MSC 35G20, 49J20 DOI: 10.14529/jcem220207

ALGORITHM FOR NUMERICAL SOLUTION OF THE OPTIMAL CONTROL PROBLEM FOR THE MATHEMATICAL MODEL OF SHALLOW WATER WAVE PROPAGATION

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The article is devoted to the algorithm for numerical solution of the optimal control problem in a mathematical model of wave propagation in shallow water. The mathematical model is based on the IMBq equation (improved modified Boussinesq equation) and Dirichlet boundary conditions. The IMBq equation belongs to the semilinear Sobolev type equations of the second order. As it is known, the Cauchy problem for a Sobolev type equation is not solvable for arbitrary initial values. We consider a mathematical model with Showalter – Sidorov initial conditions that are more natural for it, making references to the Cauchy problem where necessary. The article also provides examples of computational experiments.

Keywords: mathematical model; modified Boussinesq equation; optimal control problem; numerical research; semilinear Sobolev type equation of second order.

Dedicated to the 70^{th} birthday of Professor A.L. Shestakov

Introduction

Consider the inhomogeneous modified Boussinesq equation (IMBq equation)

$$(\lambda-\Delta)x(s,t)_{tt}-\alpha^2\Delta x(s,t)-\Delta(x^3(s,t))=u(s,t),\quad (s,t)\in(0,l)\times(0,T) \qquad (1)$$

with a homogeneous Dirichlet boundary condition

$$x(0,t)=x(l,t)=0,\quad t\in\times(0,T) \qquad (2)$$

and the initial Showalter – Sidorov conditions

$$(\lambda-\Delta)(x(s,0)-x_0(s))=0,\quad (\lambda-\Delta)(x_t(s,0)-x_1(s))=0,\quad s\in(0,l) \qquad (3)$$

or initial Cauchy conditions

$$x(s,0)=x_0(s),\quad x_t(s,0)=x_1(s),\quad s\in(0,l), \qquad (4)$$

where $\lambda,\alpha\in\mathbb{R}$.

Equation (1) has many applications in various fields of natural science. For example, it models the propagation of waves in shallow water, taking into account capillary effects. In this case, the function $x=x(s,t)$ determines the height of the wave. In [1], a linear and nonlinear mathematical model of shallow water wave propagation was constructed. In [2], the properties of solutions of the Cauchy problem for a nondegenerate IMBq equation in a one-dimensional domain were studied. The existence of a unique global solution of the Cauchy problem for equation (1) was proved in [3] for $\lambda = 1$, $\alpha = 1$. In [4], conditions of blow-up solutions were obtained.

For the mathematical model (1)–(3) (or (1), (2), (4)) we set the optimal control problem. To do this, we introduce the control space $\mathfrak{U} = L^2([0,l] \times [0,T])$ and single out a non-empty, closed, and convex set $\mathfrak{U}_{ad}$ in it, which is called the set of admissible controls

$$J(x,u) \to \inf, \quad u \in \mathfrak{U}_{ad}. \tag{5}$$

The penalty functional is given by formula

$$J(x,u) = \beta \int\limits_0^T \sum_{k=0}^{1} \|x^{(k)}(t) - z^{(k)}(t)\|^4_{L^4} dt + (1-\beta) \int\limits_0^T \|u(t)\|^2_{\mathfrak{U}} dt, \tag{6}$$

here $z(t)$ is the desired state of the system, $\beta \in (0,1)$ is the weighting factor.

The optimal control problem is solved as a classical optimization problem. It allows to balance between the proximity to the desired state and the volume of energy costs. The optimal control problem for semilinear Sobolev type equation of the first order was studied earlier in [5].

1. Numerical Study Algorithm

Let us describe the developed information processing algorithm for finding the optimal control function for the mathematical model of wave propagation in shallow water in steps corresponding to the blocks shown in Figure 1. A similar algorithm was successfully applied for semilinear Sobolev type equation of the first order in [6].

Beginning of the program.

