Modifications of the BDF-method as applied to the calculation of trajectory of the moving elements of piston machines were analyzed. On the example of the calculation of the connecting-rod bearing of "Ruston & Hornsby 6VEB-X MK-III" engine as an international standard for testing we estimated the effectiveness of proposed algorithms for solving the equations of motion.

Keywords: tribosystem of piston machines, differential equations, order of BDF-method, hydromechanical characteristics.

Introduction

To solve the equations of motion of the movable elements of tribosystems well known such techniques as implicit BDF-method [1]. BDF-methods are based on the so-called backward differentiation formulas, when the derivative, for example, \(du/d\tau\) in the point \(\tau_{n+1}\) is calculated according value of \(u_{n+1}\) and values of \(u(\tau)\) in previous \(k\) points \(u_n, u_{n-1}, ..., u_{n+1-k}\), where \(k\) defines the order of method.

Exact solution \(u(\tau)\) for the interval \((\tau_{n+1-k}; \tau_{n+1})\) is replaced by a polynomial for unequal intervals on the basis of which we forecast vector of unknown values.

BDF-method was studied in detail with reference to the specific case - for analyzing of the dynamics of autonomous supports. The traditional way is the reduction of the equations of motion of the second order to equations of the first order. Will consider this algorithm in a more general way for non-autonomous supports with misalignment of the shaft.

1. Differential equations of the first order

Represent the equations of motion of the movable elements of tribosystems, which described in papers [1,2,3], in the matrix form

\[
\hat{M} \hat{\dot{U}} = \hat{F}(U, \tau) \tag{1}
\]

where

\[
\hat{M} = \begin{bmatrix}
\hat{E} & \hat{E} & \hat{E} & \hat{E} \\
\hat{E} & \hat{O} & \hat{O} & \hat{O} \\
\hat{O} & \hat{mE} & \hat{O} & \hat{O} \\
\hat{O} & \hat{O} & \hat{E} & \hat{O}
\end{bmatrix} ; \quad U = \begin{bmatrix}
u \\
\gamma \\
v
\end{bmatrix} ; \quad \hat{F}(U, \tau) = \begin{bmatrix}
v \\
-P + f(\tau) \\
-Ma + \tilde{M}(\tau)
\end{bmatrix} ;
\]
\[ \hat{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

For discretization of system (1) we write the vector of displacement and derivatives in the next time step \( \tau_{n+1} = \tau_n + \Delta \tau \) using functions \( \dot{U}_{n+1} = \varphi(U_{n+1}); \quad U_{n+1} = \xi(U_{n+1}) \). Then instead of continuous expression we obtain a next discrete task:

\[
\hat{M} \varphi(U_{n+1}) = \hat{F}(\xi(U_{n+1}), \tau) . 
\] (2)

For BDF-method of \( k \)-order the functions \( \varphi, \xi \) for first approximation were determined by polynomials

\[
U_{n+1}^{(1)} = L(\tau_{n+1}) = U_n + \sum_{i=0}^{k} \beta_i \Delta U_{n-i}, 
\] (3)

\[
\dot{U}_{n+1}^{(1)} = \dot{L}_1(\tau_{n+1}) = \sum_{i=-1}^{k-1} \alpha_i \Delta U_{n-i} ,
\] (4)

where coefficients \( \beta_i, \alpha_i \) were determined by polynomials \( L, \dot{L}_1 \). Put (3), (4) to (2) for \( \tau = \tau_n + 1 \) we obtain

\[
\hat{M} \sum_{i=-1}^{k-1} \alpha_i \Delta U_{n-i} = \hat{F}(U_{n+1}, \tau_{n+1}) .
\] (5)

The system (5) is an implicit difference scheme, which can be efficiently solved by Newton’s method. Will find a solution for \( U_{n+1} \), writing the equation (5) in the form of Newton.

\[
\left( \alpha_{-1} \hat{M} - \left( \frac{\partial \hat{F}}{\partial U} \right)_{n+1}^{(s)} \right) \left( U_{n+1}^{(s)} - U_{n}^{(s)} \right) = \hat{F} \left( U_{n+1}^{(s)}, \tau_{n+1} \right) - \alpha_{-1} \hat{M} \left( U_{n+1}^{(s)} - U_{n}^{(s)} \right) - \sum_{i=0}^{k-1} \alpha_i \hat{M} \Delta U_{n-i} 
\] (6)

