MSC 35R30 DOI: 10.14529/jcem220205

IDENTIFICATION OF THERMOPHYSICAL PARAMETERS IN MATHEMATICAL MODELS OF HEAT AND MASS TRANSFER

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Under consideration are mathematical models of heat and mass transfer. We consider inverse problems of recovering coefficients in the main part of a parabolic equation occurring simultaneously in a Robin-type boundary condition. The overdetermination conditions are values of a solution at some collection of points lying inside the domain. In particular, in the class of these inverse problems the classical problems of recovering the thermal conductivity tensor are included. The main attention is paid to existence, uniqueness, and stability estimates for solutions to inverse problems of this type. The problem is reduced to an operator equation which is studied with the use of the fixed point theorem and a priori estimates. The method of the proof is constructive and it can be used in developing new numerical algorithms for solving the problem.

Keywords: inverse problem; heat and mass transfer; heat conductivity; parabolic equation.

Dedicated to anniversary of Professor A.L. Shestakov

Introduction

We examine the question of recovering heat conductivity and other thermophysical characteristics in mathematical models of heat and mass transfer. Let G be a domain in $\mathbb{R}^n$ with boundary Γ of class C^2. Assign $Q = (0,T) \times G$. Under consideration is the parabolic equation

$$u_t + A(t,x,D)u = \sum_{i=1}^{r} b_i(t,x)q_i(t) + f, \quad (t,x) \in Q, \tag{1}$$

where A is an elliptic operator of the second order representable as

$$A(t,x,D)u = -\sum_{i,j1}^{n} a_{ij}(t,x)u_{x_ix_j} + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0 u,$$

The equation (1) is furnished with the initial and boundary conditions

$$u|_{t=0} = u_0, \;\; Bu|_S = \sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j}(t,x)\nu_i + \sigma(t,x)u(t,x)|_S = g(t,x), \; S = (0,T)\times\Gamma, \tag{2}$$

where $\vec{\nu} = (\nu_1,\ldots,\nu_n)$ is the outward unit normal to Γ. The unknowns in (1), (2) are a solution u and the function $q_i(t)$ $(i = 1,2,\ldots,s)$ occurring into the right-hand side of (1) and the operator A as coeffcients. If the functions $\{q_i\}_{i=1}^{s}$ occur into the main part of the operator A then we arrive at the classical problem of recovering the thermal conductivity tensor [1, 2]. The overdetermination conditions for recovering the functions $\{q_i\}_{i=1}^s$ are as follows:

$$u|_{x=y_i} = \psi_i(t), \quad i = 1, 2, \ldots, s, \tag{3}$$

where $\{y_i\}$ is a collection of points in G. The problems (1)-(3) arise in many fields. The continuous emergence of new materials with complex structures (anisotropic, multilayers, porous, and heterogeneous) in various industrial sectors appeals their thermal characterization to ensure the control and the modelling of the heat transfers through the processes. Essential attention has been paid to the identification of thermophysical properties of such materials for many years. Many articles were devoted to estimation of diffusivity, effusivity, conductivity, heat capacity and other parameters. The problems of estimating the thermophysical parameters arise also in description of heat regimes of permafrost zone [3]. The development of permafrost regions occurs with the disturbance of natural conditions (deforestation, removal of snow and ground cover, development of vegetation), which lead to a change in geocryological conditions and the development of negative cryogenic processes. Assessment of the thermal state of frozen soils under climate change and anthropogenic impacts is very important in northen regions.

The theoretical result devoted to the problems (1), (2), (3) are mainly connected with the one-dimensional case. In particular, we can refer to [4, Sect.4.3], [5]-[8], where existence and uniqueness theorems in Hölder spaces are established in the case of the heat conductivity depending on time. In the case of the heat conductivity depending only on the space variable, some numerical methods for solving the problem are described in [9, 10]. The main overdetermination conditions are the additional Dirichlet data at the boundary points of the interval, on which the equation is considered or the integral of a solution over this interval with some weight. The articles [11, 12] deal with the multidimensional problems with integral and pointwise overdetermination conditions, respectively. In contrast to [11, 12], the main peculiarity of the problem (1)-(3) is that the unknowns appear in the equation (1) and the boundary condition (2) as well and this fact essentially complicates the problem. It is often the case when the heat conductivity or capacity depends on temperature. As we know, there are no serious theoretical results in this case but a lot of articles are devoted to a numerical methods for solving these inverse problems We can refer to the articles [13]-[16], where the heat conductivity or capacity are restored with the use of additional boundary data (for instance, additional Dirichlet data on the boundary) and to [17]-[23] where the unknowns are recovered with the use of temperature measurements at some points in the domain. It is often the case that the results of numerical calculations not the heat conductivity $k(u)$ but the value of this functions at some collection of points. Sometimes $k(u)$ is a polynomial in u, in this case its coefficients are to be determined. We also refer to the known monographs [24]-[29], where the reader can find the known results and the bibliography concerning with parabolic inverse problems.