Step 1. Introduce all the parameters of the problem: the parameters of the equation λ, α, the ranges of variables l and T, the initial state $x_0(s)$ and the initial velocity $x_1(s)$; functional parameters: the desired state $z(s,t)$, weight coefficient β, admissible control domain U_{ad}, number M of terms in the Galerkin sum, and number N of term in the Ritz sum; the weighting factor for the decomposition method θ, the penalty factor r and the marginal error parameter δ. At this step, the parameter θ is chosen from the interval $(0,1)$, and the parameter r is as large as possible so that the solution and the auxiliary are sufficiently close ($r = \frac{1}{\varepsilon}$, $\varepsilon \to 0$).

Step 2. Find the eigenfunctions φ_i and the eigenvalues λ_i of the homogeneous Dirichlet problem for the Laplace operator.

Step 3. Apply the decomposition method. Linearize the equation and introduce an auxiliary function $y = y(s,t)$, obtain

$$(\lambda - \Delta)x_{tt} - \alpha^2 \Delta x - \Delta(y^3) = u(s,t),$$
$$y = x.$$

Now the IMBq equation is linear with respect to the function x.

Step 4. Define the unknown functions in the form of Galerkin sums

$$x(s,t) = \sum_{i=1}^{M} x_i(t)\varphi_i(s), \quad y(s,t) = \sum_{i=1}^{M} y_i(t)\varphi_i(s), \quad u(s,t) = \sum_{i=1}^{M} u_i(t)\varphi_i(s).$$

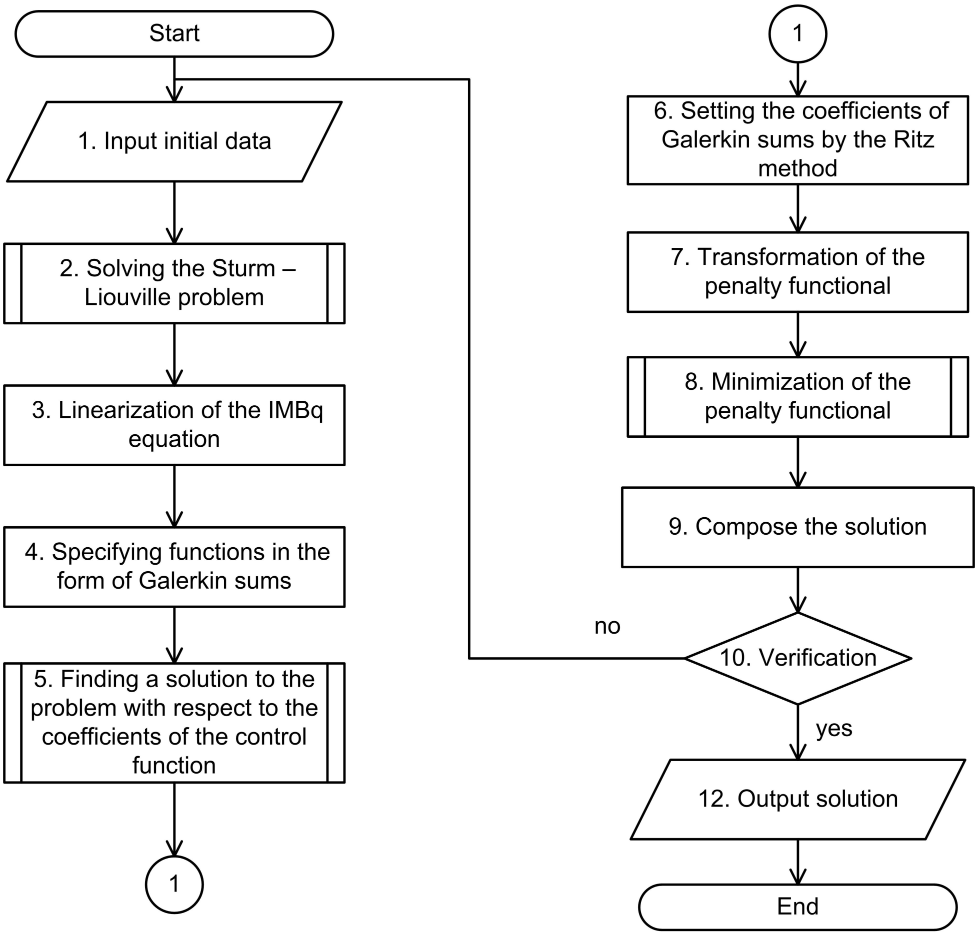


Fig. 1. Diagram of the algorithm

Step 5. A diagram illustrating the operation of this step is shown in Figure 2.