We introduce the notation:

\[
\left( \alpha_{-1} \hat{M} - \left( \frac{\partial \hat{F}}{\partial U} \right)_{n+1}^{(s)} \right) = A_{n+1}^{(s)} ;
\]

\[
\hat{F} \left( U_{n+1}^{(s)}, \tau_{n+1} \right) - \alpha_{-1} \hat{M} \left( U_{n+1}^{(s)} - U_{n}^{(s)} \right) - \sum_{i=0}^{k-1} \alpha_i \hat{M} \Delta U_{n-i} = G_{n+1}^{(s)} .
\]
then

\[
A^{(s)}_{n+1} = \begin{bmatrix}
\alpha_{-1} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_{-1} & 0 & -1 & 0 & 0 & 0 & 0 \\
-\frac{\partial R_x}{\partial x} & -\frac{\partial R_y}{\partial y} & ma_{-1} & -\frac{\partial R_x}{\partial x} & -\frac{\partial R_y}{\partial y} & -\frac{\partial R_x}{\partial x} & -\frac{\partial R_y}{\partial y} & -\frac{\partial R_x}{\partial x} \\
-\frac{\partial \bar{M}}{\partial x} & -\frac{\partial \bar{M}}{\partial y} & -\frac{\partial \bar{M}}{\partial x} & -\frac{\partial \bar{M}}{\partial y} & -\frac{\partial \bar{M}}{\partial x} & -\frac{\partial \bar{M}}{\partial y} & -\frac{\partial \bar{M}}{\partial x} & -\frac{\partial \bar{M}}{\partial y} \\
\end{bmatrix}
\]

\[
G^{(s)}_{n+1} = \begin{bmatrix}
x_{n+1} - \alpha_{-1} \Delta x_{n+1} - \alpha_0 \Delta x_n \\
y_{n+1} - \alpha_{-1} \Delta y_{n+1} - \alpha_0 \Delta y_n \\
(f_x + R_x)_{n+1} = m (\alpha_{-1} \Delta \dot{x}_{n+1} - \alpha_0 \Delta \dot{x}_n) \\
(f_y + R_y)_{n+1} = m (\alpha_{-1} \Delta \dot{y}_{n+1} - \alpha_0 \Delta \dot{y}_n) \\
\alpha'_{n+1} - \alpha_{-1} \Delta \alpha'_{n+1} - \alpha_0 \Delta \alpha'_n \\
\beta'_{n+1} - \alpha_{-1} \Delta \beta'_{n+1} - \alpha_0 \Delta \beta'_n \\
(\bar{M} + \bar{M}_N)_{n+1} - J (\alpha_{-1} \Delta \alpha'_{n+1} - \alpha_0 \Delta \alpha'_n) \\
(\bar{M} + \bar{M}_N)_{n+1} - J (\alpha_{-1} \Delta \beta'_{n+1} - \alpha_0 \Delta \beta'_n)
\end{bmatrix}
\]

Solving system (6) by Gauss method we obtain:

\[
\Delta x_{n+1}^{(s+1)} , \quad \Delta y_{n+1}^{(s+1)} , \quad \Delta \alpha_{n+1}^{(s+1)} , \quad \Delta \beta_{n+1}^{(s+1)} .
\]

According to coordinate’s increments in the polar coordinates we find:

\[
\lambda^{(s+1)}_{n+1} , \quad \alpha^{(s+1)}_{n+1} , \quad \beta^{(s+1)}_{n+1} , \quad \alpha^{(s+1)}_{n+1} , \quad \epsilon^{(s+1)}_{n+1} , \quad \sigma^{(s+1)}_{n+1} , \quad \xi^{(s+1)}_{n+1} , \quad \tau^{(s+1)}_{n+1} , \quad \zeta^{(s+1)}_{n+1} .
\]

The iterative process at the next step of the integration is carried out till next condition

\[
\tilde{E}_{n+1}^{s+1} \leq \varepsilon,
\]

where

\[
\tilde{E}_{n+1}^{s+1} = \max \left\{ \frac{|U_{n+1}^{s+1} - U_{n+1}^s|}{|U_{n+1}^{s+1}|}, \text{ if } |U_{n+1}^{s+1}| > 1; \frac{|U_{n+1}^{s+1} - U_{n+1}^s|}{|U_{n+1}^s|}, \text{ if } |U_{n+1}^{s+1}| \leq 1 \right\}.
\]

The value of error \( \varepsilon \) was equal 0.0001.