We look for the unknown coefficients of the operator A and the source function in the form $\sum_{i=1}^m q_i(t)\phi_i(t,x)$, where $\phi_i(t,x)$ are some basis functions and q_i are unknown Fourier coefficients. In this case the operator A is representable as

$$\begin{gathered} A = \textstyle\sum_{i=r+1}^s q_i(t)A_i + A_{s+1}, \quad \vec{q} = (q_{r+1}, \ldots, q_s), \\ A_k u = -\textstyle\sum_{i,j=1}^n a_{ij}^k(t,x)u_{x_i x_j} + \sum_{i=1}^n a_i^k(t,x)u_{x_i} + a_0^k u \quad (i = r+1, \ldots, s+1), \end{gathered} \tag{4}$$

where the coefficients a_{ij}^k are known functions. In the present article we prove existence and uniqueness theorems for solutions to the problem (1)-(3). The first section contains conditions on the data of the problem and auxiliary results. Main results are presented in the second section.

1. Preliminaries

Let E be a Banach space. The symbol $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) stands for the space of strongly measurable functions defined on G with values in E and a finite norm $\|\|u(x)\|_E\|_{L_p(G)}$ [15]. The notations of the Sobolev spaces $W_p^s(G;E)$ and $W_p^s(Q;E)$ are conventional (see [30, 31]). If $E=\mathbb{R}$ or $E=\mathbb{R}^n$ then the latter space is denoted by $W_p^s(Q)$. The definitions of the Hölder spaces $C^{\alpha,\beta}(\overline{Q})$, $C^{\alpha,\beta}(\overline{S})$ can be found, for example, in [32]. The function spaces and coefficients of (1) are assumed real. By the norm of a vector, we mean the sum of the norms of its coordinates. Given an interval $J=(0,T)$, put $W_p^{s,r}(Q)=W_p^s(J;L_p(G))\cap L_p\big(J;W_p^r(G)\big)$ and $W_p^{s,r}(S)=W_p^s(J;L_p(\Gamma))\cap L_p\big(J;W_p^r(\Gamma)\big)$. Let $(u,v)=\int\limits_G u(x)v(x)\,dx$.

Let $B_\delta(b)$ be a ball of radius δ centered at b. A parameter $\delta>0$ is called *admissible* if $\overline{B_\delta(y_i)}\cap\Gamma=\emptyset$, $\overline{B_\delta(y_i)}\cap\overline{B_\delta(y_j)}=\emptyset$ for $i\neq j$, $i,j=1,2,\ldots,s$. Introduce the following notations: $Q^\tau=(0,\tau)\times G$, $G_\delta=\bigcup\limits_i B_\delta(b_i)$, $S^\tau=(0,\tau)\times\Gamma$. Considering the problem (1)-(3), we assume that $\Gamma\in C^2$ (see the definition in [32]). Endow the space $W_p^s(0,\beta;E)$ ($s\in(0,1)$, $\beta>0$, E is a Banach space) with the norm

$$\|q(t)\|_{W_p^s(0,\beta;E)}=\big(\|q\|^p_{L_p(0,\beta;E)}+\langle q\rangle^p_{s,\beta}\big)^{1/p},\ \ \langle q\rangle^p_{s,\beta}=\int\limits_0^\beta\int\limits_0^\beta\frac{\|q(t_1)-q(t_2)\|_E^p}{|t_1-t_2|^{1+sp}}\,dt_1dt_2.$$

If $E=\mathbb{R}$ then we obtain the usual space, $W_p^s(0,\beta)$. Given $s\in(0,1)$, denote $\tilde W_p^s(0,\beta;E)=\big\{q\in W_p^s(0,\beta;E):t^{-s}q(t)\in L_p(0,\beta;E)\big\}$. Endow this space with the norm

$$\|q(t)\|^p_{\tilde W_p^s(0,\beta;E)}=\Big\|\frac{q}{t^s}\Big\|^p_{L_p(0,\beta;E)}+\langle q\rangle^p_{s,\beta}.$$

If $s>1/p$ and $q\in\tilde W_p^s(0,\beta;E)$ then $q(0)=0$ and this norm and the usual norm $\|\cdot\|_{W_p^s(\alpha,\beta;E)}$ for functions $q(t)$ such that $q(0)=0$ are equivalent (see [15, Subsect. 3.2.6, Lemma 1]). The spaces $\tilde W_p^s(0,\beta;L_p(G))$ and $\tilde W_p^{s,2s}(Q^\beta)=\tilde W_p^s(0,\beta;L_p(G))\cap L_p(0,\beta;W_p^{2s}(G))$ for $s\neq1/p$ comprise the functions $v(t,x)$ in $W_p^s(0,\beta;L_p(G))$ and $W_p^{s,2s}(Q^\beta)$, respectively, such that $v(0,x)=0$ for $s>1/p$. The norms $\|\cdot\|_{\tilde W_p^{s,2s}(Q^\beta)}$ and $\|\cdot\|_{\tilde W_p^s(0,\beta;L_p(G))}$ are defined naturally, i. e.,

$$\|u\|_{\tilde W_p^{s,2s}(Q^\beta)}=\big(\|u\|^p_{\tilde W_p^s(0,\beta;L_p(G))}+\|u\|^p_{L_p(0,\beta;W_p^{2s}(G))}\big)^{1/p}.$$

The spaces $\tilde W_p^s(0,\beta;L_p(\Gamma))$ and $\tilde W_p^{s,2s}(S^\beta)$ are defined by analogy.