Step 5.1. Initial data from the main program is passed to the input.

Step 5.2. Check equation (1) for degeneracy if $\lambda = -\left(\frac{k\pi}{l}\right)^2$, for some $k \in \mathbb{N}$, then the equation is degenerate go to step 5.3, otherwise the equation is nondegenerate, go to step 5.8.

Step 5.3. If the Cauchy initial conditions are given, then go to Step 5.4, otherwise go to Step 5.6.

Step 5.4. Check whether the initial functions (the initial state and initial velocity) belong to the phase space of equation (1). If they do not belong to the phase space, then go to step 5.5, otherwise go to step 5.6.

Step 5.5. The problem has no solutions. Completion of the subprogram.

Step 5.6. In the cycle over i from 1 to M, multiply the equation and initial conditions by the eigenfunctions (Step 2) in the sense of the inner product in $L^2[0,l]$. At the k-th step, an algebraic equation appears, and the initial conditions for $x_k(t)$ are excluded from further consideration. Then the ordinary differential equations with initial conditions and the algebraic equation are combined into an algebraic differential system.

Step 5.7. Solve the algebraic differential system [7].

Step 5.8. In the cycle over i from 1 to M, multiply the equation and initial

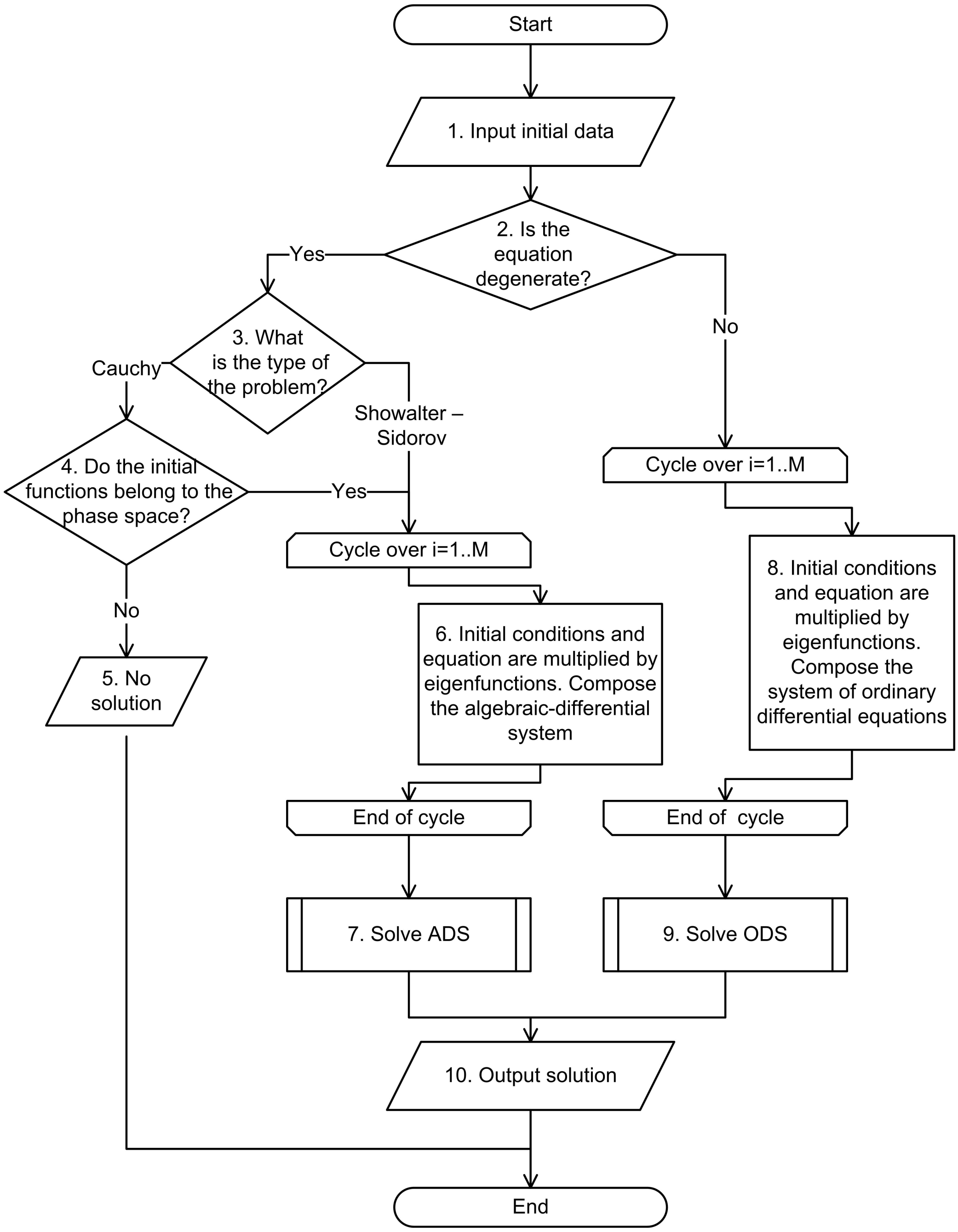


Fig. 2. Diagram of the step 5

conditions by the eigenfunctions (Step 2) in the sense of the inner product in $L^2[0, l]$. Since the equation is non-degenerate, the result is a system of ordinary linear differential equations.