The value of error we estimated for each components of vector and chosen the maximal value.

If we had \( \tilde{E}_{n+1}^{s+1} > \varepsilon \) after \( i \) iterations the solving process was finished and we decreased the step of the integration. After that we calculated new values of \( U_{n+1}^s \).

Since it is necessary to determine the forces \( R(u, \bar{u}, \gamma, \bar{\gamma}) \) at each time step to integrate the partial differential equation of second order to determine the pressure in the lubricating layer (Reynolds equation [2] or Elrod [4,5]), the time required are so large that in some cases it is impossible to perform parametric research. This is particularly evident in the tasks of nonlinear dynamics of journal bearings with supply of lubricating fluid through the
holes and grooves, as in such cases the equation for the pressure distribution are integrated numerically.

On this basis, appear relevant research and development of methods and algorithms for solving of the equations of motion directly without reducing them to equations of the first order, which halves the dimension of the task.

2. Differential equations of the second order

Consider another method of solving of the equations of motion without reducing them to a system of equations of the first order. It allows to reduce the number of integrations of Reynolds equation [5]. This is especially important when we use numerical methods of integration, since the calculated time is essentially increasing. Consider it an example of solutions of equation of motion for autonomous tribosystem, written in the form

\[ m \ddot{u} = f(\tau) - P(u, \dot{u}). \]

For discretization of the system (8) we replace the vector of derivatives in the next moment of time \( \tau_{n+1} = \tau_n + \Delta\tau \) by the functions \( \dot{u}_{n+1} = \varphi(u_{n+1}), \ddot{u} = \psi(u_{n+1}) \). Then, instead of continuous writing obtain

\[ m \cdot \psi(u_{n+1}) = f_{n+1} - P(u_{n+1}, \varphi(u_{n+1})). \]

The system of equations (9) is an implicit difference scheme, which is solved by Newton’s method. At each iteration step is required to find the solution of algebraic equations in the form

\[ J^{(s)}_{n+1}(u_{n+1}^{(s)}, \ddot{u}_{n+1}) = f^{(s)}_{n+1}, \]

where

\[ J^{(s)}_{n+1}(u_{n+1}^{(s)}, \ddot{u}_{n+1}) = \left[ m \frac{\partial \dot{u}}{\partial u} + \frac{\partial P}{\partial \dot{u}} \cdot \partial \ddot{u} \right]_{n+1}; \]

\[ F^{(s)}_{n+1} = -m\ddot{u}_{n+1} - P(u_{n+1}, \dot{u}_{n+1}) + f_{n+1}. \]

Having constructed the same as in the previous method, interpolation polynomials according to (3) and (4) we can to determine the value of the function and its first derivative \( \dot{u}^{(1)}_{n+1} \) at the point \( \tau_{n+1} \). The value of the second derivative of the function \( L_1(\tau) \) we obtain from the expression

\[ \ddot{u}_{n+1} = \ddot{L}_1(\tau_{n+1}) = \sum_{i=-1}^{k-1} \beta_i \Delta u_{n-i}, \]

where coefficients \( \beta_i \) determines from \( L_1(\tau) \)

Substituting (3), (4), (10) in the equation of motion (8) for moment of time \( \tau_{n+1} \) we obtain

\[ m \sum_{i=-1}^{k-1} \beta_i \Delta u_{n-i} + P(u_{n+1}, \sum_{i=-1}^{k-1} \alpha_i \Delta u_{n-i}) = f_{n+1}. \]

The solution of equation (11) for the unknown value \( u_{n+1} \) we find by Newton’s iterative method

\[ \left[ m \beta_{-1} \ddot{E} + \left( \frac{\partial P}{\partial U} \right)_{n+1}^{(s)} + \alpha_{-1} \left( \frac{\partial P}{\partial U} \right)_{n+1}^{(s)} \right] \left( u_{n+1}^{(s+1)} - u_{n+1}^{(s)} \right) = F_{n+1}^{(s)}. \]

6 Journal of Computational and Engineering Mathematics
where \( F_{n+1}^{(s)} = f_{n+1} - P \left( u_{n+1}^{(s)} \sum_{i=1}^{k-1} \alpha_i \Delta u_{n-i}^{(s)} \right) - m \sum_{i=1}^{k-1} \beta_i \Delta u_{n-i}^{(s)}; \hat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; u_{n+1}^{(1)}, \dot{u}_{n+1}^{(s)} \).