The following lemma is known (see [33], Lemmas 1–4).

Lemma 1. *Assume that G is a bounded domain with boundary of the class C^2, $Q^\tau=(0,\tau)\times G$, and $S^\tau=(0,\tau)\times\partial G$. There exists a constant C independent of $\tau\in(0,T]$ such* *that*

$$\|v\|_{\tilde W_p^{s_1,2s_1}(S^\tau)} + \left\|\frac{\partial v}{\partial \nu}\right\|_{\tilde W_p^{s_0,2s_0}(S^\tau)} \le C\|v\|_{W_p^{1,2}(Q^\tau)}, \quad s_0 = \frac{1}{2} - \frac{1}{2p},\ s_1 = 1 - \frac{1}{2p},$$

for all $v \in W_p^{1,2}(Q^\tau)$ *such that* $v(x,0) = 0$. *Here* $\frac{\partial v}{\partial \nu}$ *is a derivative with respect to the outward normal to* ∂G.

The following lemma can be found in [33, lemma 2].

Lemma 2. *Assume that* $s \in ((n+2)/2p, 1)$ *and* $p > n+2$. *Then the product* qv *of the functions in* $W_p^{s,2s}(Q^\tau)$ $(\tau \in (0,T])$ *belongs to* $W_p^{s,2s}(Q^\tau)$, *and if* $q \in \tilde W_p^{s,2s}(Q^\tau)$ *and* $v \in W_p^{s,2s}(Q^\tau)$ *then* $qv \in \tilde W_p^{s,2s}(Q^\tau)$ *and the following estimate holds:*

$$\|qv\|_{\tilde W_p^{s,2s}(Q^\tau)} \le c_0\|q\|_{\tilde W_p^{s,2s}(Q^\tau)}(\|v\|_{W_p^{s,2s}(Q^\tau)} + \|v\|_{L_\infty(Q^\tau)}).$$

If $v \in W_p^{s,2s}(Q)$ *then the last inequality can be rewritten in the form*

$$\|qv\|_{\tilde W_p^{s,2s}(Q^\tau)} \le c_1\|q\|_{\tilde W_p^{s,2s}(Q^\tau)}\|v\|_{W_p^{s,2s}(Q)},$$

and if $v \in \tilde W_p^{s,2s}(Q^\tau)$ *then*

$$\|qv\|_{\tilde W_p^{s,2s}(Q^\tau)} \le c_2\|q\|_{\tilde W_p^{s,2s}(Q^\tau)}\|v\|_{\tilde W_p^{s,2s}(Q^\tau)}.$$

where the constants c_i, $i = 0,1,2$, *are independent of* q, v *and* $\tau \in (0,T]$. *The set* Q^τ *can be replaced* S^τ *and the claim is valid for* $s \in ((n+1)/2p, 1)$. *If* q *depends on only one variable* t *then the norm of* q *in* $\tilde W_p^{s,2s}(Q^\tau)$ *in this inequalities is replaced with the norm of* q *in* $\tilde W_p^s(0,\tau)$. *If both functions are independent of* x *then the claim remains valid but the norms* $\tilde W_p^{s,2s}(Q^\tau)$ *and* $W_p^{s,2s}(Q^\tau)$ *in the above inequalities are replaced with the norms in* $\tilde W_p^s(0,\tau)$ *and* $W_p^s(0,\tau)$, *respectively. In the case of* $q(t), v(t) \in \tilde W_p^{s_0}(0,\tau)$ *we have the estimate*

$$\|qv\|_{\tilde W_p^{s_0}(0,\tau)} \le \|q\|_{L_\infty(0,\tau)}\|v\|_{\tilde W_p^{s_0}(0,\tau)} + \|v\|_{L_\infty(0,\tau)}\|q\|_{\tilde W_p^{s_0}(0,\tau)}.$$

In what follows, we assume that an admissible parameter $\delta > 0$ is fixed.