Step 5.9. Solve a system of ordinary differential equations by varying an arbitrary constant.

Step 5.10. The resulting solution is output and passed to the main program. Completion of the subprogram.

Step 6. The coefficients of the Galerkin sums for the auxiliary and control functions according to the Ritz method are represented as polynomials

$$y_i(t) = \sum_{j=1}^{N} b_{ij}(t)t^j, \quad u_i(t) = \sum_{j=1}^{N} c_{ij}(t)t^j.$$

It must be taken into account that

$$y_i(0) = x_i(0), \quad \frac{\partial y_i}{\partial t}(0) = \frac{\partial x_i}{\partial t}(0).$$

Step 7. Transform the penalty functional to take into account the introduced auxiliary function

$$\begin{gathered} J(x,u) = \beta\theta \int_0^T \sum_{k=0}^{1} \|x^{(k)}(s,t) - z^{(k)}(s,t)\|_{L^4}^4 dt + \\ +\beta(1-\theta) \int_0^T \sum_{k=0}^{1} \|y^{(k)}(s,t) - z^{(k)}(s,t)\|_{L^4}^4 dt + \\ +(1-\beta) \int_0^T \|u(t)\|_{\mathfrak{U}}^2 dt + r \int_0^T \sum_{k=0}^{1} \|y^{(k)}(s,t) - x^{(k)}(s,t)\|_{L^4}^4 dt. \end{gathered} \tag{7}$$

Step 8. Using the branch and bound method built into Maple, the minimum values of the functional and the minimum point $c_{ij}, i = \overline{1,M}, j = \overline{0,N}$ and $b_{ij}, i = \overline{1,M}, j = \overline{0,N}$ are found.

Step 9. Substituting the found values into the Ritz expansions (step 5) and then into the Galerkin sums (step 4), obtain an approximate solution of problem (1)–(3), (5) (or (1), (2), (4), (5)).