\( \dot{u}_{n+1}^{(s)} \) – are calculated by formulas (3), (4), (10).

In general, for the coordinates \( x, y \) we can be written solution of the system (11) as follows:

\[
x_{n+1}^{(s)} = x_{n+1} + c_1 b_2 - c_2 b_1 \\
y_{n+1}^{(s)} = y_{n+1} + c_1 c_2 - c_2 c_1
\]

where

\[
a_1 = m \beta_{-1} - \frac{\partial R_x}{\partial x} - \alpha_{-1} \frac{\partial R_x}{\partial \dot{x}}; \\
b_1 = -\frac{\partial R_x}{\partial y} - \alpha_{-1} \frac{\partial R_x}{\partial \dot{y}}; \\
a_2 = -\frac{\partial R_y}{\partial x} - \alpha_{-1} \frac{\partial R_y}{\partial \dot{x}}; \\
b_2 = m \beta_{-1} - \frac{\partial R_y}{\partial y} - \alpha_{-1} \frac{\partial R_y}{\partial \dot{y}}; \\
c_1 = f_{n+1,x} + R_x^{(s)} - m \sum_{i=1}^{k-1} \beta_i \Delta x_{n-i}^{(s)}; \\
c_2 = f_{n+1,x} + R_y^{(s)} - m \sum_{i=1}^{k-1} \beta_i \Delta y_{n-i}^{(s)};
\]

We named this approach – algorithm 1.

Numerical experiments have shown that a significant increase in effectiveness of BDF-method can be achieved if instead of Newton’s method in the form of (12) to apply it’s modification

\[
\left[ m \beta_{-1} \hat{E} + \left( \frac{dP}{du} \right)_{n+1}^{(s)} \right] \cdot \left[ u_{n+1}^{(s+1)} - u_{n+1}^{(s)} \right] = F_{n+1}^{(s)},
\]

where

\[
F_{n+1}^{(s)} = f_{n+1} - m \sum_{i=1}^{k-1} \beta_i \cdot \Delta u_{n-i}^{(s)} - P \left( u_{n+1}^{(s)}, \sum_{i=1}^{k-1} \alpha_i \Delta u_{n-i}^{(s)} \right); \\
\Delta u_{n-i}^{(s)} = \Delta u_{n-i}, (i = 0, 1, ..., k - 1).
\]

The expression \( \frac{dP}{du} \) is a matrix of full derivatives of function

\[
P(u) = P \left( u_{n+1}, \sum_{i=1}^{k-1} \alpha_i \Delta u_{n-i} \right).
\]

Thus, in the solution is not necessary to compute partial derivatives of functions \( P \) by \( \dot{u} \), so the calculation time is essentially reduced. We named this approach - fast BDF-method or algorithm 2. In the available literature this modification of the method could not be found.

To determine the partial derivatives we used difference relations:

\[
\frac{\partial R_x}{\partial x} = \frac{R_x \left( \tilde{u}_{n+1}^{(s)}, \sum_{i=1}^{k-1} \alpha_i \Delta \tilde{u}_{n-i} \right) - R_x \left( u_{n+1}^{(s)}, \sum_{i=1}^{k-1} \alpha_i \Delta u_{n-i}^{(s)} \right)}{\Delta x},
\]

where

\[
\tilde{u}_{n+1}^{(s)} = \begin{pmatrix} x_{n+1}^{(s)} + \Delta x \\ y_{n+1}^{(s)} \end{pmatrix}; \\
\Delta \tilde{u}_{n+1}^{(s)} = \begin{pmatrix} x_{n+1}^{(s)} + \Delta x - x_n \\ y_{n+1}^{(s)} - y_n \end{pmatrix}, \\
\Delta \tilde{u}_{n+1}^{(s)} = \Delta u_{n-1}, (i = 0, 1, ..., k - 1).
\]
Solution of system (14) we found by formulas (13), where
\[ a_1 = m_\beta - 1 + \frac{\partial R_x}{\partial x}; \quad a_2 = \frac{\partial R_y}{\partial x}; \quad b_1 = \frac{\partial R_x}{\partial y}; \quad b_2 = m_\beta - 1 + \frac{\partial R_y}{\partial y}; \]
\[ c_1 = (R^{(s)}_{n+1})_x; \quad c_2 = (R^{(s)}_{n+1})_y. \]

It is known that Newton’s method converges for the right choice of the initial approximation. To select the initial positions and velocities of the shaft we use a polynomial of the second degree, and for accelerations - third-degree polynomial.