The smoothness and consistency conditions on the data can be written as follows:

$$u_0(x) \in W_p^{2-2/p}(G),\ g \in W_p^{s_0,2s_0}(S),\ s_0 = 1/2 - 1/2p,\ f \in L_p(Q); \tag{5}$$

$$u_0(x) \in W_p^{4-2/p}(G_\delta),\ f \in L_p(0,T; W_p^2(G_\delta)); \tag{6}$$

$$\psi_j(t) \in W_p^{3/2-1/2p}(0,\tau), \quad \psi_j(0) = u_0(x_j),\ j = 1,2,\dots,s; \tag{7}$$

$$a_{ij}^k \in C(\overline{Q}),\ a_i^k \in L_p(Q),\ \sigma, a_{ij}^k|_S \in W_p^{s_0,2s_0}(S), \quad p > n+2; \tag{8}$$

$$a_{ij}^k \in L_\infty(0,T; W_p^2(G_\delta)), \quad a_l^k \in L_p(0,T; W_p^2(G_\delta)),$$
$$i,j = 1,2,\dots,n,\ l = 0,1,\dots,n,\ k = r+1,\dots,s; \tag{9}$$

$$b_j(t,x) \in L_p(Q), \quad b_j(t,x) \in L_p(0,T; W_p^2(G_\delta)), \quad (j = 1,2,\dots,r). \tag{10}$$

We look for the functions q_i in the class $W_p^{s_0}(0,T)$. So we need additional smoothness conditions

$$a_{ij}^k(t,y_l),\ a_q^k(t,y_l), f(t,y_l), b_m(t,y_l) \in W_p^{s_0}(0,T), \tag{11}$$

for all $q=0,1,\dots,n$, $i,j=1,2,\dots,n$, $k=r+1,1,\dots,s$, $m=1,2,\dots,r$, $l=1,\dots,s$. In view of the conditions (9), the traces $b_m(t,y_p), a_l^k(t,y_j)$ are defined and $b_m(t,y_p), a_l^k(t,y_j) \in L_p(0,T)$; even more we have that $b_m(t,y_p), a_l^k(t,x) \in C(\overline{G_\delta}; L_p(0,T))$ after a possible modification of these functions on sets of zero measure.

Introduce a matrix $B(t)$ of dimension $s\times s$ with rows

$$(-b_1(t,x), -b_2(t,x), \dots, -b_r(t,x), A_{r+1}u_0(x), \dots, A_s u_0(x))|_{x=y_j}. \tag{12}$$

Under the above conditions, it is easy to demonstrate that the entries of this matrix belong to the space $W_p^{s_0}(0,T)$. We require that there exists a constant $\delta_0>0$ such that

$$|\det B(t)| \geq \delta_0 \ \ \forall t\in[0,T]. \tag{13}$$

Consider the system

$$B(t)\vec{q}^{\,0} = \vec{g},\ \vec{q}^{\,0} = \left(q_1^0, q_2^0, \dots, q_s^0\right)^T,$$

where $\vec{g}$ is a vector with the coordinates $f(t,y_j) - A_{s+1}u_0(y_j) - \psi_{jt}(t)$ $(j=1,2,\dots,s)$. Under the condition (11), this system has a unique solution $\vec{q}^{\,0} = (q_1^0,\dots,q_s^0) = (B(t))^{-1}\vec{g}(t)$. The above conditions on the data and Lemma 2 ensure that $\vec{q}^{\,0} \in W_p^{s_0}(0,T) \subset C([0,T])$.

Introduce the operators $A_0(t,x,D_x) = -\sum_{i,j=1}^n a_{ij}^0(t,x)u_{x_ix_j} + \sum_{i=1}^n a_i^0(t,x)u_{x_i} + a_0^0 u = \sum_{i=r+1}^s q_i^0(t)A_i + A_{s+1}$, $B_0 u = \sum_{i,j=1}^n a_{ij}^0(t,x)u_{x_j}\nu_i + \sigma u$ and assume that there exists a constant $\delta_0>0$ such that

$$\sum_{i,j=1}^n a_{ij}^0(t,x)\xi_i\xi_j \geq \delta_0|\xi|^2, \forall \xi\in\mathbb{R}^n,\ \forall (t,x)\in Q.$$

We need also the consistency condition

$$B_0u_0(0,x)|_\Gamma = g(0,x). \tag{14}$$

The former part of the following theorem results from Theorem 2.1 in [31] and the proof of the latter is quite similar to those in Theorems 1.1 in [34] and Theorem 2 in [35]. Additional interior smoothness of solutions is established in many books, for instance, we can refer to Ch. 3,5 in [32].