Step 10. This step of the algorithm is informative and consists in checking the proximity of the solution to problem (1)–(3) (or (1), (2), (4)) $x(s,t)$ obtained above and the solution of the same problem $w(s,t)$ obtained under the assumption that the control function is known. The solution $w(s,t)$ can be found using the algorithm described in [7]. If the error

$$err = \left(\int_0^\pi \sum_{k=0}^{m} \sum_{t_i=0}^{T} (u(x,t_i) - w(x,t_i))^4 ds \right)^{\frac{1}{4}} \tag{8}$$

is greater than the given δ, then it is necessary to return to step 1 and increase one or more of the parameters M, N, r. If $err \leq \delta$, then go to the next step.

Step 11. The functions $x(s,t)$, $u(s,t)$ and their graphs are displayed.

End of the program.

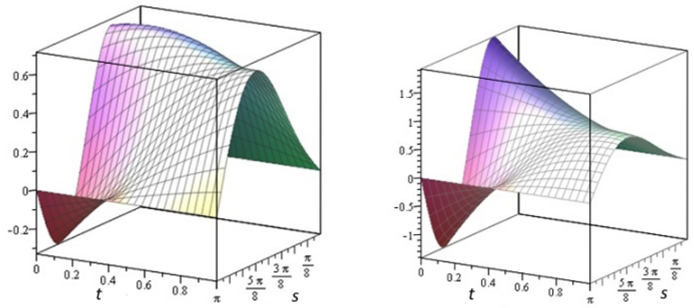
2. Computational Experiments

Let us present the results of information processing according to the developed algorithm, which was implemented in the Maple environment. Information processing was carried out on the basis of computational experiments.

Example 1. Let the following input information be given: $\lambda = -1$, $\alpha = 1$, $T = 1$, $l = \pi$, $r = 10$, $\beta = 0.5$, $\theta = 0.5$, $z(s,t) = s(\pi - s)$, $M = 2$, $N = 2$, $x_0(s) = 0.25\sin(s) +$

$0.5\sin(2s)$, $x_1(s) = \sin(s) - 0.5\sin(2s)$. It is required to determine the optimal control over the solutions of the Showalter – Sidorov problem for a mathematical model of shallow water wave propagation, such that the marginal error δ will not exceed 0.01. The domain of admissible controls $\mathfrak{U}_{ad} = \{u \in L^2 : \|u(s,t)\|^2_{L^2} \leq 100\}$.

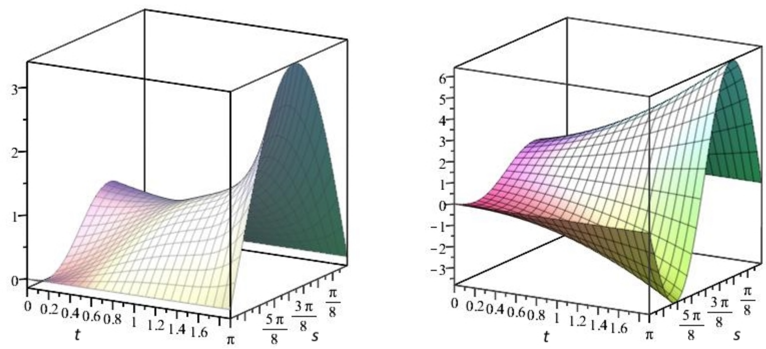
From this input information, it is obvious that the equation is degenerate. Using the developed algorithm, the information was processed and the minimum value of the functional $J_{min} = 4.41$ was obtained, but the marginal error was 0.03. Turn to beginning of the algorithm and increase the penalty parameter to $r = 100$, obtain $J_{min} = 4.86$ and $err = 0.008$. On figure 3 shows graphs of the solution.



a) b)

Fig. 3. Graph: a) of function $x(s,t)$; b) of function $u(s,t)$

Example 2. Let the following input information be given: $\lambda = -1$, $\alpha = 1$, $T = 1.9$, $l = \pi$, $r = 100$, $\beta = 0.5$, $\theta = 0.5$, $z(s,t) = 2\sin(s)$, $M = 2$, $N = 2$, $x_0(s) = 0.25\sin(s) + 0.5\sin(2s)$, $x_1(s) = \sin(s) - 0.5\sin(2s)$. The domain of admissible controls is the same. In this example, we have increased the time span to 1.9 and got the minimal value $J_{min} = 4119$. On Figure 4 shows the graphs of the control function and the system state function.



a) b)

Fig. 4. Graph: a) of function $x(s,t)$; b) of function $u(s,t)$

Figure 5 shows changing the deviation of the state of the system from the desired. The graph shows that the best approximation is achieved in the interval $t \in [1.8, 1.9]$.