We introduce a new coordinate axis \( T \). Positive direction which is opposite to the direction of the axis \( \tau \), the beginning corresponding to the current point in time \( \tau_{n+1} \).

Discrete values \( T_i \) are determined by the values \( \tau_{n-i} \) according to the expression
\[ T_i = \tau_{n+1} - \tau_{n-i}, (i = -1, 0, 1, ..., k - 1, k). \]

In the coordinate system \( T \) we can write the expression (3), (4) for polynomials of second degree \( (k = 2) \)
\[ u_{n+1} = u_n + \gamma_0 \Delta u_n + \gamma_1 \Delta u_{n-1}, \quad (16) \]
\[ \dot{u}_{n+1} = \alpha_{-1} \Delta u_{n+1} + \alpha_0 \Delta u_n, \quad (17) \]
where
\[ \gamma_0 = \frac{T_0}{T_1 - T_0} \left( 1 + \frac{T_1}{T_2 - T_0} \right); \quad \gamma_1 = \frac{T_0 T_1}{(T_2 - T_0)(T_2 - T_1)}; \]
\[ \alpha_{-1} = \frac{1}{T_0} + \frac{1}{T_1}; \quad \alpha_0 = \frac{T_0}{T_1 (T_1 - T_0)}. \]

For determination \( \ddot{u}_{n+1} \) (10) we use polynomials of third degree \( (k = 3) \)
\[ \ddot{u}_{n+1} = \beta_{-1} \Delta u_{n+1} + \beta_0 \Delta u_n + \beta_1 \Delta u_{n-1}, \quad (18) \]
where
\[ \beta_{-1} = \frac{2(T_0 + T_1 + T_2)}{T_0 T_1 T_2}; \]
\[ \beta_0 = -\frac{2(T_2 - T_0)(T_2 + T_0) + T_1(T_2 + T_1)}{T_1 T_2 (T_1 - T_0)(T_2 - T_0)}; \]
\[ \beta_1 = \frac{2(T_0 + T_1)}{T_2(T_2 - T_1)(T_2 - T_0)}. \]

If we use the interpolation polynomial of the second degree \( L_1(\tau) \), then the approximation of the derivative of the second order becomes a constant, which reduces the accuracy and stability of the method.

The value of the integration step was limited by the range of variation \( T_{0_{\text{min}}} < T_0 < T_{0_{\text{max}}} \), where \( T_{0_{\text{min}}}, T_{0_{\text{max}}} \) respectively the minimum and maximum size of discretization of \( \tau \), which selected experimentally in solving test tasks. To correct time step in the algorithm we used Fowler procedure [5].

The algorithm 1 uses an iterative Newton’s scheme. At each iteration is required to make nine integrations of Reynolds equation. The algorithm 2, due to the approximate calculation of the Jacobian at each iteration of Newton’s method, requires only five integrations of Reynolds equation. It significantly decreases time of calculation of hydromechanical characteristics of tribosystems.
3. Results and calculations

Studies, which performed for calculation of trajectories of movement parts of connecting rod bearings for engines Ch 12/12 and ChN 21/21 by rapid BDF-method shown that the computation time compared with the BDF method for equations of the 1st order twice decreased, while the number of integrations of Reynolds equation was respectively: 1531 and 5931, (engine Ch 12/12); 1497 and 5621 (engine ChN 21/21).

Results of calculation of the connecting-rod bearing of "Ruston & Hornsby 6VEBX MK-III" engine as an international standard for testing [5] are presented in Table. We analyzed main output hydromechanical characteristics of journal bearings for piston engines, which include: the minimal (\(h_{\text{min}}\)) and average (\(h^*_{\text{min}}\)) film thicknesses of the loading cycle; the maximum (\(p_{\text{max}}\)) and average (\(p^*_{\text{max}}\)) hydrodynamic pressures in a lubricant film; the integrated average friction losses \(N^*\) for a loading cycle. These parameters indirectly characterize wear resistance, durability and reliability of tribosystems of piston engines.

As can be seen from Table, algorithm 2 is the most efficient. Compared with the classical algorithm 1 solution time is reduced more than two time, and the number of integrations of Reynolds equation for calculation step – by 2.5 times.

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