Theorem 1. *Assume that G is a bounded domain with boundary of class C^2 and the conditions* (5), (8), (14) *hold. Then there exists a unique solution $u\in W_p^{1,2}(Q)$ to the problem*

$$u_t + A_0(t,x,D_x)u = f,\ u|_{t=0} = u_0(x), \quad B_0u|_S = g \tag{15}$$

satisfying the estimate

$$\|u\|_{W_p^{1,2}(Q)} \leq c\left[\|f\|_{L_p(Q)} + \|g\|_{W_p^{s_0,2s_0}(S)} + \|u_0\|_{W_p^{2-2/p}(G)}\right], \tag{16}$$

with c a constant independent of f, g, u_0 and a solution u. If additionally $u_0=0, g(0,x)=0$ then a solution u satisfies the estimate

$$\|u\|_{W_p^{1,2}(Q^\tau)} \leq c(\|f\|_{L_p(Q^\tau)} + \|g\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)}), \tag{17}$$

where a constant c is independent of τ. If in addition the conditions (6), (9) *hold then* $u \in L_p(0,T;W_p^4(G_{\delta_1}))$, $u_t \in L_p(0,T;W_p^2(G_{\delta_1}))$ *for every* $\delta_1 \in (0,\delta)$. *If* $u_0 = 0, g(0,x) = 0$ *and* $\delta_1 \in (0,\delta)$ *then a there exists a constant* $c > 0$ *such that*

$$\|u\|_{W_p^{1,2}(Q^\tau)} + \|u\|_{L_p(0,\tau;W_p^4(G_{\delta_1}))} + \|u_t\|_{L_p(0,\tau;W_p^2(G_{\delta_1}))} \leq c(\|f\|_{L_p(Q^\tau)} + \|g\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)} + \|f\|_{L_p(0,T;W_p^2(G_\delta))}), \quad (18)$$

where the constant c is independent of $\tau \in (0,T]$.

2. Main results

Our main result is the following theorem

Theorem 2. *Assume that the conditions* (5)-(11), (13), (14) *are fulfilled. Then on some interval* $(0,\tau_0)$ *there exists a unique solution* $(u, q_1, \ldots, q_s)$ *to the problem* (1)-(3) *of the class*

$$u \in W_p^{1,2}(Q^{\tau_0}): \quad u \in L_p(0,\tau_0;W_p^4(G_{\delta_1})), u_t \in L_p(0,\tau_0;W_p^2(G_{\delta_1}))\ \forall \delta_1 \in (0,\delta),$$
$$q_j \in W_p^{s_0}(0,\tau_0),\ j = 1,2,\ldots,s.$$

Proof. Let u be a solution to the problem (1)-(3). Consider an auxiliary problem

$$\Phi_t + A_0\Phi = f_0, \quad \Phi(0,x) = u_0(x),\ B_0\Phi|_S = g, \quad (19)$$

where

$$f_0 = f + \sum_{i=1}^{r} b_i(t,x)q_i^0(t), \quad B_0\Phi = \sum_{i,j=1}^{n} a_{ij}^0(t,x)\Phi_{x_j}(t,x)\nu_i + \sigma(t,x)u(t,x).$$

By Theorem 1, a solution Φ to this problem exists and

$$\Phi \in W_p^{1,2}(Q), \Phi \in L_p(0,\tau_0;W_p^4(G_{\delta_1})), \Phi_t \in L_p(0,\tau_0;W_p^2(G_{\delta_1}))\ \forall \delta_1 \in (0,\delta).$$

After the change of variables $u = v + \Phi$, we arrive at the problem

$$v_t + A_0v + A(\vec{\mu})v = -A(\vec{\mu})\Phi + \sum_{i=1}^{r} b_i(t,x)\mu_i(t),\ \mu_i(t) = q_i(t) - q_i^0(t). \quad (20)$$

$$v|_{t=0} = 0, \quad B_0v + B(\vec{\mu})v|_S = -B(\vec{\mu})\Phi, \quad (21)$$

$$v(t,y_j) = \psi_j - \Phi(t,y_j) = \tilde{\psi}_j, \quad j = 1,2,\ldots,s, \quad (22)$$

where $A(\vec{\mu})v = \sum_{k=r+1}^{s}\mu_k(t)A_k$, $B(\vec{\mu})v = \sum_{k=r+1}^{s}\mu_k(t)a_{ij}^k v_{x_j}\nu_i$, $\vec{\mu} = (\mu_1,\ldots,\mu_s)$. Thus, we reduce the problem (1)-(3) to an equivalent problem (20)-(22). Fixing the functions $\mu_j \in \tilde{W}_p^{s_0}(0,\tau)$ and constructing a solution v to the problem (20)-(21) on $(0,\tau)$, we obtain a mapping $v = v(\vec{\mu})$. Study its properties. By definition, $\|\vec{\mu}\|_{\tilde{W}_p^{s_0}(0,\tau))} = \sum_{i=1}^{s}\|\mu_i\|_{\tilde{W}_p^{s_0}(0,\tau)}$. Denote by a_{ij}^τ the higher-order coefficients of the operator $A_0 + A(\vec{\mu})$. In view of the structure of this operator, there exist a parameter μ_0 such that, for $\|\vec{\mu}\|_{C([0,T])} \leq \mu_0$,

$$\sum_{i,j=1}^{n} a_{ij}^\tau\xi_i\xi_j \geq \delta_0|\xi|^2/2\ \forall \xi \in \mathbb{R}^n,\ \forall (t,x) \in Q.$$