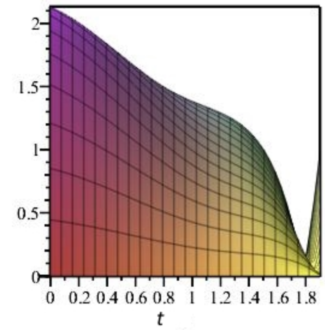


Fig. 5. Graph of function $|x(s,t) - z(s,t)|$

Acknowledgments. *This work was supported by a grant from the Ministry of Education and Science of the Russian Federation № FENU-2020-0022 (2020072GZ).*

References

1. Whitham G.B. *Linear and Nonlinear Waves.* NY, Wiley, 1974.

2. Wang Y. On Properties of Solutions to the Improved Modified Boussinesq Equation. *Journal of Nonlinear Science and Applications,* 2016, vol. 9, no. 12, pp. 6004–6020. DOI: 10.22436/jnsa.009.12.08

3. Runzhang Xu, Yacheng Liu. Global Existence and Blow-up of Solutions for Generalized Pochhammer–Chree equations. *Acta Mathematica Scientia*, 2010, vol. 30, no. 5, pp. 1793–1807. DOI: 10.1016/S0252-9602(10)60173-7

4. Korpusov M.O., Sveshnikov A.G. Sufficient Conditions for the Blow-up of a Solution to the Boussinesq Equation Subject to a Nonlinear Neumann Boundary Condition. *Computational Mathematics and Mathematical Physics*, 2008, vol. 48, no. 11. pp. 2077–2080. DOI: 10.1134/S0965542508110122

5. Sviridyuk G.A., Manakova N.A. An Optimal Control Problem for the Hoff Equation. *Journal of Applied and Industrial Mathematics,* 2007, vol. 1, no. 2, pp. 247–253. DOI: 10.1134/S1990478907020147

6. Gavrilova O.V. Numerical Study of the Unique Solvability of the Showalter – Sidorov Problem for a Mathematical Model of the Propagation of Nerve Impulses in the Membrane. *Journal of Computational and Engineering Mathematics,* 2021, vol. 8, no. 3, pp. 32–48. DOI: 10.14529/jcem210303

7. Zamyshlyaeva A.A., Bychkov E.V. On the Numerical Study of the Mathematical Model of Wave Propagation in Shallow Water. *Mathematical notes of YaSU,* 2013, vol. 20, no. 1, pp. 27–34. (in Russian)

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Received May 15, 2022.

УДК 517.95, 517.97 **DOI: 10.14529/jcem220207**

АЛГОРИТМ ЧИСЛЕННОГО РЕШЕНИЯ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ РАСПРОСТРАНЕНИЯ ВОЛН НА МЕЛКОЙ ВОДЕ

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В статье обсуждается алгоритм численного решения задачи оптимального управления решениями в математической модели распространения волн на мелкой воде. В основе математической модели лежит IMBq уравнение (улучшенное модифицированное уравнение Буссинеска) и краевые условия Дирихле. Уравнение IMBq уравнение к полулинейным уравнениям соболевского типа второго порядка. Как известно, задача Коши для уравнения соболевского типа не разрешима при произвольных начальных значениях. Мы будем рассматривать математическую модель с более естественными для нее начальными условиями Шоуолтера – Сидорова, делая отсылки к задаче Коши там, где это необходимо. В статье также приведены примеры вычислительных экспериментов.

Ключевые слова: математическая модель; модифицированное уравнение Буссинеска; задача оптимального управления; численное исследование; полулинейное уравнение соболевского типа второго порядка.

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Поступила в редакцию 15 мая 2022 г.