematical expectation of the corresponding section, i.e. the averaged trajectory (realization) obtained as a result of processing m experiments. The sample mean variance and standard deviation will be functions of $\mathbf{D}(\tilde{\eta}(s,t))$ and $\sigma_{\tilde{\eta}}(t)$, which for any value of t are equal to the variance and standard deviation of the corresponding sections of the random process.

As a result, with a probability of 0,997 we can use the estimate

$$\|\eta(s,t) - \mathbf{E}(\tilde{\eta}(s,t))\| < 3\sigma_{\tilde{\eta}}(t). \tag{11}$$

As an example of processing information obtained as a result of a series of computational experiments, we will use the results of ten experiments CE1 – CE10. Table presents the values of random variables of computational experiments.

Table

Values of random variables in computational experiments CE1 – CE10

	$\eta_{01}(t)$	$\eta_{02}(t)$	$\eta_{03}(t)$	$\eta_{04}(t)$	$\eta_{05}(t)$
CE1	3,133285	1,253314	$1,110223 \cdot 10^{-16}$	2,506628	3,759942
CE2	2,351236	0,681369	0,458532	0,942530	0,900221
CE3	2,215488	0,562147	0,493288	0,671034	0,304110
CE4	2,137601	0,518318	0,498064	0,515260	0,084965
CE5	2,085840	0,501462	0,498692	0,411737	0,000682
CE6	2,050346	0,494856	0,498748	0,340750	−0,032346
CE7	2,025767	0,492238	0,498738	0,291592	−0,045437
CE8	2,008669	0,491190	0,498727	0,257396	−0,050675
CE9	0,025985	0,490767	0,498720	0,233544	−0,052793
CE10	1,988412	0,490593	0,498716	0,216882	−0,053660

On Figure 4 shows graphs of the function $\tilde{\eta}(s,t)$, $i = 1, 5, 10$. Graphs are placed one by one according to the experiment number. On Figure 5 shows the execution of the estimate (11), the red lines represent the graphs of the functions $\tilde{\eta}(s,0)$, $i = 1,\ldots,10$, the green lines represent the functions $\mathbf{E}(\tilde{\eta}(s,0)) + 3\sigma_{\tilde{\eta}}(0)$ and $\mathbf{E}(\tilde{\eta}(s,0)) - 3\sigma_{\tilde{\eta}}(0)$ obtained numerically. On Fig. 6 combines several graphs of functions. The red lines represent the graphs of the functions $\tilde{\eta}(s,0,5)$, $i = 1,\ldots,10$, the green lines represent the functions $\mathbf{E}(\tilde{\eta}(s,0,5)) + 3\sigma_{\tilde{\eta}}(0,5)$ and $\mathbf{E}(\tilde{\eta}(s,0,5)) - 3\sigma_{\tilde{\eta}}(0,5)$ obtained numerically. Note that the evaluation (11) is performed. On Figure 7 shows the graphs of the functions $\tilde{\eta}(s,1)$, $i = 1,\ldots,10$, $\mathbf{E}(\tilde{\eta}(s,1)) + 3\sigma_{\tilde{\eta}}(1)$ and $\mathbf{E}(\tilde{\eta}(s,1)) - 3\sigma_{\tilde{\eta}}(1)$. Graphs of $\tilde{\eta}(s,1)$, $i = 1,\ldots,10$ are represented by green lines, the red lines represent functions $\mathbf{E}(\tilde{\eta}(s,1)) + 3\sigma_{\tilde{\eta}}(1)$ and $\mathbf{E}(\tilde{\eta}(s,1)) - 3\sigma_{\tilde{\eta}}(1)$ obtained numerically, the estimate (11) is satisfied.

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АЛГОРИТМ ЧИСЛЕННОГО МЕТОДА НАХОЖДЕНИЯ РЕШЕНИЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ НЕЛИНЕЙНОЙ ФИЛЬТРАЦИИ СО СЛУЧАЙНЫМ НАЧАЛЬНЫМ УСЛОВИЕМ ШОУОЛТЕРА–СИДОРОВА

Н.А. Манакова, К.В. Перевозчикова, О.В. Гаврилова, И.М. Манаков

Статья посвящена исследованию нелинейной модели фильтрации жидкости, основанной на стохастическом уравнении Осколкова. Предполагается, что на экспериментальные начальные данные влияет «шум», который приводит к исследованию стохастической модели с производной Нельсона–Гликлиха. Построены достаточные условия существования решений исследуемой модели с начальным условием Шоуолтера–Сидорова. Построен алгоритм численного метода решения и представлен вычислительный эксперимент.

Ключевые слова: уравнения соболевского типа; стохастическая модель нелинейной фильтрации; производная Нельсона–Гликлиха.

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