Theorem 1 yields that a solution to the problem (20), (21) exists and the corresponding estimates that of Theorem 1 hold. Without loss of generality, we can assume that constants from these estimates are independent of $\tau \in (0,T]$. Thus, we have

$$\|v\|_{W_p^{1,2}(Q^\tau)} \leq c(\| -A(\vec{\mu})\Phi + \sum_{i=1}^{r} b_i(t,x)\mu_i(t)\|_{L_p(Q^\tau)} + \\ +\|B(\vec{\mu})\Phi\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)} \leq c_1\|\vec{\mu}\|_{W_p^{s_0,2s_0}(0,\tau)}, \tag{23}$$

where a constant c is independent of τ and we employ Lemmas 1, 2. Fix $\delta_1 < \delta_2 < \delta$ and write out the last estimate of Theorem 1, where we take the set G_{δ_2} rather than G_δ on the right-hand side. Similarly, we obtain that

$$\|v\|_{W_p^{1,2}(Q^\tau)} + \|v\|_{L_p(0,\tau;W_p^4(G_{\delta_2}))} + \|v_t\|_{L_p(0,\tau;W_p^2(G_{\delta_2}))} \leq c_2\|\vec{\mu}\|_{W_p^{s_0,2s_0}(0,\tau)}, \tag{24}$$

where c_2 is independent of τ but it depends on $\delta_1,\ \delta_2$. Let $\vec{\mu} \in B_M = \{\vec{\mu} : \|\vec{\mu}\|_{\tilde{W}_p^{s_0}(0,\tau))} \leq M\}$. The parameter $M \leq \mu_0$ is chosen below. We have the embeddings (see Sect. 6.3 in [36] and Theorem 18.9 in [37], the last inclusion is well known [38]).

$$W_p^{s_0,2s_0}(S) \subset C^{1/2-(n+2)/2p,1-(n+2)/p}(\overline{S}),\ W_p^{1,2}(Q) \subset C^{1-(n+2)/2p,2-(n+2)/p}(\overline{Q}), \\ W_p^{s_0}(0,T) \subset C^{1/2-3/2p}([0,T]). \tag{25}$$

For functions vanishing at $t = 0$, the corresponding embedding estimates are valid in S^τ and Q^τ with constants independent of τ. This statement follows from the possibility of zero extensions of these functions for $t < 0$ preserving the class. But the norm on the right-hand side must be replaced with the norm of the space $\tilde{W}_p^{s_0,2s_0}(S^\tau)$ and $\tilde{W}_p^{s_0}(0,\tau)$, respectively, in the case of the first and the last embeddings. For example, we have the inequality

$$\|u\|_{C^{1/2-(n+2)/2p,1-(n+2)/p}(\overline{S^\tau})} \leq c\|u\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)}, \tag{26}$$

with c independent of τ. The following inequality results from the last embedding

$$\|u(t)\|_{C([0,\tau])} \leq c_1\tau^{s_0-1/p}\|u\|_{\tilde{W}_p^{s_0}(0,\tau)}, \tag{27}$$

with c_1 independent of τ. In order to prove this inequality, we should consider the function $u(\xi\tau) \in \tilde{W}_p^{s_0}(0,1)$, write out the inequality

$$\|u(\xi\tau)\|_{C([0,1])} \leq c_1\|u(\xi\tau)\|_{\tilde{W}_p^{s_0}(0,1)},$$

and make the change of variables $\xi\tau = t$. The last inequality in lemma 2 yields

$$\|q(t)u(t,x)\|_{\tilde{W}_p^{s_0}(0,\tau)} \leq c_2(\|q(t)\|_{L_\infty(0,\tau)}\|u(t,x)\|_{\tilde{W}_p^{s_0}(0,\tau)} + \|u(t,x)\|_{L_\infty(0,\tau)}\|q\|_{\tilde{W}_p^{s_0}(0,\tau)})$$

and the definition of the norm in the space $W_p^{s_0,2s_0}(S^\tau)$ and the estimate (27) ensures the estimate

$$\|q(t)u(t,x)\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)} \leq c_3\tau^{s_0-1/p}\|q\|_{\tilde{W}_p^{s_0}(0,\tau)}\|u\|_{\tilde{W}_p^{s_0,2s_0}(S^\tau)}, \tag{28}$$

Now we obtain some additional estimates for solutions. Let v_1, v_2 be two solutions to the problem (20)-(21) relating to two different vectors $\vec{\mu}^1, \vec{\mu}^2 \in B_M$. Extracting equalities

(20)-(21) for different $\vec{\mu}^i$, we obtain

$$w_t + A_0 w + A(\vec{\mu}_1)w = -A(\vec{\mu}_2)(v_1+v_2)/2 + \sum_{i=1}^{r} b_i(t,x)(\mu_i^1(t) - \mu_i^2(t)),$$
$$\vec{\mu}_2(t) = \vec{\mu}^1 - \vec{\mu}^2, \vec{\mu}_1(t) = (\vec{\mu}^1 + \vec{\mu}^2)/2,\ w = v_1 - v_2. \quad (29)$$

$$w|_{t=0} = 0,\quad B_0 w + B(\vec{\mu}_1)w|_S = -B(\vec{\mu}_2)(v_1+v_2)/2|_S - B(\vec{\mu}_2)\Phi. \quad (30)$$

The analogs of the estimates (23), (24) written for solutions to this problem and these estimates themselves imply the estimate

$$\|w\|_{W_p^{1,2}(Q^\tau)} + \|w\|_{L_p(0,\tau;W_p^4(G_{\delta_1}))} + \|w_t\|_{L_p(0,\tau;W_p^2(G_{\delta_1}))} \le c_4 \|\vec{\mu}_2\|_{W_p^{s_0,2s_0}(0,\tau)}, \quad (31)$$

where the constant c_4 depends on known quantities and independent of τ. Proceed with the construction of a solution to our inverse problem. Let v be a solution to the problem (20), (21). The embedding theorems ensure the existence of the traces at $x = y_j$ of all functions occurring in (20) and we have

$$\tilde{\psi}_{jt} + A_0 v(t,y_j) + A(\vec{\mu})v(t,y_j) = -A(\vec{\mu})\Phi(t,y_j) + \\ + \sum_{i=1}^{r} b_i(t,y_j)\mu_i(t),\ j = 1,2,\ldots,s. \quad (32)$$

These equalities can be written in the form

$$\tilde{B}\vec{\mu} = F + S(\vec{\mu}), \quad (33)$$

where the rows of the matrix $\tilde{B}(t)$ are as follows:

$$(-b_1(t,x), -b_2(t,x), \ldots, -b_r(t,x), A_{r+1}\Phi(t,x), \ldots, A_s\Phi(t,x))|_{x=y_j}.$$

In view of (11), all entries of $\tilde{B}(t)$ are continuous. We have $\tilde{B}(0) = B(0)$ and there exists a parameter τ_1 such that $|det\,\tilde{B}(t)| > \delta_1 > 0$ on $[0,\tau_1]$, where δ_1 is a positive constant. The right-hand side of (33) contains the vector-function $S(\vec{\mu})$ whose j-th coordinate is written as $-A_0 v(t,y_j) - A(\vec{\mu})v(t,y_j)$, with $v = v(\vec{\mu})$) a solution to the problem (20), (21). The j-th coordinate F_j of the vector F is just a function $-\tilde{\psi}_{jt}$. Define the parameter $M = \|B^{-1}F\|_{W_p^{s_0}(0,T)}/2$. As we have proven, the operator $\vec{\mu} \to v(\vec{\mu})$ is defined for the vectors $\vec{\mu} \in B_M$. We can determine the vector $\vec{\mu}$ as a solution to the system (33). Demonstrate that there exists a constant $\tau_2 \le \tau_1$ such that the operator $B^{-1}F + B^{-1}S(\vec{\mu}) : \tilde{W}_p^{s_0}(0,\tau_2) \to \tilde{W}_p^{s_0}(0,\tau_2)$ takes the ball B_M into itself and is a contraction. The definition of the quantities q_j^0 implies that $F_j(0) = 0$. As before, let v_1, v_2 be solutions to the problem (20)-(21) relating to two different vectors $\vec{\mu}^1, \vec{\mu}^2 \in B_M$. We need to estimate the norm $\|B^{-1}S(\vec{\mu^1}) - B^{-1}S(\vec{\mu^2})\|_{\tilde{W}_p^{s_0}(0,\tau)}$. To this end, we consider the expression

$$-A_0 v_1(t,y_j) - A(\vec{\mu}^1)v_1(t,y_j) + A_0 v_2(t,y_j) + A(\vec{\mu}^2)v_2(t,y_j) = \\ -A_0 w(t,y_j) - A(\vec{\mu}_1)w(t,y_j) - A(\vec{\mu}_2)(v_1+v_2)(t,y_j)/2, \quad (34)$$

Lemma 1 yields ($\tau \le \tau^1$)

$$\|B^{-1}S(\vec{\mu}^1)(t) - B^{-1}S(\vec{\mu}^2)(t)\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c_1(\sum_{j=1}^{s}(\|A_0(v_1-v_2)(t,y_j)\|_{\tilde{W}_p^{s_0}(0,\tau)}+$$
$$+\|A(\vec{\mu}_2)(v_1+v_2)(t,y_j)\|_{\tilde{W}_p^{s_0}(0,\tau)} + \|A(\vec{\mu}_1)(v_1-v_2)(t,y_j)\|_{\tilde{W}_p^{s_0}(0,\tau)}). \quad (35)$$

Next, involving the conditions on the coefficients, Lemma 2, and the inequality (28), we can estimate the right-hand side by the quantity

$$c_2\sum_{j=1}^{s}\sum_{|\alpha|\le 2}\|D^\alpha(v_1-v_2)(t,y_j\|_{\tilde{W}_p^{s_0}(0,\tau)}+$$
$$+c_3\tau^{s_0-1/p}\|\vec{\mu}^1-\vec{\mu}^2\|_{\tilde{W}_p^{s_0}(0,\tau)}\sum_{j=1}^{s}\sum_{|\alpha|\le 2}\|D^\alpha(v_1+v_2)(t,y_j\|_{\tilde{W}_p^{s_0}(0,\tau)} = J. \quad (36)$$

Next, we refer to Sect.5.5 in [36]. Given a function $v \in W_p^{1,2}(Q)$ with $v(0,x)=0$, we have the estimate

$$\|v(t,y_j)\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\tau^{s_1-s_0}\|v(t,y_j\|_{\tilde{W}_p^{s_1}(0,\tau)} \le \|v(t,x)\|_{\tilde{W}_p^{s_2,2s_2}(Q^\tau_{\delta_2})}, \quad (37)$$

where we take $s_0 < s_1 < s_2 < 1-n/2p$. Note that (see Sect.5.5 in [36]) if $v \in W_p^{1,2}(Q^\tau_{\delta_2})$ then $v(t,y_j) \in \tilde{W}_p^{1-n/2p}(0,\tau)$ and the corresponding estimate holds. Using the estimate (37) in (36) we obtain that

$$J \le c\tau^\beta(\|v_1-v_2\|_{L_p(0,\tau;W_p^4(G_{\delta_2}))} + \|v_{1t}-v_{2t}\|_{L_p(0,\tau;W_p^2(G_{\delta_2}))} + \|\vec{\mu}^1-\vec{\mu}^2\|_{\tilde{W}_p^{s_0}(0,\tau)}),$$

where β is a positive constant. This inequality, the inequalities (31), (35) imply the estimate

$$\|B^{-1}S(\vec{\mu}_1)(t) - B^{-1}S(\vec{\mu}_2)(t)\|_{\tilde{W}_p^{s_0}(0,\tau)} \le c\tau^\beta\|\vec{\mu}^1-\vec{\mu}^2\|_{\tilde{W}_p^{s_0}(0,\tau)},$$

where the constant c is independent of τ. If $\tau^\beta < 1/2c$ then the operator $B^{-1}F+B^{-1}S(\vec{\mu}) : \tilde{W}_p^{s_0}(0,\tau) \to \tilde{W}_p^{s_0}(0,\tau)$ takes the ball B_M into itself and is a contraction. Choose a parameter $\tau_2 \le \tau_1$ so that $\tau_2^\beta < 1/2c$. The fixed point theorem ensures the existence of a solution to the system (33) in the ball B_M.

Let $v = v(\vec{\mu})$. Show that this function satisfies (22). By construction, v is a solution to the problem (20)-(21). Taking $x = y_j$ in (20), we infer

$$v_t(t,y_j) + A_0v(t,y_j) + A(\vec{\mu})v(t,y_j) = -A(\vec{\mu})\Phi(t,y_j)+$$
$$\textstyle\sum_{i=1}^{r} b_i(t,y_j)\mu_i(t),\ j=1,2,\ldots,s.$$

Subtracting these equalities from (32), we obtain that $(v(t,y_j)-\tilde{\psi}_j))_t = 0$ for all j and thus the equality (22) holds.

□

This research was funded by the Russian Science Foundation and the government of Khanty-Mansiysk autonomous okrug—Yugra (Grant 22-11-20031).

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Received May 7, 2022.

УДК 517.956 **DOI: 10.14529/jcem220205**

ОПРЕДЕЛЕНИЕ ТЕРМОФИЗИЧЕСКИХ ПАРАМЕТРОВ В МАТЕМАТИЧЕСКИХ МОДЕЛЯХ ТЕПЛОМАССОПЕРЕНОСА

С. Г. Пятков

Мы рассматриваем математические модели тепломассопереноса. Исследуются обратные задачи определения коэффицентов в главной части параболического уравнения одновременно входящих и в граничное условие типа Робина. Условия переопределения – значения решения в некотором наборе точек, лежащих внутри области. В частности, в класс рассматриваемых задач входят классические задачи восстановления тензора теплопроводности. Главное внимание уделяется вопросам существования, единственности и оценкам устойчивости решений обратных задач этого типа. Задача сводится к операторному уравнению которое исследуется при помощи теоермы о неподвижной точке и априорных оценок. Метод доказательства является конструктивным и может быть использован при построении численных алгоритмов решения задачи.

Ключевые слова: обратная задача; тепломассоперенос; теплопроводность; параболическое уравнение.

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Поступила в редакцию 7 мая 2022 г.